DIAGRAMMATIC OBJECTS PINE FOR SPECIALIZATIONS. AND THE TURAEV COBRACKET?

Let Σ be a compact oriented surface with first homology $H = H_1(\Sigma; \mathbb{Q})$. The $\mathbb{Q}$-vector space spanned by free homotopy classes of loops on Σ carries the Goldman bracket $\{\ ,\ \}$ and the Turaev cobracket δ, forming the Goldman–Turaev Lie bialgebra. After I-adic completion and passage to the associated graded with respect to the augmentation ideal of $\mathbb{Q}\pi_1(\Sigma)$, one obtains a graded Lie bialgebra on

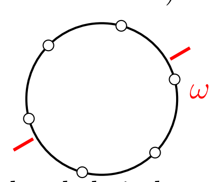
$$\mathrm{Cyc}(H) := T(H)/[T(H), T(H)],$$

where the grading is by word length.

1. The graded cobracket. Write $\mathrm{Cyc}^m(H)$ for cyclic words of length m. The leading term

$$\delta_{-1} : \mathrm{Cyc}^m(H) \longrightarrow \bigoplus_{p+q=m-1} \mathrm{Cyc}^p(H) \otimes \mathrm{Cyc}^q(H)$$

is of degree -1: it lowers total length by one. For genus 0, δ_{-1} is the necklace cobracket of Schedler: it is given by summing over ordered pairs of positions in a cyclic word, inserting the intersection form and cutting the word into two cyclic pieces. (Any explicit formula we write should reflect this degree -1 behavior.)



The graded cobracket cuts a beaded circle and contracts one pair of beads, lowering degree by 1.

2. What is known: noncommutative avatars. For suitable group-like ("special") expansions, the Goldman bracket and Turaev cobracket admit algebraic descriptions in terms of Van den Bergh-style double brackets and a noncommutative divergence cocycle. In genus 0, Alekseev–Kawazumi–Kuno–Naef show that δ can be reconstructed from such data, and that choosing expansions compatible with the framed Turaev cobracket is essentially equivalent to solving the Kashiwara–Vergne problem. Relatedly, Kawazumi and Hain interpret parts of regular-homotopy and framed versions of δ via divergence-type cocycles.

3. What is missing. These descriptions are powerful but not yet a single, canonical "geometric divergence operator on a moduli space" that one could point to and say, without qualifiers,

"the Turaev cobracket *is* this specialization".

The graded Turaev cobracket sits as a natural diagrammatic operation of degree -1, beautifully constrained and algebraically understood in several frameworks, but its interpretation as a straightforward geometric or Lie-theoretic operation remains, at best, indirect and expansion-dependent.

4. Tagline. *Diagrammatic objects pine for specializations. For the Turaev cobracket, the wait is not convincingly over.*