Partial Quadratics, their Pushwards, and Signature Invariants for Tangles

Abstract. Following a general discussion of the computation of zombians of unfinished columbaria (with examples), I will tell you about my recent joint work with Jessica Liu on what we feel is the "textbook" extension of knot signatures to tangles, which for unknown reasons, is not in any of the textbooks that we know.

Jessica Liu

Columbaria in an East Sydney Cemetery

Image: Freepik.com

Jacobian, Hamiltonian, Zombian

```
SetAttributes[Bndry, Orderless];

CF[b_Bndry] :=
 RotateLeft[#, First@Ordering[#] - 1] & /@ b

Kas[P[i_, j_]] :=
 Kas[CF@Bndry[{i, j}], 0,
  PQ[Subspace[{y_{i}, y_{j}}, {y_{i}, y_{j}}], 0]]

Kas[X[i_, j_, k_, l_]] := If[PositiveQ@X[i, j, k, l],
   Kas[CF@Bndry[{-i, j, k, -l}], 0,
    PQ[Subspace[{y_{-i}, y_{j}, y_{k}, y_{-l}}, {y_{-i}, y_{j}, y_{k}, y_{-l}}],
     1/2 (η_{-i}^2 + 2 u η_{-i} η_{j} + v η_{j}^2 + 2 η_{-i} η_{k} + 2 u η_{j} η_{k} + η_{k}^2 +
        2 u η_{-i} η_{-l} + 2 η_{j} η_{-l} + 2 u η_{k} η_{-l} + v η_{-l}^2)]],
   Kas[CF@Bndry[{-i, -j, k, l}], 0,
    PQ[Subspace[{y_{-j}, y_{k}, y_{l}, y_{-i}}, {y_{-j}, y_{k}, y_{l}, y_{-i}}],
     1/2 (-v η_{-i}^2 - 2 u η_{-i} η_{-j} - η_{-j}^2 - 2 η_{-i} η_{k} - 2 u η_{-j} η_{k} -
        v η_{k}^2 - 2 u η_{-i} η_{l} - 2 η_{-j} η_{l} - 2 u η_{k} η_{l} - η_{l}^2)]]
 ]

Kas /: Kas[b1_, σ1_, pq1_] ⋃ Kas[b2_, σ2_, pq2_] :=
  Kas[CF@Join[b1, b2], σ1 + σ2, pq1 ⊕ pq2];

(* FM for FaceMerge *)
Kas[Bndry[{li___, i_, ri___}, {lj___, j_, rj___},
    bs___], σ_, PQ[Subspace[vs_, gs_], Q_]] //
  FM_{i_, j_} := Module[{ϕ},
  ϕ = Echo@LT[{y_{0}} ⋃ Complement[vs, {y_{i}, y_{j}}], vs,
     {y_{0} → y_{i} + y_{j}} ⋃
      Table[v → v, {v, Complement[vs, {y_{i}, y_{j}}]}]];
  Kas[CF@Bndry[{ri, li, i, rj, lj, j}, bs], σ,
   PQ[Subspace[vs, gs], Q] // ϕ^* // Echo // ϕ_⋆]
 ]
```

```
RowRed[Subspace[vs_, gens_]] :=
  RowReduce[
   Join[Table[Coefficient[g, v], {g, gens},
     {v, Sort[vs]}], IdentityMatrix[Length@gens],
    2]];
```

```
CF[Subspace[{}, {0...}]] := Subspace[{}, {}];
CF[Subspace[vs_, {}]] := Subspace[Sort[vs], {}];
CF[Subspace[vs_, gens_]] := Module[{cvs = Sort[vs]},
   Subspace[cvs,
    DeleteCases[
     (RowRed[Subspace[vs, gens]][[All, ;; Length@vs]]).
      cvs, 0]
   ]];
CF[lt_LT] := Sort /@ lt
CFSteps[Subspace[{}, {0...}]] := {};
CFSteps[Subspace[vs_, {}]] := {};
CFSteps[sub_] :=
  RowRed[sub][[All, -Length@RowRed[sub] ;;]];
```

```
Eval[Q_, v_, w_] :=
  Expand[Q v w] //. {η_i__ y_i__ :→ 1, η_i__^2 y_i__^2 :→ 2} /.
   (η | y)__ → 0;
Eval[ϕ_, v_] :=
  Expand[ϕ v] /. {η_i__ y_i__ :→ 1, η_i__^2 y_i__ :→ 2 η_i} /.
   y__ → 0;
```

```
Pivot[v_Plus] := v[[1]]; Pivot[v_] := v;
y_i__^* := η_i; η_i__^* := y_i;
(vs_List)^* := Table[v^*, {v, vs}];
```

```
CF[PQ[sub_Subspace, Q_]] := Module[{csub, cvs, cgens},
  {cvs, cgens} = List @@ (csub = CF[sub]);
  PQ[csub, Sum[Eval[Q, v, w] Pivot[v]^* Pivot[w]^* / 2,
    {v, cgens}, {w, cgens}]]
 ]
```

```
Perp[Subsp_] := Module[{pp, cvs, cgens},
  {cvs, cgens} = List @@ CF@Subsp;
  pp = Complement[cvs, Pivot /@ cgens]^*;
  CF@Subspace[cvs^*,
    Table[p - Sum[Coefficient[g, p^*] Pivot[g]^*,
       {g, cgens}], {p, pp}]
   ]
 ]
```

```
Id[vs_] := LT[vs, vs, Table[v → v, {v, vs}]]
```

```
LT[dom_, ran_, rs_]^*[Subspace[ran_, gens_]] :=
 Perp@CF@Subspace[dom^*, Table[
     Sum[Eval[p, v /. rs] v^*, {v, dom}],
     {p, Perp[Subspace[ran, gens]][[2]]}
    ]]
```

```
LT[dom_, ran_, rs_]^*[PQ[sub_, Q_]] := CF@PQ[
   LT[dom, ran, rs]^*[sub],
   Sum[Eval[Q, v1 /. rs, v2 /. rs] v1^* v2^* / 2,
    {v1, dom}, {v2, dom}]
  ]
```

```
Subspace /: Subspace[v1s_, gen1s_] ⊕
    Subspace[v2s_, gen2s_] :=
   CF@Subspace[v1s ⋃ v2s, gen1s ⋃ gen2s];
Subspace /: Subspace[vs_, gen1s_] +
    Subspace[vs_, gen2s_] :=
   CF@Subspace[vs, gen1s ⋃ gen2s];
Subspace /: sub1_Subspace ⋂ sub2_Subspace :=
  Perp[Perp[sub1] + Perp[sub2]];
Subspace /: v_ ∈ Subspace[vs_, gens_] :=
   (Subspace[vs, gens] ⋂ Subspace[vs, {v}])[[2]] =!= {};
```

```
PQ /: PQ[sub1_, Q1_] ⊕ PQ[sub2_, Q2_] :=
   CF@PQ[sub1 ⊕ sub2, Q1 + Q2];
PQ /: CirclePlus[PQ1_PQ, PQs__PQ] :=
   CirclePlus @@ Join[{PQ1 ⊕ First[{PQs}]},
     {PQs}[[2 ;;]]];
```

```
Ann_PQ[𝒟_Subspace,Q_][Subspace[vs_, gens_]] :=
 𝒟 ⋂ Perp@Subspace[vs^*, Table[Eval[Q, g], {g, gens}]]
```

```
Ker[LT[{}, _, _]] := Subspace[{}, {}];
Ker[LT[dom_, {}, _]] := Subspace[dom, dom];
Ker[LT[dom_, ran_, rs_]] := Module[{ns},
   ns = NullSpace[Table[Coefficient[d /. rs, r],
      {r, ran}, {d, dom}]];
   If[Length@ns > 0, CF@Subspace[dom, ns.dom],
    Subspace[dom, {}]]
  ]
```

```
(*will return a LT that is a section on the image*)
Section[LT[dom_, ran_, rs_]] :=
 Module[{im = Subspace[ran, dom /. rs], newrs = {}},
  newrs =
   Thread[Pivot /@ (CF[im][[2]]) →
     (CFSteps[im].dom)[[ ;; Length@CF[im][[2]]]]];
  LT[ran, dom,
   Join[newrs,
    Thread[Complement[ran, Pivot /@ (CF[im][[2]])] →
      0]]]
  ]
```

```
Section[LT[Subspace[vs_, gens_], ran_, rs_]] :=
  Section[LT[gens, ran, rs]];
```

```
(LT[dom_, ran_, rs_])_*[Subspace[dom_, gens_]] :=
 CF@Subspace[ran, gens /. rs]
```

```
lt_LT_*[PQ[sub_, Q_]] :=
  CF@Section[CF[lt]]^*[
    CF@PQ[Ann_PQ[sub,Q][Ker[CF[lt]]], Q]];
```

```
Sig[PQ[Subspace[vs_, gens_], Q_]] :=
  Plus @@
   Sign@Eigenvalues[Table[Eval[Q, v, w],
      {v, gens}, {w, gens}]];
```