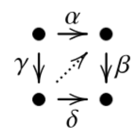
Pensieve header: The linear algebra preliminaries for the partial quadratic signature formalism for tangles; with Jessica Liu.

230109 **Def.** Given a v.s. V, a Partial Quadratic (PQ) Q on V is a symmetric bilinear form Q on a subspace $\mathcal{D}(Q) \subset V$. For $U \subset \mathcal{D}(Q)$, denote $\operatorname{ann}_Q(U) := \{v \in \mathcal{D}(Q) : Q(U, v) = 0\}$.
Def. $Q_1 + Q_2$ is with $\mathcal{D}(Q_1 + Q_2) = \mathcal{D}(Q_1) \cap \mathcal{D}(Q_2)$.
Def. Given a linear $\psi\colon V \to W$ and a PQ Q on W, the pullback is $(\psi^* Q)(v_1, v_2) = Q(\psi v_1, \psi v_2)$ with $\mathcal{D}(\psi^* Q) = \phi^{-1}(\mathcal{D}(Q))$.
Def. Given $\phi\colon V \to W$ and a PQ Q on V the pushforward $\phi_* Q$ is with $\mathcal{D}(\phi_* Q) = \phi(\operatorname{ann}_Q(\mathcal{D}(Q) \cap \ker \phi))$ and $(\phi_* Q)(w_1, w_2) = Q(v_1, v_2)$, where v_i are s.t. $\phi(v_i) = w_i$ and $Q(v_i, \operatorname{rad} Q|_{\ker \phi}) = 0$.

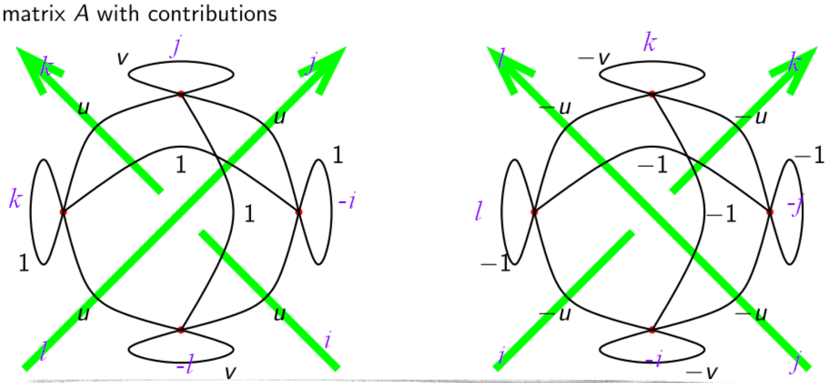
Thm(?). ψ^* and ϕ_* are well-defined and functorial, and if $\alpha /\!/ \beta = \gamma /\!/ \delta$, then $\gamma^* /\!/ \alpha_* = \delta_* /\!/ \beta^*$. ψ^* is additive but ϕ_* isn't.

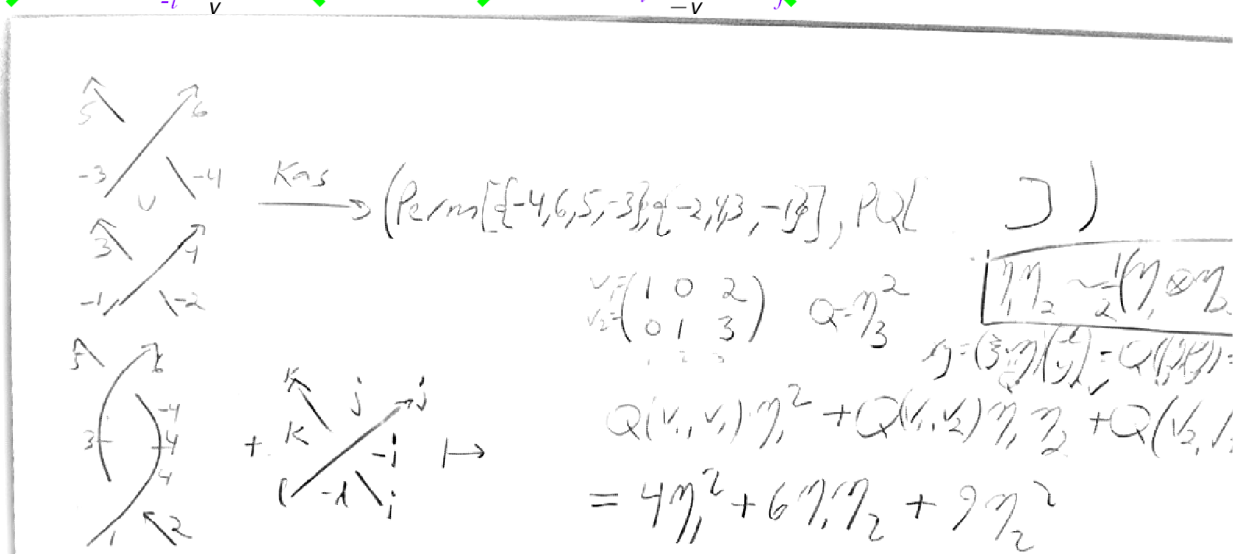


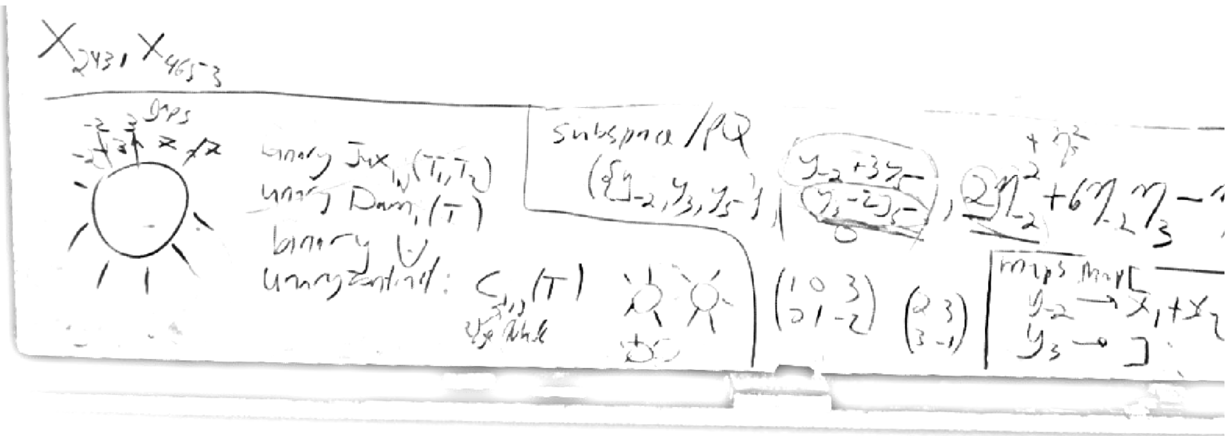
Thm(?). Over $\mathbb{R}$, given $\phi\colon V \to W$ and PQs Q on V and C on W,

$$\operatorname{sign}_V(Q + \phi^* C) = \operatorname{sign}_{\ker \phi}(\iota^* Q) + \operatorname{sign}_W(C + \phi_* Q).$$

For a knot K and a complex unit ω set $u = \Re(\omega^{1/2})$, $v = \Re(\omega)$, make an $F \times F$ matrix A with contributions







pdf

In[∘]:=
```
RowRed[Subspace[vs_, gens_]] :=
  RowReduce[Join[Table[Coefficient[g, v], {g, gens}, {v, Sort[vs]}],
    IdentityMatrix[Length@gens], 2]];
```

In[∘]:=
```
RowRed[Subspace[{y, z, x, w}, {x + y, x - y + z, x + 2 y + w}]] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

pdf

In[∘]:=
```
CF[Subspace[{}, {0 ...}]] := Subspace[{}, {}];
CF[Subspace[vs_, {}]] := Subspace[Sort[vs], {}];
CF[Subspace[vs_, gens_]] := Module[{cvs = Sort[vs]},
   Subspace[cvs,
    DeleteCases[(RowRed[Subspace[vs, gens]][[All, ;; Length@vs]]).cvs, 0]
   ]];
CF[lt_LT] := Sort /@ lt
CFSteps[Subspace[{}, {0 ...}]] := {};
CFSteps[Subspace[vs_, {}]] := {};
CFSteps[sub_] := RowRed[sub][[All, -Length@RowRed[sub] ;;]];
```

In[∘]:=
```
CF[Subspace[{y, z, x, w}, {x + y, x - y + z, x + 2 y + w}]]
```

Out[∘]=

$$\text{Subspace}\left[\{w, x, y, z\}, \left\{w + \frac{z}{2}, x + \frac{z}{2}, y - \frac{z}{2}\right\}\right]$$

In[∘]:=
```
CFSteps[Subspace[{y, z, x, w}, {x + y, x - y + z, x + 2 y + w}]] // MatrixForm
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{3}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

pdf

In[∘]:=
```
Eval[Q_, v_, w_] := Expand[Q v w] //. {η_i__ y_i__ :> 1, η_i__^2 y_i__^2 :> 2} /. (η | y)__ → 0;
Eval[ϕ_, v_] := Expand[ϕ v] /. {η_i__ y_i__ :> 1, η_i__^2 y_i__ :> 2 η_i} /. y__ → 0;
```

In[∘]:=
```
Eval[u η1^2 + v η1 η2, y1 + y2]
```
Out[∘]=

$2\,u\,\eta_1 + v\,\eta_1 + v\,\eta_2$

In[∘]:=
```
Eval[Eval[u η1^2 + v η1 η2, y1], y1 + y2]
```
Out[∘]=

$2\,u + v$

In[∘]:=
```
Eval[u η1^2 + v η1 η2, y1 + y2, y1]
```
Out[∘]=

$2\,u + v$

pdf

In[∘]:=
```
Pivot[v_Plus] := v[[1]]; Pivot[v_] := v;
y_i__^* := η_i; η_i__^* := y_i; (vs_List)^* := Table[v^*, {v, vs}];
```

pdf

In[∘]:=
```
CF[PQ[sub_Subspace, Q_]] := Module[{csub, cvs, cgens},
  {cvs, cgens} = List @@ (csub = CF[sub]);
  PQ[csub, Sum[Eval[Q, v, w] Pivot[v]^* Pivot[w]^* / 2, {v, cgens}, {w, cgens}]]
 ]
```

In[∘]:=
```
CF[PQ[Subspace[{y1, y2, y3}, {y1 + 2 y3, y2 + 3 y3}], η3^2]]
```
Out[∘]=

$PQ\left[Subspace[\{y_1, y_2, y_3\}, \{y_1 + 2\,y_3, y_2 + 3\,y_3\}], 4\,\eta_1^2 + 12\,\eta_1\,\eta_2 + 9\,\eta_2^2\right]$

In[∘]:=
```
Eval[η3^2, y1 + 2 y3, y2 + 3 y3]
```
Out[∘]=

12

In[∘]:=
```
Eval[4 η1^2 + 12 η1 η2 + 9 η2^2, y1 + 2 y3, y2 + 3 y3]
```
Out[∘]=

12

In[∘]:=
```
Eval[4 η₁² + 12 η₁ η₂ + 9 η₂², y₁, y₂]
```
Out[∘]=
```
12
```

In[∘]:=
```
Eval[12 η₁ η₂, y₁, y₂]
```
Out[∘]=
```
12
```

pdf

In[∘]:=
```
Perp[Subsp_] := Module[{pp, cvs, cgens},
  {cvs, cgens} = List @@ CF@Subsp;
  pp = Complement[cvs, Pivot /@ cgens]*;
  CF@Subspace[cvs*,
    Table[p - Sum[Coefficient[g, p*] Pivot[g]*, {g, cgens}], {p, pp}]
   ]
 ]
```

In[∘]:=
```
Perp@Subspace[{y₁, y₂, y₃}, {y₁ - y₂}]
```
Out[∘]=
```
Subspace[{η₁, η₂, η₃}, {η₁ + η₂, η₃}]
```

In[∘]:=
```
Perp@Perp@Subspace[{y₁, y₂, y₃}, {y₁ - y₂}]
```
Out[∘]=
```
Subspace[{y₁, y₂, y₃}, {y₁ - y₂}]
```

pdf

In[∘]:=
```
Id[vs_] := LT[vs, vs, Table[v → v, {v, vs}]]
```

In[∘]:=
```
Id[{y₁, y₂}]
```
Out[∘]=
```
LT[{y₁, y₂}, {y₁, y₂}, {y₁ → y₁, y₂ → y₂}]
```

pdf

In[∘]:=
```
LT[dom_, ran_, rs_]*[Subspace[ran_, gens_]] := Perp@CF@Subspace[dom*, Table[
     Sum[Eval[p, v /. rs] v*, {v, dom}],
     {p, Perp[Subspace[ran, gens]][[2]]}
    ]]
```

In[∘]:=
```
LT[{y₋₁, y₋₂, y₋₃}, {y₁, y₂, y₃}, {y₋₁ → y₁ + 2 y₃, y₋₂ → 2 y₂ - y₃, y₋₃ → y₃}]*[
 Subspace[{y₁, y₂, y₃}, {y₁ - y₂} ]]
```
Out[∘]=

$$\text{Subspace}\left[\{y_{-3}, y_{-2}, y_{-1}\}, \left\{y_{-3} + \frac{y_{-2}}{5} - \frac{2\, y_{-1}}{5}\right\}\right]$$

pdf

In[∘]:=
```
LT[dom_, ran_, rs_]*[PQ[sub_, Q_]] := CF@PQ[
   LT[dom, ran, rs]*[sub],
   Sum[Eval[Q, v1 /. rs, v2 /. rs] v1* v2* / 2, {v1, dom}, {v2, dom}]
  ]
```

In[∘]:= Id[{y_1, y_2, y_3}]*[PQ[Subspace[{y_1, y_2, y_3}, {$y_1 + 2 y_3$, $y_2 + 3 y_3$}], $4 \eta_1^2 + 12 \eta_1 \eta_2 + 9 \eta_2^2$]]

Out[∘]=

PQ[Subspace[{y_1, y_2, y_3}, {$y_1 + 2 y_3$, $y_2 + 3 y_3$}], $4 \eta_1^2 + 12 \eta_1 \eta_2 + 9 \eta_2^2$]

In[∘]:= LT[{y_{-1}, y_{-2}, y_{-3}}, {y_1, y_2, y_3}, {$y_{-1} \to y_1 + 2 y_3$, $y_{-2} \to 2 y_2 - y_3$, $y_{-3} \to y_3$}]*[
PQ[Subspace[{y_1, y_2, y_3}, {$y_1 + 2 y_3$, $y_2 + 3 y_3$}], $4 \eta_1^2 + 12 \eta_1 \eta_2 + 9 \eta_2^2$]]

Out[∘]=

PQ[Subspace[{y_{-3}, y_{-2}, y_{-1}}, {$y_{-3} + \frac{y_{-2}}{7}$, y_{-1}}], $\frac{36 \eta_{-3}^2}{49} + \frac{24}{7} \eta_{-3} \eta_{-1} + 4 \eta_{-1}^2$]

In[∘]:= Eval[$\frac{36 \eta_{-3}^2}{49} + \frac{24}{7} \eta_{-3} \eta_{-1} + 4 \eta_{-1}^2$, $y_{-3} + \frac{y_{-2}}{7}$, $y_{-3} + \frac{y_{-2}}{7}$]

Out[∘]=

$\frac{72}{49}$

In[∘]:= Eval[$4 \eta_1^2 + 12 \eta_1 \eta_2 + 9 \eta_2^2$,
$y_{-3} + \frac{y_{-2}}{7}$ /. {$y_{-1} \to y_1 + 2 y_3$, $y_{-2} \to 2 y_2 - y_3$, $y_{-3} \to y_3$},
$y_{-3} + \frac{y_{-2}}{7}$ /. {$y_{-1} \to y_1 + 2 y_3$, $y_{-2} \to 2 y_2 - y_3$, $y_{-3} \to y_3$}]

Out[∘]=

$\frac{72}{49}$

pdf

In[∘]:=
```
Subspace /: Subspace[v1s_, gen1s_] ⊕ Subspace[v2s_, gen2s_] :=
  CF@Subspace[v1s ⋃ v2s, gen1s ⋃ gen2s];
Subspace /: Subspace[vs_, gen1s_] + Subspace[vs_, gen2s_] :=
  CF@Subspace[vs, gen1s ⋃ gen2s];
Subspace /: sub1_Subspace ⋂ sub2_Subspace := Perp[Perp[sub1] + Perp[sub2]];
Subspace /: v_ ∈ Subspace[vs_, gens_] :=
  (Subspace[vs, gens] ⋂ Subspace[vs, {v}])[[2]] =!= {};
```

In[∘]:= Subspace[{y_1, y_2}, {$y_1 - 3 y_2$}] ⊕ Subspace[{y_{-1}, y_{-2}, y_{-3}}, {y_{-3}, $y_{-1} + y_{-2}$}]

Out[∘]=

Subspace[{y_{-3}, y_{-2}, y_{-1}, y_1, y_2}, {y_{-3}, $y_{-2} + y_{-1}$, $y_1 - 3 y_2$}]

In[∘]:= Subspace[{y_1, y_2, y_3}, {$y_1 + 2 y_3$}] + Subspace[{y_1, y_2, y_3}, {$3 y_3$}]

Out[∘]=

Subspace[{y_1, y_2, y_3}, {y_1, y_3}]

In[∘]:= $\text{Subspace}[\{y_1, y_2, y_3\}, \{y_1 + 2 y_3\}] \bigcap \text{Subspace}[\{y_1, y_2, y_3\}, \{2 y_3, y_1\}]$

Out[∘]=

$\text{Subspace}[\{y_1, y_2, y_3\}, \{y_1 + 2 y_3\}]$

In[∘]:= $y_3 \in \text{Subspace}[\{y_1, y_2, y_3\}, \{y_1 + 2 y_3\}]$

Out[∘]=

False

In[∘]:= $y_3 \in \text{Subspace}[\{y_1, y_2, y_3\}, \{y_1 + 2 y_3, y_1 + y_3\}]$

Out[∘]=

True

pdf

In[∘]:=

```
PQ /: PQ[sub1_, Q1_] ⊕ PQ[sub2_, Q2_] := CF@PQ[sub1 ⊕ sub2, Q1 + Q2];
PQ /: CirclePlus[PQ1_PQ, PQs__PQ] :=
  CirclePlus @@ Join[{PQ1 ⊕ First[{PQs}]}, {PQs}[[2 ;;]]];
```

In[∘]:= $\text{PQ1} = \text{PQ}\left[\text{Subspace}[\{y_1, y_2\}, \{y_2 + 2 y_1\}], 4 \eta_1^2 + 12 \eta_1 \eta_2 + 9 \eta_2^2\right];$

$\text{PQ2} = \text{PQ}\left[\text{Subspace}\left[\{y_{-3}, y_{-2}, y_{-1}\}, \left\{y_{-3} + \frac{y_{-2}}{7}, y_{-1}\right\}\right], \frac{36 \eta_{-3}^2}{49} + \frac{24}{7} \eta_{-3} \eta_{-1} + 4 \eta_{-1}^2\right];$

$\text{PQ3} = \text{PQ}\left[\text{Subspace}[\{y_3\}, \{y_3\}], \eta_3^2\right];$

In[∘]:= CirclePlus[PQ1, PQ2, PQ3]

Out[∘]=

$\text{PQ}\left[\text{Subspace}\left[\{y_{-3}, y_{-2}, y_{-1}, y_1, y_2, y_3\}, \left\{y_{-3} + \frac{y_{-2}}{7}, y_{-1}, y_1 + \frac{y_2}{2}, y_3\right\}\right], \frac{36 \eta_{-3}^2}{49} + \frac{24}{7} \eta_{-3} \eta_{-1} + 4 \eta_{-1}^2 + \frac{49 \eta_1^2}{4} + \eta_3^2\right]$

pdf

In[∘]:=

```
Ann_{PQ[𝒟_Subspace,Q_]}[Subspace[vs_, gens_]] :=
  𝒟 ⋂ Perp@Subspace[vs*, Table[Eval[Q, g], {g, gens}]]
```

In[∘]:= $\text{Ann}_{\text{PQ}[\text{Subspace}[\{y_1,y_2,y_3\},\{y_1+2 y_3, y_2+3 y_3\}], 4 \eta_1^2+12 \eta_1 \eta_2+9 \eta_2^2]}[\text{Subspace}[\{y_1, y_2, y_3\}, \{y_1 + 2 y_3\}]]$

Out[∘]=

$\text{Subspace}\left[\{y_1, y_2, y_3\}, \left\{y_1 - \frac{2 y_2}{3}\right\}\right]$

In[∘]:= $y_1 - \frac{2 y_2}{3} \in \text{Subspace}[\{y_1, y_2, y_3\}, \{y_1 + 2 y_3, y_2 + 3 y_3\}]$

Out[∘]=

True

In[∘]:= $\text{Eval}\left[4 \eta_1^2 + 12 \eta_1 \eta_2 + 9 \eta_2^2, y_1 - \frac{2 y_2}{3}, y_1 + 2 y_3\right]$

Out[∘]=

0

pdf

```
Ker[LT[{}, _, _]] := Subspace[{}, {}];
Ker[LT[dom_, {}, _]] := Subspace[dom, dom];
Ker[LT[dom_, ran_, rs_]] := Module[{ns},
  ns = NullSpace[Table[Coefficient[d /. rs, r], {r, ran}, {d, dom}]];
  If[Length@ns > 0, CF@Subspace[dom, ns.dom], Subspace[dom, {}]]
 ]
```

In[∘]:= $\text{Ker}[\text{LT}[\{y_{-1}, y_{-2}, y_{-3}\}, \{y_1, y_2, y_3\}, \{y_{-1} \to y_1 + 2 y_3, y_{-2} \to 2 y_2 - y_3, y_{-3} \to y_3\}]]$

Out[∘]= $\text{Subspace}[\{y_{-1}, y_{-2}, y_{-3}\}, \{\}]$

In[∘]:= $\text{Ker}[\text{LT}[\{y_{-1}, y_{-2}, y_{-3}\}, \{y_1, y_2, y_3\}, \{y_{-1} \to y_1 + 2 y_3, y_{-2} \to -y_3, y_{-3} \to y_3\}]]$

Out[∘]= $\text{Subspace}[\{y_{-3}, y_{-2}, y_{-1}\}, \{y_{-3} + y_{-2}\}]$

pdf

```
(*will return a LT that is a section on the image*)
Section[LT[dom_, ran_, rs_]] := Module[{im = Subspace[ran, dom /. rs], newrs = {}},
  newrs = Thread[Pivot /@ (CF[im][[2]]) → (CFSteps[im].dom)[[ ;; Length@CF[im][[2]]]]];
  LT[ran, dom, Join[newrs, Thread[Complement[ran, Pivot /@ (CF[im][[2]])] → 0]]]
 ]
```

In[∘]:= $\text{Section}[\text{LT}[\{y_{-1}, y_{-2}, y_{-3}\}, \{y_1, y_2, y_3\}, \{y_{-1} \to y_1 + 2 y_3, y_{-2} \to 2 y_2 - y_3, y_{-3} \to y_3\}]]$

Out[∘]= $\text{LT}\left[\{y_1, y_2, y_3\}, \{y_{-1}, y_{-2}, y_{-3}\}, \left\{y_1 \to -2 y_{-3} + y_{-1}, y_2 \to \frac{y_{-3}}{2} + \frac{y_{-2}}{2}, y_3 \to y_{-3}\right\}\right]$

In[∘]:= $\text{Section}[\text{LT}[\{y_{-1}, y_{-2}, y_{-3}\}, \{y_1, y_2, y_3\}, \{y_{-1} \to 0, y_{-2} \to 2 y_2 - y_3, y_{-3} \to y_3\}]]$

Out[∘]= $\text{LT}\left[\{y_1, y_2, y_3\}, \{y_{-1}, y_{-2}, y_{-3}\}, \left\{y_2 \to \frac{y_{-3}}{2} + \frac{y_{-2}}{2}, y_3 \to y_{-3}, y_1 \to 0\right\}\right]$

pdf

```
Section[LT[Subspace[vs_, gens_], ran_, rs_]] := Section[LT[gens, ran, rs]];
```

pdf

```
(LT[dom_, ran_, rs_])*[Subspace[dom_, gens_]] := CF@Subspace[ran, gens /. rs]
```

In[∘]:= $\text{LT}[\{y_1, y_2, y_3\}, \{y_1, y_2\}, \{y_1 \to 0, y_2 \to y_1, y_3 \to y_2\}]_*[\text{Subspace}[\{y_1, y_2, y_3\}, \{y_1, y_3\}]]$

Out[∘]= $\text{Subspace}[\{y_1, y_2\}, \{y_2\}]$

pdf

```
lt_LT*[PQ[sub_, Q_]] := CF@Section[CF[lt]]*[CF@PQ[Ann_PQ[sub,Q][Ker[CF[lt]]], Q]];
```

In[∘]:=
```
LT[{y_{-1}, y_{-2}, y_{-3}}, {y_1, y_2, y_3}, {y_{-1} → y_1 + 2 y_3, y_{-2} → 2 y_2 - y_3, y_{-3} → y_3}]_*[
  PQ[Subspace[{y_{-1}, y_{-2}, y_{-3}}, {y_{-1} + 2 y_{-3}, y_{-2} + 3 y_{-3}}], η_{-3}^2]]
```

Out[∘]=

$$PQ\left[Subspace[\{y_1, y_2, y_3\}, \{y_1 + 4 y_3, y_2 + y_3\}], 4 \eta_1^2 + 6 \eta_1 \eta_2 + \frac{9 \eta_2^2}{4}\right]$$

pdf

In[∘]:=
```
Sig[PQ[Subspace[vs_, gens_], Q_]] :=
  Plus @@ Sign@Eigenvalues[Table[Eval[Q, v, w], {v, gens}, {w, gens}]];
```

In[∘]:=
```
Sig[PQ[Subspace[{y_1, y_2, y_3}, {y_1 + 4 y_3, y_2 + y_3}], 4 η_1^2 + 6 η_1 η_2 + 9 η_2^2/4]]
```

Out[∘]=

1