Introducing the q-Casimir w to all orders and expressing and computing the invariant in terms of it.

Pensieve header: By Roland, June 20, 2020.

In[ ]:=
```
Once[<< KnotTheory`];
```

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[ ]:=
```
PP_ = Identity; $k = 0; γ = 1; ℏ = 1;
```

The "Speedy" Engine

Internal Utilities

Canonical Form:

In[ ]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CCF[x]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := Module[
  {vs = Cases[ℰ, (y | b | t | a | w | x | η | β | τ | α | ω | ξ)_, ∞] ⋃
     {y, b, t, a, w, x, η, β, τ, α, ω, ξ}},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] (Times @@ vs^ps)]
  ];
CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

In[ ]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

In[ ]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

```
In[•]:= {t*, b*, y*, a*, w*, x*, z*} = {τ, β, η, α, ω, ξ, ζ};
        {τ*, β*, η*, α*, ω*, ξ*, ζ*} = {t, b, y, a, w, x, z}; (u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

```
In[•]:= U2l = {B_i_^p_. :→ e^(-p ℏ γ b_i), B^p_. :→ e^(-p ℏ γ b), T_i_^p_. :→ e^(-p ℏ t_i),
          T^p_. :→ e^(-p ℏ t), A_i_^p_. :→ e^(p γ α_i), A^p_. :→ e^(p γ α), Ω_i_^p_. :→ e^(p ω_i), Ω^p_. :→ e^(p ω)};
        l2U = {e^(c_. b_i_+d_.) :→ B_i^(-c/(ℏ γ)) e^d, e^(c_. b+d_.) :→ B^(-c/(ℏ γ)) e^d,
          e^(c_. t_i_+d_.) :→ T_i^(-c/ℏ) e^d, e^(c_. t+d_.) :→ T^(-c/ℏ) e^d,
          e^(c_. α_i_+d_.) :→ A_i^(c/γ) e^d, e^(c_. α+d_.) :→ A^(c/γ) e^d,
          e^(c_. ω_i_+d_.) :→ Ω_i^c e^d, e^(c_. ω+d_.) :→ Ω^c e^d,
          e^ℰ_ :→ e^(Expand@ℰ)};
```

Derivatives in the presence of exponentiated variables:

```
In[•]:= D_b[f_] := ∂_b f - ℏ γ B ∂_B f; D_b_i_[f_] := ∂_b_i f - ℏ γ B_i ∂_B_i f;
        D_t[f_] := ∂_t f - ℏ T ∂_T f; D_t_i_[f_] := ∂_t_i f - ℏ T_i ∂_T_i f;
        D_α[f_] := ∂_α f + γ A ∂_A f; D_α_i_[f_] := ∂_α_i f + γ A_i ∂_A_i f;
        D_ω[f_] := ∂_ω f + Ω ∂_Ω f; D_ω_i_[f_] := ∂_ω_i f + Ω_i ∂_Ω_i f;
        D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f; D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
        D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

Finite Zips:

```
In[•]:= collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
        collect[ℰ_, ζ_] := Collect[ℰ, ζ];
        Zip_{}[P_] := P;
        Zip_ζs_[Ps_List] := Zip_ζs /@ Ps;
        Zip_{ζ_,ζs___}[P_] :=
          (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :→ (D_{ζ*,d}[f])) /. ζ* → 0 /.
           ((ζ* /. {b → B, t → T, α → A, ω → Ω}) → 1)
```

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}|\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i}\Big|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde{q}|e^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

In[∘]:=
```
QZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
   zs = Table[ζ*, {ζ, ζs}];
   c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
   ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → CF[qt.(zs + ys)]];
   ζrule = Thread[ζs → ζs + ηs.qt];
   CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]] ];
```

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^Q as a single"P". Here the z's are b and α and the ζ's are β and a.

In[∘]:=
```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
   (*Print["LZip"];*)
   zs = Table[ζ*, {ζ, ζs}];
   Zs = zs /. {b → B, t → T, α → 𝒜, ω → Ω};
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   Zrule = Join[zrule, zrule /.
      r_Rule :> ((U = r[[1]] /. {b → B, t → T, α → 𝒜, ω → Ω}) → (U /. U2l /. r //. l2U))];
   ζrule = Thread[ζs → ζs + ηs.lt];
   Q1 = Q /. (Zrule ⋃ ζrule);
   EEQ[ps___] := EEQ[ps] =
     (CF[e^{-Q1} D_{Thread[{zs,{ps}}]}[e^{Q1}]] /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
   CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
     Det[lt] (Zip_{ζs}[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
        Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1) ]];
```

In[∘]:=
```
TZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, Zs, c, ys, ηs,
    lt, zrule, Zrule, ζrule, Q1, EEQ, EQ, Lnew = L, Qnew = Q, Pnew = P},
   zrule = Table[ζ* → Coefficient[L, ζ], {ζ, ζs}];
   (*Print["Tzip"];*)
   ζrule = Table[ζ → 0, {ζ, ζs}];
   Lnew = L /. U2l /. zrule /. ζrule;
   Qnew = Q /. U2l /. zrule /. ζrule; (**)
   Pnew = P /. U2l /. zrule /. ζrule;
   CF@(𝔼[Lnew, Qnew, Pnew] //. l2U)
  ];
```

In[◦]:=
```
B{} [L_, R_] := L R;
B{is__} [L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : b | B | t | T | a | w | x | y)i → vn@i, {i, {is}}],
     R /. Table[(v : β | τ | α | 𝒜 | ω | Ω | ξ | η)i → vn@i, {i, {is}}]
    ] // TZipJoin@@Table[{τn@i},{i,{is}}] // LZipJoin@@Table[{wn@i,βn@i,an@i},{i,{is}}] //
   QZipJoin@@Table[{ξn@i,yn@i},{i,{is}}] ];(**)
Bis___ [L_, R_] := B{is} [L, R];
```

𝔼 morphisms with domain and range.

In[◦]:=
```
Bis_List [𝔼d1_→r1_ [L1_, Q1_, P1_], 𝔼d2_→r2_ [L2_, Q2_, P2_]] :=
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ Bis [𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼d1_→r1_ [L1_, Q1_, P1_] // 𝔼d2_→r2_ [L2_, Q2_, P2_] :=
  Br1⋂d2 [𝔼d1→r1 [L1, Q1, P1], 𝔼d2→r2 [L2, Q2, P2]];
𝔼d1_→r1_ [L1_, Q1_, P1_] ≡ 𝔼d2_→r2_ [L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼d1_→r1_ [L1_, Q1_, P1_] 𝔼d2_→r2_ [L2_, Q2_, P2_] ^:=
  𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼dr_ [L_, Q_, P_]$k_ := 𝔼dr @@ 𝔼[L, Q, P]$k;
𝔼_[ℰ___][i_] := {ℰ}[[i]];
```

𝔼[Λ]

In[◦]:=
```
𝔼dr_ [Λ_] := CF@
  Module[{L, Λ0 = Limit[Λ, ϵ → 0]}, 𝔼dr[L = Λ0 /. (η | y | ξ | x)_ → 0, Λ0 - L, e^(Λ-Λ0)]$k /. l2U]
```

“Define” Code

Define[lhs = rhs, …] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

In[◦]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[opnisp,$k_Integer, Block[{i, j, k}, opisp,$k = ℰ; opnis,$k]];
      SD[opisp, op{is},$k]; SD[opsis__, op{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

Symmetric Algebra Objects

In[∘]:=
```
sm_{i_,j_→k_} := 𝔼_{{i,j}→{k}}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) + y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_{i_→j_,k_} := 𝔼_{{i}→{j,k}}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) + η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_{i_} := 𝔼_{{i}→{i}}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sϵ_{i_} := 𝔼_{{}→{i}}[0];
sη_{i_} := 𝔼_{{i}→{}}[0];
```

In[∘]:=
```
sσ_{i_→j_} := 𝔼_{{i}→{j}}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sϒ_{i_→j_,k_,l_,m_} := 𝔼_{{i}→{j,k,l,m}}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

Booting Up QU

In[∘]:=
```
Define[aσ_{i→j} = 𝔼_{{i}→{j}}[a_j α_i + x_j ξ_i], bσ_{i→j} = 𝔼_{{i}→{j}}[b_j β_i + y_j η_i]]
```

In[∘]:=
```
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k + (𝒜_j^{-1} ξ_i + ξ_j) x_k],
  bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k + (η_i + e^{-ϵ β_i} η_j) y_k]]
```

Three types of inverses appear below!

$\overline{R}$ is the inverse of R in the algebra $\mathbb{B}\otimes\mathbb{A}$.

P is the inverse of R as a quadratic form, like how an element of $V^*\otimes V^*$ can be the inverse of an element of $V\otimes V$.

$\overline{\text{aS}}$ is the inverse of aS as an operator form, like how an element of $V^*\otimes V$ can be the inverse of another element of $V^*\otimes V$.

In[∘]:=
```
Define[R_{i,j} = 𝔼_{{}→{i,j}}[ℏ a_j b_i + Sum[((1 - e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k) / (k (1 - e^{k γ ϵ ℏ})), {k, 1, $k+1}]],
  R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i, 1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_{$k}[3] -
      (((R̄_{{i,j},0})_{$k} R_{1,2} (R̄_{{3,4},$k-1})_{$k}) // (bm_{i,1→i} am_{j,2→j}) // (bm_{i,3→i} am_{j,4→j}))[3]]],
  P_{i,j} = 𝔼_{{i,j}→{}}[β_i α_j / ℏ, η_i ξ_j / ℏ, 1 + If[$k == 0, 0, (P_{{i,j},$k-1})_{$k}[3] -
     (R_{1,2} // ((P_{{1,j},0})_{$k} (P_{{i,2},$k-1})_{$k}))[3]]]]
```

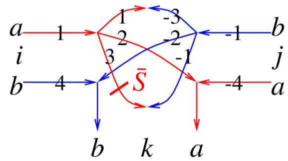
In[∘]:=
```
Define[aS_i = (aσ_{i→2} R̄_{1,i}) // P_{1,2},
  aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i, 1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_{$k}[3] -
      ((aS̄_{{i},0})_{$k} // aS_i // (aS̄_{{i},$k-1})_{$k})[3]]]]
```

(was $\text{aS}_j = \overline{R}_{i,j} \sim B_i \sim P_{i,j}$).

In[∘]:=
```
Define[bS_i = bσ_{i→1} R_{i,2} // aS_2 // P_{1,2},
  bS̄_i = bσ_{i→1} R_{i,2} // aS̄_2 // P_{1,2},
  aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{3,i},
  bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{i,3}]
```

(was $\mathrm{bS}_i = R_{i,1} \sim B_1 \sim \mathrm{aS}_1 \sim B_1 \sim P_{i,1}$, $\overline{\mathrm{bS}}_i = R_{i,1} \sim B_1 \sim \overline{\mathrm{aS}}_1 \sim B_1 \sim P_{i,1}$).

The Drinfel'd double:



In[∘]:=

$$\mathtt{Define}\Big[\mathtt{dm}_{i,j\to k} = \Big(\big(\mathtt{s\Upsilon}_{i\to 4,4,1,1} \;//\; \mathtt{a\Delta}_{1\to 1,2} \;//\; \mathtt{a\Delta}_{2\to 2,3} \;//\; \overline{\mathtt{aS}}_3\big)\;\big(\mathtt{s\Upsilon}_{j\to -1,-1,-4,-4} \;//\; \mathtt{b\Delta}_{-1\to -1,-2} \;//\; \mathtt{b\Delta}_{-2\to -2,-3}\big)\Big) \;//\; \big(\mathtt{P}_{-1,3}\,\mathtt{P}_{-3,1}\,\mathtt{am}_{2,-4\to k}\,\mathtt{bm}_{4,-2\to k}\big)\Big]$$

In[∘]:=

$$\begin{aligned}&\mathtt{Define}\Big[\mathtt{d\sigma}_{i\to j} = \mathtt{a\sigma}_{i\to j}\,\mathtt{b\sigma}_{i\to j},\\ &\mathtt{d\epsilon}_i = \mathtt{s\epsilon}_i,\ \ \mathtt{d\eta}_i = \mathtt{s\eta}_i,\\ &\mathtt{dS}_i = \mathtt{s\Upsilon}_{i\to 1,1,2,2} \;//\; \big(\overline{\mathtt{bS}}_1\,\mathtt{aS}_2\big) \;//\; \mathtt{dm}_{2,1\to i},\\ &\overline{\mathtt{dS}}_i = \mathtt{s\Upsilon}_{i\to 1,1,2,2} \;//\; \big(\mathtt{bS}_1\,\overline{\mathtt{aS}}_2\big) \;//\; \mathtt{dm}_{2,1\to i},\\ &\mathtt{d\Delta}_{i\to j,k} = \big(\mathtt{b\Delta}_{i\to 3,1}\,\mathtt{a\Delta}_{i\to 2,4}\big) \;//\; \big(\mathtt{dm}_{3,4\to k}\,\mathtt{dm}_{1,2\to j}\big)\Big]\end{aligned}$$

In[∘]:=

$$\begin{aligned}&\mathtt{Define}\Big[\mathtt{C}_i = \mathbb{E}_{\{\}\to\{i\}}\big[0,\,0,\,\mathtt{B}_i^{1/2}\,e^{-\hbar\,\epsilon\,a_i/2}\big]_{\$k},\\ &\overline{\mathtt{C}}_i = \mathbb{E}_{\{\}\to\{i\}}\big[0,\,0,\,\mathtt{B}_i^{-1/2}\,e^{\hbar\,\epsilon\,a_i/2}\big]_{\$k},\\ &\mathtt{Kink}_i\ =\ \big(\mathtt{R}_{1,3}\,\overline{\mathtt{C}}_2\big) \;//\; \mathtt{dm}_{1,2\to 1} \;//\; \mathtt{dm}_{1,3\to i},\\ &\overline{\mathtt{Kink}}_i\ =\ \big(\overline{\mathtt{R}}_{1,3}\,\mathtt{C}_2\big) \;//\; \mathtt{dm}_{1,2\to 1} \;//\; \mathtt{dm}_{1,3\to i}\Big]\end{aligned}$$

Note. $t = -\epsilon \mathrm{a} + \gamma \mathrm{b}$ and $b = t/\gamma + \epsilon \mathrm{a}/\gamma$

In[∘]:=

$$\begin{aligned}&\mathtt{Define}\Big[\mathtt{b2t}_i = \mathbb{E}_{\{i\}\to\{i\}}\big[\alpha_i\,a_i + \beta_i\,(\epsilon\,a_i + t_i)\,/\,\gamma + \xi_i\,x_i + \eta_i\,y_i\big],\\ &\mathtt{t2b}_i = \mathbb{E}_{\{i\}\to\{i\}}\big[\alpha_i\,a_i + \tau_i\,(-\epsilon\,a_i + \gamma\,b_i) + \xi_i\,x_i + \eta_i\,y_i\big]\Big]\end{aligned}$$

The t-Tensors

In[∘]:=

$$\begin{aligned}&\mathtt{Define}\Big[\mathtt{tR}_{i,j} = \mathtt{R}_{i,j} \;//\; \big(\mathtt{b2t}_i\,\mathtt{b2t}_j\big),\\ &\overline{\mathtt{tR}}_{i,j} = \overline{\mathtt{R}}_{i,j} \;//\; \big(\mathtt{b2t}_i\,\mathtt{b2t}_j\big),\\ &\mathtt{tm}_{i,j\to k} = \big(\big(\mathtt{t2b}_i\,\mathtt{t2b}_j\big) \;//\; \mathtt{dm}_{i,j\to k} \;//\; \mathtt{b2t}_k\big),\\ &\mathtt{tC}_i = \big(\mathtt{C}_i \;//\; \mathtt{b2t}_i\big),\\ &\overline{\mathtt{tC}}_i = \big(\overline{\mathtt{C}}_i \;//\; \mathtt{b2t}_i\big),\\ &\mathtt{tKink}_i = \mathtt{Kink}_i \;//\; \mathtt{b2t}_i,\\ &\overline{\mathtt{tKink}}_i = \overline{\mathtt{Kink}}_i \;//\; \mathtt{b2t}_i,\\ &\mathtt{t\Delta}_{i\to j,k} = \mathtt{t2b}_i \;//\; \mathtt{d\Delta}_{i\text{->}j,k} \;//\; \big(\mathtt{b2t}_j\,\mathtt{b2t}_k\big),\\ &\mathtt{tS}_i = \mathtt{t2b}_i \;//\; \mathtt{dS}_i \;//\; \mathtt{b2t}_i\Big]\end{aligned}$$

Use the central variable $\mathrm{w} = \frac{1}{2} + \mathrm{a} + \frac{\mathrm{x\,y}}{1-\mathrm{T}}$

In[∘]:= $\text{a2w}_{i\_} := \mathbb{E}_{\{i\}\to\{i\}}\left[\tau_i\, t_i + \alpha_i \left(\frac{-1}{2} + w_i\right),\ \left(e^{-\alpha_i} - 1\right) \frac{y_i\, x_i}{1 - T_i} + \xi_i\, x_i + \eta_i\, y_i,\ 1\right]$

$\text{w2a}_{i\_} := \mathbb{E}_{\{i\}\to\{i\}}\left[\tau_i\, t_i + \left(a_i + \frac{1}{2}\right) \omega_i,\ \frac{\left(1 - e^{-\omega_i}\right)}{1 - T_i}\, y_i\, x_i + \xi_i\, x_i + \eta_i\, y_i,\ 1\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{i,j\}}[0, 0, x_i\, y_j - x_j\, y_i]$ // $\text{wm}_{i,j\to k}$

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}[0, 0, 1]$

Up to some notational annoyance the kink is exp(tw)

In[∘]:= tKink_i // a2w_i
$\overline{\text{tKink}}_i$ // a2w_i

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[-\frac{t_i}{2} + t_i\, w_i,\ 0,\ \frac{1}{\sqrt{T_i}} + O[\epsilon]^1\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[\frac{t_i}{2} - t_i\, w_i,\ 0,\ \sqrt{T_i} + O[\epsilon]^1\right]$

The R-matrix becomes complicated! I also rescaling x by xnew = $(1 - T)^{-1}$xold will help

In[∘]:= $\text{tR}_{i,j}$ // a2w_i // a2w_j
$\overline{\text{tR}}_{i,j}$ // a2w_i // a2w_j
w2a_i // w2a_j // $\text{tm}_{i,j\to k}$ // a2w_k
tC_i // a2w_i
$\overline{\text{tC}}_i$ // a2w_i

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[-\frac{t_i}{2} + t_i\, w_j,\ x_j\, y_i + \frac{\left(1 - T_i\right) x_j\, y_j}{-1 + T_j},\ 1 + O[\epsilon]^1\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[\frac{t_i}{2} - t_i\, w_j,\ -\frac{x_j\, y_i}{T_i} + \frac{\left(-1 + T_i\right) x_j\, y_j}{-T_i + T_i\, T_j},\ 1 + O[\epsilon]^1\right]$

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\left[t_k\, \tau_i + t_k\, \tau_j + w_k\, \omega_i + w_k\, \omega_j,\ y_k\, \eta_i + y_k\, \eta_j + x_k\, \xi_i + \left(1 - T_k\right) \eta_j\, \xi_i + x_k\, \xi_j,\ 1 + O[\epsilon]^1\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \sqrt{T_i} + O[\epsilon]^1\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \frac{1}{\sqrt{T_i}} + O[\epsilon]^1\right]$

So let's define the newly found building blocks independently: (recall T = exp(-t) so the annoying $-\frac{t}{2}$ in the L part is really Sqrt[T] in the P part.)
(not quite sure I did the rescaling correctly (why would it not be T_j?))

```
Define[
  wR_{i,j} = 𝔼_{{}→{i,j}}[t_i w_j, (1 - T_i) x_j y_i - (1 - T_i) x_j y_j, Sqrt[T_i]],
  wR̄_{i,j} = 𝔼_{{}→{i,j}}[-t_i w_j, (-(1 - T_i) x_j y_i + (1 - T_i) y_j x_j)/T_i, 1/Sqrt[T_i]],
  wC_i = 𝔼_{{}→{i}}[0, 0, Sqrt[T_i]],
  wC̄_i = 𝔼_{{}→{i}}[0, 0, 1/Sqrt[T_i]],
  wm_{i,j→k} = 𝔼_{{i,j}→{k}}[t_k τ_i + t_k τ_j + w_k ω_i + w_k ω_j, y_k η_i + y_k η_j + x_k ξ_i + η_j ξ_i + x_k ξ_j, 1 + O[ϵ]^1]
]
```

Almost matching Γ calculus

Checking Reidemeister 1: (it is satisfied up to an overall factor of $e^{+-w\,t}$)

```
In[•]:= wR_{1,2} wC̄_3 // wm_{1,3→1} // wm_{1,2→1}
        wR̄_{1,2} wC_3 // wm_{1,3→1} // wm_{1,2→1}
        wR̄_{1,2} wC̄_3 // wm_{2,3→2} // wm_{2,1→1}
        wR_{1,2} wC_3 // wm_{2,3→2} // wm_{2,1→1}
Out[•]= 𝔼_{{}→{1}}[t_1 w_1, 0, 1 + O[ϵ]^1]
Out[•]= 𝔼_{{}→{1}}[-t_1 w_1, 0, 1 + O[ϵ]^1]
Out[•]= 𝔼_{{}→{1}}[-t_1 w_1, 0, 1 + O[ϵ]^1]
Out[•]= 𝔼_{{}→{1}}[t_1 w_1, 0, 1 + O[ϵ]^1]
```

Checking Reidemeister 2:

```
In[•]:= wR_{1,2} wR̄_{3,4} // wm_{1,3->1} // wm_{2,4→2}
Out[•]= 𝔼_{{}→{1,2}}[0, 0, 1 + O[ϵ]^1]
```

Checking Reidemeister 3:

```
In[•]:= (wR_{1,2} wR_{4,3} wR_{5,6} // wm_{1,4→1} // wm_{2,5→2} // wm_{3,6→3}) ≡
         (wR_{2,3} wR_{1,6} wR_{4,5} // wm_{1,4→1} // wm_{2,5→2} // wm_{3,6→3})
Out[•]= True
```

Trefoil knot

```
In[•]:= (wR_{5,1} wR_{2,6} wR_{7,3} wC_4 // wm_{1,2→1} // wm_{1,3→1} // wm_{1,4→1} // wm_{1,5→1} // wm_{1,6→1} // wm_{1,7→1})
Out[•]= 𝔼_{{}→{1}}[3 t_1 w_1, 0, T_1/(1 - T_1 + T_1^2) + O[ϵ]^1]
```

Let's look at the product in Γ calculus style. Caution: variables γ and ϵ are in use, use g and e instead.
Taking the opposite product gives Γ calc. Provided the matrix A of Γ calculus is written as A=I+Q, where Q is the quadratic
actually used in Gaussian calculus.
First let's check out the crossings $wR_{1,2}$ and $\overline{wR}_{1,2}$ turn into the Γ calc values for the crossings. Except for

the annoying? Sqrt[T] factor.
but that's ok.

```
In[∘]:= Table[Coefficient[wR1,2[[2]], yi xj], {i, {1, 2}}, {j, {1, 2}}] + IdentityMatrix[2] //
          FullSimplify // MatrixForm
        Table[Coefficient[wR̄1,2[[2]], yi xj], {i, {1, 2}}, {j, {1, 2}}] + IdentityMatrix[2] //
          FullSimplify // Expand // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 1-T_1 \\ 0 & T_1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 1-\frac{1}{T_1} \\ 0 & \frac{1}{T_1} \end{pmatrix}$$

In[∘]:= (*We start with the matrix *) $A = \begin{pmatrix} \alpha & \beta & \theta \\ g & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}$; (*and scalar Omega*)

```
In[∘]:= (*Now form the relevant Q =A-I*)
        Q = {ya, yb, yS}.(A - IdentityMatrix[3]).{xa, xb, xS} // Expand
```

Out[∘]= $-x_a y_a + \alpha x_a y_a + \beta x_b y_a + \theta x_S y_a + g x_a y_b - x_b y_b + \delta x_b y_b + \epsilon x_S y_b + \phi x_a y_S + \psi x_b y_S - x_S y_S + \Xi x_S y_S$

```
In[∘]:= (*Compute the product in the Gaussian way but OPPOSITE*)
        ProdResult = 𝔼{}→{a,b,S}[0, Q, Omega] // wmb,a→c
```

Out[∘]= $\mathbb{E}_{\{\}\to\{c,S\}}\Big[0, \frac{1}{-1+\beta}(x_c y_c - g x_c y_c - \beta x_c y_c + g\beta x_c y_c - \alpha\delta x_c y_c - \epsilon x_S y_c + \beta\epsilon x_S y_c - \delta\theta x_S y_c - \phi x_c y_S + \beta\phi x_c y_S - \alpha\psi x_c y_S + x_S y_S - \beta x_S y_S - \Xi x_S y_S + \beta\Xi x_S y_S - \theta\psi x_S y_S), -\frac{\text{Omega}}{-1+\beta} + O[\epsilon]^1\Big]$

```
In[∘]:= (*Extract the resulting newQ*)
        NewQ = Table[Coefficient[ProdResult[[2]], yi xj], {i, {c, S}}, {j, {c, S}}];
        NewQ // MatrixForm;
        (*Form the newA = NewQ+I*)
        NewA = NewQ + IdentityMatrix[2] // FullSimplify;
        NewA // MatrixForm (*et voila, the golden standard comes out.*)
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} g-\frac{\alpha\delta}{-1+\beta} & \epsilon-\frac{\delta\theta}{-1+\beta} \\ \phi-\frac{\alpha\psi}{-1+\beta} & \Xi-\frac{\theta\psi}{-1+\beta} \end{pmatrix}$$