DIAGRAMMATIC OBJECTS, EXPANSIONS, AND THE GRADED TURAEV COBRACKET AS DIVERGENCE

Let Σ be a compact oriented surface with first homology $H = H_1(\Sigma; \mathbb{Q})$ carrying its intersection form $\omega \in \Lambda^2 H$. The vector space spanned by free homotopy classes of loops on Σ carries the *Goldman–Turaev Lie bialgebra* structure: the bracket records signed intersections, and the cobracket records self–intersections. After completing and grading by powers of the augmentation ideal of $\pi_1(\Sigma)$, a *symplectic expansion*

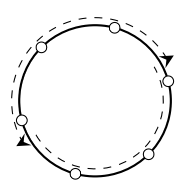
$$Z : \widehat{\mathbb{Q}[\pi_1(\Sigma)]} \longrightarrow \widehat{T}(H)$$

identifies the associated graded with cyclic words $\mathrm{Cyc}(H) = T(H)/[T(H), T(H)]$.

1. The graded cobracket. Under Z, the graded Turaev cobracket is the operator

$$\delta_T(a_1 \cdots a_n) = \sum_{i<j} \omega^{ij}\,(a_{i+1} \cdots a_j) \otimes (a_{j+1} \cdots a_i),$$

where ω^{ij} are the components of ω^{-1} in a symplectic basis. Diagrammatically, δ_T "cuts" a beaded circle and inserts ω at the cut.



A beaded circle split into two by inserting the symplectic form ω.

2. The interpretation map. For a quadratic Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$, each $h \in H$ acts by contraction with the corresponding infinitesimal holonomy variable. This yields an *interpretation map*

$$I : \mathrm{Cyc}(H) \longrightarrow \mathrm{Inv}\big(S(\mathfrak{g})\big),$$

sending a cyclic word $a_1 \cdots a_n$ to the invariant tensor obtained by replacing each a_i with the corresponding element of $\mathfrak{g}$ and contracting indices using $\langle\ ,\ \rangle$ and ω.

3. The differential–geometric avatar. A theorem of Alekseev–Kawazumi–Kuno–Naef (*Invent. Math.* 2023) asserts that, for a symplectic expansion Z, the diagram

$$\begin{array}{ccc} \mathrm{Cyc}(H) & \xrightarrow{\delta_T} & \mathrm{Cyc}(H)\otimes\mathrm{Cyc}(H) \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle I\otimes I} \\ \mathrm{Inv}\big(S(\mathfrak{g})\big) & \xrightarrow{\mathrm{div}} & \mathrm{Inv}\big(S(\mathfrak{g})\big)\otimes\mathrm{Inv}\big(S(\mathfrak{g})\big) \end{array}$$

commutes, where div is the *divergence operator*—the trace of the covariant derivative with respect to the canonical symplectic connection on the moduli of flat $\mathfrak{g}$–connections.

4. Aphorism. *Diagrammatic objects pine for specializations: under the interpretation map, the beads of the Turaev cobracket become tensor indices, and the cobracket itself becomes the divergence operator.*