Thanks for inviting me to Les Diablerets!

ωεβ:=http://drorbn.net/ld27/

# 2D Intersection Theory and its Homological Model

**Abstract.** When you intersect half-dimensional cycles in 4D or higher, you get a count. In 2D, half-dimensional cycles can be replaced by paths and they and their intersections carry homotopical information. After inquiring why we should care, we ponder if this intersection theory has a graded model, and an expansion, much as the Taylor series is an expansion of smooth functions into their graded model, the space of power series.

The tentative affirmative answer uses Drinfeld associators and/or solutions of the KV equations, and makes drunken port calls at "emergent" knots in 3D which barely know knottedness, at truly knotted knots in 4D, at a theory of convolutions on finite dimensional Lie groups and algebras, and maybe in the theory of quantum groups. God was having a good laugh when she threw all this at us without a user's manual.

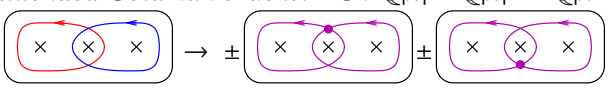


With Z. Dancso and others. Thanks, T. Hogan, Y. Kuno, D. Lin, J. Liu, N. Odins-Jones, J. Richardson, & N. Scherich.

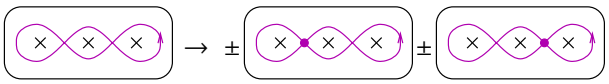
**Disclaimer.**

**2D Intersection Theory** carries homotopy information and not just counting! Let $D_p$ be the disk with $p$ punctures, let $\pi := \pi_1(D_p) = FG(\gamma_1,\dots,\gamma_p)$, and let $|\pi| := \pi/(g \sim h^{-1}gh)$ and $|\pi\times\pi| := \pi\times\pi/((g_1,g_2)\sim(h^{-1}g_1h, h^{-1}g_2h))$.

The *extended Goldman bracket* $EG\colon \mathbb{Q}|\pi|\times\mathbb{Q}|\pi| \to \mathbb{Q}|\pi\times\pi|$ is:



The *extended Turaev cobracket* $ET\colon \mathbb{Q}|\pi| \to \mathbb{Q}|\pi\times\pi|$ is:



Well defined*!

**Applications.** $\emptyset$ *Kinks & R1 aside.

**The Conway Quotient** is defined by $\times \to \wr$, giving the *Goldman bracket* $G\colon \mathbb{Q}|\pi|\times\mathbb{Q}|\pi| \to \mathbb{Q}|\pi|$ and the *Turaev cobracket* $T\colon \mathbb{Q}|\pi| \to \mathbb{Q}|\pi|\times\mathbb{Q}|\pi|$ which together make the *Goldman-Turaev Lie bialgebra*. Lie bialgebras are super cool! Yet,

**Applications.** $\sim\emptyset$ (though check [Ch])

**The Taylor Expansion.** Consider $\mathcal{R} = C^\infty(\mathbb{R}^d)$, $\mathcal{I} = \{f : f(0) = 0\}$, $\mathcal{F}_k = \mathcal{I}^k$: ($\operatorname{gr}\mathcal{R} := \prod \mathcal{F}_k/\mathcal{F}_{k+1}$)

$$m \circlearrowright C^\infty(\mathbb{R}^d) = \mathcal{R} \underset{Z}{\overset{\operatorname{gr}}{\rightleftarrows}} \operatorname{gr}\mathcal{R} = \mathbb{R}[\![x_1,\dots,x_d]\!] \circlearrowright m$$

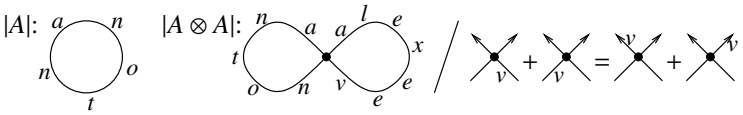
Here $Z$ can be the Taylor expansion. This is a repeating theme!

**The Free Group.** For $\mathcal{R} = \mathbb{Q}\pi$ with $\mathcal{I} = \langle\gamma_i - 1\rangle$, get

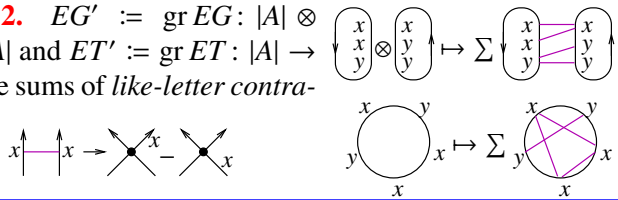
$$\iota \circlearrowright \mathcal{R} \underset{Z}{\overset{\operatorname{gr}}{\rightleftarrows}} \operatorname{gr}\mathcal{R} = FA\langle x_1,\dots,x_p\rangle = FA\langle H_1(D_p)\rangle =: A \circlearrowright \iota$$

Here $Z_1 : \gamma_i \mapsto 1 + x_i$ is famed ("the Magnus expansion") but not homomorphic and $Z_2 : \gamma_i \mapsto e^{x_i}$ fixes that.

**TTheorem 1.** We have that $\operatorname{gr}\mathbb{Q}|\pi| = |A| := FA(x_i)/(wx = xw)$ is "cyclic words", and $\operatorname{gr}\mathbb{Q}|\pi\times\pi| = |A\otimes A| := A\otimes A/(xu\otimes v + u\otimes xv = ux\otimes v + u\otimes vx)$.



**TTheorem 2.** $EG' := \operatorname{gr} EG\colon |A|\otimes|A| \to |A\otimes A|$ and $ET' := \operatorname{gr} ET\colon |A| \to |A\otimes A|$ are the sums of *like-letter contractions:*



**TTheorem 3.** There exists expansions $Z_1\colon \mathbb{Q}|\pi| \to |A|$ and $Z_2\colon \mathbb{Q}|\pi\times\pi| \to |A\otimes A|$ that intertwine the extended Goldman-Turaev operations:

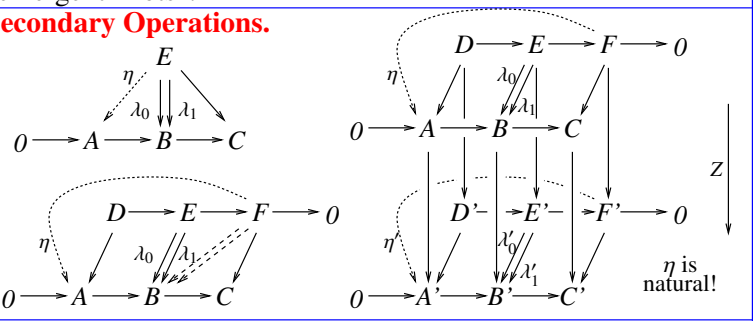
$$\begin{array}{ccc} \mathbb{Q}|\pi|\otimes\mathbb{Q}|\pi| & \xrightarrow{EG} & \mathbb{Q}|\pi\otimes\pi| \\ {\scriptstyle Z_1\otimes Z_1}\downarrow & & \downarrow{\scriptstyle Z_2} \\ |A|\otimes|A| & \xrightarrow{EG'} & |A\otimes A| \end{array} \qquad \begin{array}{ccc} \mathbb{Q}|\pi| & \xrightarrow{ET} & \mathbb{Q}|\pi\otimes\pi| \\ {\scriptstyle Z_1}\downarrow & & \downarrow{\scriptstyle Z_2} \\ |A| & \xrightarrow{ET'} & |A\otimes A| \end{array}$$

Furthermore, such expansions exist that descend to the Conway quotient and hence define an expansion of the Goldman-Turaev Lie bialgebra into the Schedler necklace Lie bialgebra, reproducing results of [Ma] and [AKKN].

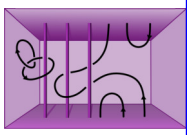
**Applications.** $\emptyset$

Our "pole dancing" group [BDHLS] reproduced the [Ma]/[AKKN] results using "secondary operations" and "emergent knots".

**Secondary Operations.**

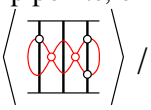



**Emergent Knotting.** Note that the statement "Dror has emergent French proficiency" means "Dror's French is just barely above 0". All knotting occurs in $PDS_p := D_p\times I$, the pole dancing studio with $p$ poles. $S$ is a *skeleton* – on the right, it is "two circles and 3 arcs with specified endpoints".



**Definitions.** • $\mathcal{K} = \mathcal{K}_0 = \mathcal{K}_0^0 = \mathcal{K}(S) := \mathbb{Q}\langle$tangles in $PDS_p\rangle$.

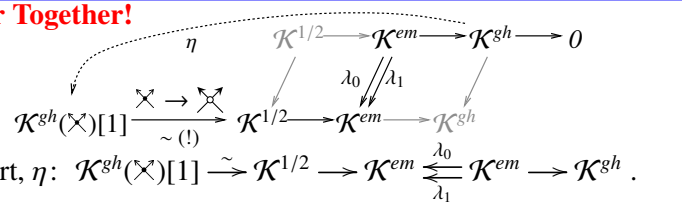
• $\mathcal{K}_t^s$ :=(the image via $\times \to \times - \times$ of tangles in $PDS_p$ that have $t$ flip points, of which $s$ are strand-strand).

E.g., $\mathcal{K}_5^2(\bigcirc) = \left\langle \right.$  $\left.\right\rangle /. \times \to \times - \times$

• $\mathcal{K}^{/s} := \mathcal{K}/\mathcal{K}^s$. In particular, $\mathcal{K}^{/1} =: \mathcal{K}^{gh}$ ("gh" for "ghosted") is just homotopy with $P\colon \mathcal{K}^{/1}(\bigcirc) \tilde{\to} \mathbb{Q}|\pi|$ and $P\colon \mathcal{K}^{/1}($  $) \tilde{\to} \mathbb{Q}|\pi\times\pi|$. Also, $\mathcal{K}^{/2} =: \mathcal{K}^{em}$ ("em" for "emergent").

• $\mathcal{A} := \prod \mathcal{K}_t/\mathcal{K}_{t+1}$, $\mathcal{A}^s := \prod \mathcal{K}_t^s/\mathcal{K}_{t+1}^s \subset \mathcal{A}$, $\mathcal{A}^{/s} := \mathcal{A}/\mathcal{A}^s$.

**Better Together!**

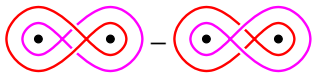


In short, $\eta\colon \mathcal{K}^{gh}(\times)[1] \xrightarrow{\sim} \mathcal{K}^{1/2} \to \mathcal{K}^{em} \underset{\lambda_1}{\overset{\lambda_0}{\rightrightarrows}} \mathcal{K}^{em} \to \mathcal{K}^{gh}$.

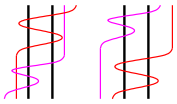
There is a similar diagram at the $\mathcal{A} = \operatorname{gr}\mathcal{K}$ level.



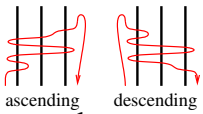

**TExample 1.** With $\gamma_1 \in |\pi|$ and $\gamma_2 \in |\pi|$ set $\lambda_0(\gamma_1,\gamma_2) = \tilde\gamma_1 \cdot \tilde\gamma_2$ and $\lambda_1(\gamma_1,\gamma_2) = \tilde\gamma_2 \cdot \tilde\gamma_1$ where $\tilde\gamma_i$ are arbitrary lifts of $\gamma_i$. Then $\eta_1$ is the extended Goldman bracket $EG$!

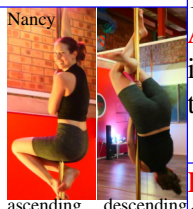


**TExample 2.** With $\gamma_1,\gamma_2 \in \pi$ and with $\lambda_0,\lambda_1$ as on the right, we get the "extended double bracket" $\eta_2 : \mathbb{Q}(\pi\times\pi) \to \mathbb{Q}(\pi\times\pi\times\pi)$.

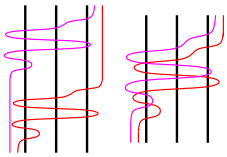


**TExample 3.** With $\gamma \in \pi$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3 : \mathbb{Q}\pi \to \mathbb{Q}(\pi\times\pi)$. Closing at the bottom and anti-symmetrizing, this is the extended Turaev cobracket $ET$.



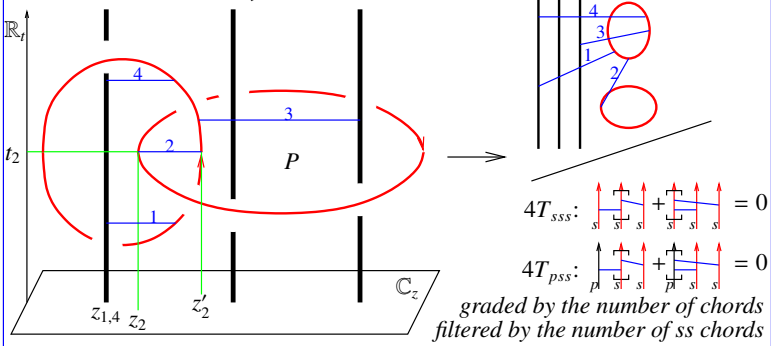

**TExample 4** [Ma, Ri]. With $\gamma \in \pi$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4 : \mathbb{Q}\pi \to \mathbb{Q}(\pi\otimes\pi\otimes\pi)$. After some massaging, it too becomes the extended Turaev cobracket $ET$.



**Kontsevich in a Pole Dance Studio.**(w/o poles? See [Ko, BN1])

$$Z = \left( \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \sum_{\substack{t_1<\ldots<t_m \\ P=\{(z_i,z_i')\}}} (-1)^{\#P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \right)^{\sim} \in \mathcal{A}$$









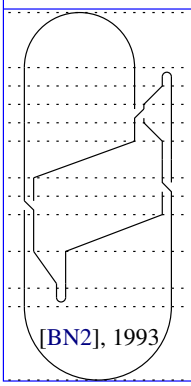
*graded by the number of chords*
*filtered by the number of ss chords*

**TProof of TTheorem 3.** With $\eta_3 \sim ET$ as in TExample 3,

$$\begin{array}{ccccccccc}
\eta_3 : \mathcal{K}^{gh}\left(\circlearrowright\right)[1] & \xrightarrow{\sim} & \mathcal{K}^{1/2}(\cap) & \longrightarrow & \mathcal{K}^{em}(\cap) & \underset{\lambda_1}{\overset{\lambda_0}{\rightleftarrows}} & \mathcal{K}^{em}(\cap) & \longrightarrow & \mathcal{K}^{gh}(\cap) \\
\downarrow Z & & \downarrow Z & & \downarrow Z & & \downarrow Z & & \downarrow Z \\
\eta_3' : \mathcal{A}^{gh}\left(\circlearrowright\right)[1] & \xrightarrow{\sim} & \mathcal{A}^{1/2}(\cap) & \longrightarrow & \mathcal{A}^{em}(\cap) & \underset{\lambda_1'}{\overset{\lambda_0'}{\rightleftarrows}} & \mathcal{A}^{em}(\cap) & \longrightarrow & \mathcal{A}^{gh}(\cap)
\end{array}$$

The ☠ means: "fails to commute yet in a controlled way". That added 7 pages to [BDHLS], but we'll whitewash it.

**Issues.** • Oh no, analysis! • The Kontsevich integral is a huge overkill; it's for knots, not just for emergent knots.

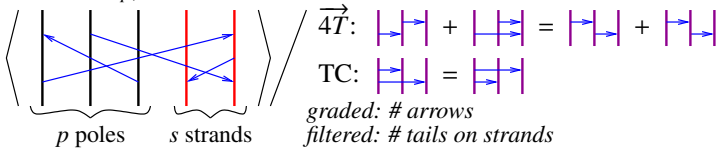



**The Obvious Solution.** Use "Associators in XXX". Associators: ⌿ ↦ Φ, must satisfy "the pentagon ⬠ and the hexagon ⬡ equations". XXX: Some simpler target. Must be rich enough to express ⬠&⬡, large enough to carry $|A|$, $|A\times A|$, and true enough to retain the commutativities. The ☠ square requires $\Phi \in \mathrm{im}(\alpha\circ\iota)$:

$$\begin{array}{ccc}
\mathcal{K}^{braids} & \xrightarrow{\ i\ } & \mathcal{K}^{em} \\
\Downarrow \mathrm{gr} & & \\
\mathcal{A}^{braids} & \xrightarrow{\ \iota\ } & \mathcal{A}^{em} \xrightarrow{\ \alpha\ } XXX
\end{array}$$

2026!

**XXX.** Let $\mathcal{A}^w_{p,s}$ be the space studied in [BD]:




*graded: # arrows*
*filtered: # tails on strands*

There is a map $\alpha : \mathcal{A} \to \mathcal{A}^w$ by $(\text{—}) \mapsto (\leftarrow) + (\rightarrow)$. Let $\mathcal{A}^{wgh} := \mathcal{A}^w/\mathcal{A}^{w/1}$ ("no tails on strands") and $\mathcal{A}^{wem} := \mathcal{A}^w/\mathcal{A}^{w/2}$ ("at most 1 tail on strands").

**ATTheorem 4.** $\mathcal{A}^{wem}$ is an XXX and an associator $\Phi \in \mathrm{im}(\alpha\circ\iota) \subset \mathcal{A}^{wem}_{2,1}$ (all we need!) is equivalent to a solution of the KV equations [KV, AT]:

(1) $F(x+y) = \mathrm{bch}(x,y)$ (2) $j(F) = a(x) + a(y) - a(\mathrm{bch}(x,y))$

**Homework.** • To what extent at the KV equations *necessary* for EG/ET expansions? • Can you fit in here the topology of 2-knots in $\mathbb{R}^4$ ([BD])? • Why is this related to convolutions on Lie algebras and Lie groups [KV]? • And to quantizations of $\mathcal{U}(I\mathfrak{g} := \mathfrak{g} \ltimes \mathfrak{g}^*)$ [OJ]?

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**AI Statement.** Since September 2025 I am unable to draw a sharp line separating the intellectual ability of humans from the intellectual ability of machines.



Image credits: ChatGPT

Seeing that the former is roughly constant and the latter is rapidly rising, very soon the latter will vastly exceed the former, if it hasn't yet. Thus soon biological brains and hence humans will no longer be necessary for those who control the machines, be it a few powerful humans or the machines themselves. Economic reality regrettably implies that those who are unnecessary will shrink and starve. I believe this is the gravest danger presently facing most humans.

I believe "safe AI" is wishful thinking, easily subverted.

Can AI and HI coexist? Like those jets that have bicycle pedals for extra power? Wait, that's absurd.

Thus I believe that the only solution, unlikely and temporary as it may be, is to pull the plug fully and completely. Yesterday is better than today.

Yet while AI is here and while I still can, I will use it to my advantage. I don't believe in sainthood and falling behind helps no one[1].

Students: Shamelessly use it when you can. Yet remember it's an enemy and dream of its demise.

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A classic prisoner's dilemma, isn't it? These are more common than most people appreciate.