

Pensieve header: Proof of invariance of ρ_2 using integration techniques.

Initialization

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Groningen-240530"];
Once[<< KnotTheory` ; << Rot.m];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/AP/Talks/Groningen-240530> to compute rotation numbers.

Initialization

```
In[*]:= CCF[ $\mathcal{E}$ _] := ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ];
CCF[ $\mathcal{E}$ _] := Factor[ $\mathcal{E}$ ];
CF[ $\omega$  .  $\mathcal{E}$ _E] := CF[ $\omega$ ] CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _List] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _] := Module[{vs = Cases[ $\mathcal{E}$ , (x | p |  $\pi$ )_ ,  $\infty$ ]  $\cup$  {x, p,  $\epsilon$ }, ps, c},
  Total[CoefficientRules[Expand[ $\mathcal{E}$ ], vs] /. (ps_  $\rightarrow$  c_)  $\Rightarrow$  CCF[c] (Times @@ vsps) ]];
```

The Basic Feynman Ring

```
In[*]:= S = {x, x_, y, z};
qx,y[f_] := (∂x,yf) /. Thread[S  $\rightarrow$  0];
 $\theta_{x,y}$  := x y;
f_  $\equiv$  0 := f === 0;
EVvs_List  $\rightarrow$  0[f_] := CF[f /. Thread[vs  $\rightarrow$  0]]
```

The ϵ Series Feynman Ring

```

In[*]:= S = {x, y, z,  $\phi$ , x_, p_,  $\underline{x}$ ,  $\underline{p}$ };
qx,y[ser_eSeries] := ( $\partial_{x,y}$ ser[[1]]) /. Thread[S  $\rightarrow$  0];
 $\theta_{x,y}$  := x y;
 $\epsilon$ Series /: D[ser_eSeries, vs___] := D[#, vs] & /@ ser;
 $\epsilon$ Series /: Plus[ss___eSeries] /; Length[{ss}] > 1 := Module[{l = Min[Length /@ {ss}]},
  eSeries @@ Total[Take[List @@ #, l] & /@ {ss}]]
 $\epsilon$ Series /: t_ + ser_eSeries := MapAt[ (# + t) &, ser, 1];
 $\epsilon$ Series /: s1_eSeries * s2_eSeries := eSeries @@ Table[
  Sum[s1[[ii + 1]] s2[[kk - ii + 1]], {ii, 0, kk}], {kk, 0, Min[Length@s1, Length@s2] - 1};
 $\epsilon$ Series /: c_ * ser_eSeries := (c #) & /@ ser;
ser_eSeries  $\equiv$  0 := And @@ ((# === 0) & /@ ser);
 $\epsilon$ Series /: Integrate[ser_eSeries, pars_] := eSeries @@ (Integrate[#, pars] & /@ ser);
 $\epsilon$ Series /: EVvs_List  $\rightarrow$  0[ser_eSeries] := ser /. Thread[vs  $\rightarrow$  0];
CF[ser_eSeries] := CF /@ ser;

```

Integration

Using Picard Iteration!

```

In[*]:= E /: E[A_] E[B_] := E[A + B]

```

```

In[*]:= E[sd_SeriesData] /; (List @@ sd) [{1, 2, 4, 6}] === { $\epsilon$ , 0, 0, 1} :=
  E[eSeries @@ PadRight[sd[[3]], sd[[5]], 0]]

```

Following a program in Projects/FullDoPeGDO/Engine.nb, we write $Z_\lambda = \sum Z[m] \lambda^m$.

```

In[*]:= Unprotect[Integrate];
Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";
∫ ω_ . E[L_] d(vs_List) := Module[{n, Q, Δ, G, a, b, m, m1, $m}, Clear[Z];
n = Length@vs;
Q = Table[q_{vs[[a]], vs[[b]] [L], {a, n}, {b, n}];
If[(Δ = CF@Det[-Q]) == 0, Message[Integrate::sing]; Return[]];
G = CF[-Inverse[Q] / 2];
Z[] = Z[0] = CF[L - Sum[Q[[a, b]] θ_{vs[[a]], vs[[b]]}, {a, n}, {b, n}] / 2];
Z[m_, a_] := Z[m, a] = CF@D[Z[m], vs[[a]];
Z[m_, a_, b_] /; a ≤ b := Z[m, a, b] = CF@D[Z[m, a], vs[[b]];
Z[m_, a_, b_] /; a > b := Z[m, b, a];
For[$m = m = 0, m ≤ 2 $m, ++m,
Z[m + 1] = CF@Sum[Sum[If[G[[a, b]] == 0, 0,
G[[a, b]] / (m + 1) (Z[m, a, b] + Sum[Z[m1, a] Z[m - m1, b], {m1, 0, m})]],
{a, n}], {b, n}];
If[!(Z[m + 1] == 0), $m = m + 1; Z[] += Z[m + 1]];
];
PowerExpand@Factor[ω Δ^{-1/2}] E[CF[EV_{vs→0}[Z[]]]];
Protect[Integrate];

```

$$\text{In[*]} := \int \mathbb{E} \left[-\mu x^2 / 2 + i \xi x \right] d\{x\}$$

Out[*]=

$$\frac{\mathbb{E} \left[-\frac{\xi^2}{2\mu} \right]}{\sqrt{\mu}}$$

$$\text{In[*]} := L = -\frac{1}{2} \{x_1, x_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{x_1, x_2\} + \{\xi_1, \xi_2\} \cdot \{x_1, x_2\};$$

$$Z12 = \int \mathbb{E}[L] d\{x_1, x_2\}$$

Out[*]=

$$\frac{\mathbb{E} \left[\frac{c \xi_1^2 - 2b \xi_1 \xi_2 + a \xi_2^2}{2(-b^2 + ac)} \right]}{\sqrt{-b^2 + ac}}$$

$$\text{In[*]} := \{Z1 = \int \mathbb{E}[L] d\{x_1\}, Z12 = \int Z1 d\{x_2\}\}$$

Out[*]=

$$\left\{ \frac{\mathbb{E} \left[-\frac{(-b^2 + ac) x_2^2}{2a} + \frac{\xi_1^2}{2a} + \frac{x_2 (-b \xi_1 + a \xi_2)}{a} \right]}{\sqrt{a}}, \text{True} \right\}$$

Integration of ϵ Series

$$\text{In[*]:= } \int \mathbb{E} \left[-x^2 / 2 + \epsilon x^3 / 6 + \mathbf{0}[\epsilon]^{13} \right] d\{\mathbf{x}\}$$

Out[*]=

$$\mathbb{E} \left[\epsilon \text{Series} \left[0, 0, \frac{5}{24}, 0, \frac{5}{16}, 0, \frac{1105}{1152}, 0, \frac{565}{128}, 0, \frac{82825}{3072}, 0, \frac{19675}{96} \right] \right]$$

The ρ_2 Integrand

Adopted from pensieve://Talks//Oaxaca-2210/Rho.nb.

```

In[*]:= S = {x_, p_};
q[s_, i_, j_] := x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^5 - 1) x_i (p_{i+1} - p_{j+1});
r1[s_, i_, j_] :=  $\frac{5}{2}$  (x_i (p_i - p_j) ((T^5 - 1) x_i p_j + 2 (1 - p_j x_j)) - 1);
r2[1, i_, j_] :=
(-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j -
6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12;
r2[-1, i_, j_] :=
(-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 - 4 (-1 + T) T p_i^2 p_j x_i^3 +
2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j -
18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j -
6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2);
gamma1[phi_, k_] := phi (1 / 2 - x_k p_k);
gamma2[phi_, k_] := -phi^2 p_k x_k / 2;
L[X_{i_, j_}[s_]] := T^{5/2} E[q[s, i, j] + epsilon r1[s, i, j] + epsilon^2 r2[s, i, j] + O[epsilon]^3];
L[C_{k_}[phi_]] := T^{phi/2} E[-x_k (p_k - p_{k+1}) + epsilon gamma1[phi, k] + epsilon^2 gamma2[phi, k] + O[epsilon]^3];
L[K_] := (2 pi)^{-Features[K][[1]]} CF[L / @ Features[K][[2]]];
vs[K_] := Union @@ Table[{p_i, x_i}, {i, Features[K][[1]]}]

```

pdf

$$\text{In[*]:= } \epsilon^2 r_2[1, i, j]$$

Out[*]=

pdf

$$\frac{1}{12} \epsilon^2 \left(-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 - \right. \\ \left. 2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j - \right. \\ \left. 6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2 \right)$$

pdf

In[*]:= $\epsilon^2 r_2[-1, i, j]$

Out[*]=

pdf

$$\frac{1}{12 T^2} \epsilon^2 \left(-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 - 4 (-1 + T) T p_i^2 p_j x_i^3 + \right. \\ \left. 2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j - 18 T^2 p_j^2 x_i x_j - \right. \\ \left. 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j - 6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2 \right)$$

pdf

In[*]:= $\epsilon^2 \gamma_2[\varphi, i]$

Out[*]=

pdf

$$-\frac{1}{2} \epsilon^2 \varphi^2 p_i x_i$$

In[*]:= **Features [Knot [3, 1]]**

 **KnotTheory**: Loading precomputed data in PD4Knots`.

Out[*]=

Features [7, C₄ [-1] X_{2,6} [-1] X_{5,1} [-1] X_{7,3} [-1]]

In[*]:= $\mathcal{L}[\text{Knot}[3, 1]]$

Out[*]=

$$\frac{1}{128 \pi^7 T^2} \mathbb{E} \left[\epsilon \text{Series} \left[-p_1 x_1 + p_2 x_1 - p_2 x_2 + \frac{p_3 x_2}{T} + \frac{(-1+T) p_7 x_2}{T} - p_3 x_3 + p_4 x_3 - p_4 x_4 + p_5 x_4 + \frac{(-1+T) p_2 x_5}{T} - p_5 x_5 + \frac{p_6 x_5}{T} - p_6 x_6 + p_7 x_6 + \frac{(-1+T) p_4 x_7}{T} - p_7 x_7 + \frac{p_8 x_7}{T}, 1 - p_2 x_2 + p_6 x_2 + \frac{(-1+T) p_2 p_6 x_2^2}{2T} - \frac{(-1+T) p_6^2 x_2^2}{2T} + p_4 x_4 + p_1 x_5 - p_5 x_5 - p_1^2 x_1 x_5 + p_1 p_5 x_1 x_5 - \frac{(-1+T) p_1^2 x_5^2}{2T} + \frac{(-1+T) p_1 p_5 x_5^2}{2T} + p_2 p_6 x_2 x_6 - p_6^2 x_2 x_6 + p_3 x_7 - p_7 x_7 - p_3^2 x_3 x_7 + p_3 p_7 x_3 x_7 - \frac{(-1+T) p_3^2 x_7^2}{2T} + \frac{(-1+T) p_3 p_7 x_7^2}{2T}, -\frac{1}{2} p_2 x_2 + \frac{p_6 x_2}{2} + \frac{(-3+T) p_2 p_6 x_2^2}{4T} - \frac{(-3+T) p_6^2 x_2^2}{4T} - \frac{(-1+T) p_2^2 p_6 x_2^3}{3T} + \frac{(-1+T) (1+5T) p_2 p_6^2 x_2^3}{6T^2} - \frac{(-1+T) (1+3T) p_6^3 x_2^3}{6T^2} - \frac{p_4 x_4}{2} + \frac{p_1 x_5}{2} - \frac{p_5 x_5}{2} - \frac{3}{2} p_1^2 x_1 x_5 + \frac{3}{2} p_1 p_5 x_1 x_5 + \frac{1}{2} p_1^3 x_1^2 x_5 - \frac{1}{2} p_1^2 p_5 x_1^2 x_5 - \frac{(-3+T) p_1^2 x_5^2}{4T} + \frac{(-3+T) p_1 p_5 x_5^2}{4T} - \frac{(1+T) p_1^3 x_1 x_5^2}{2T} + \frac{(1+2T) p_1^2 p_5 x_1 x_5^2}{2T} - \frac{1}{2} p_1 p_5^2 x_1 x_5^2 - \frac{(-1+T) (1+3T) p_1^3 x_5^3}{6T^2} + \frac{(-1+T) (1+5T) p_1^2 p_5 x_5^3}{6T^2} - \frac{(-1+T) p_1 p_5^2 x_5^3}{3T} + \frac{3}{2} p_2 p_6 x_2 x_6 - \frac{3}{2} p_6^2 x_2 x_6 - \frac{1}{2} p_2^2 p_6 x_2^2 x_6 + \frac{(1+2T) p_2 p_6^2 x_2^2 x_6}{2T} - \frac{(1+T) p_6^3 x_2^2 x_6}{2T} - \frac{1}{2} p_2 p_6^2 x_2 x_6^2 + \frac{1}{2} p_6^3 x_2 x_6^2 + \frac{p_3 x_7}{2} - \frac{p_7 x_7}{2} - \frac{3}{2} p_3^2 x_3 x_7 + \frac{3}{2} p_3 p_7 x_3 x_7 + \frac{1}{2} p_3^3 x_3^2 x_7 - \frac{1}{2} p_3^2 p_7 x_3^2 x_7 - \frac{(-3+T) p_3^2 x_7^2}{4T} + \frac{(-3+T) p_3 p_7 x_7^2}{4T} - \frac{(1+T) p_3^3 x_3 x_7^2}{2T} + \frac{(1+2T) p_3^2 p_7 x_3 x_7^2}{2T} - \frac{1}{2} p_3 p_7^2 x_3 x_7^2 - \frac{(-1+T) (1+3T) p_3^3 x_7^3}{6T^2} + \frac{(-1+T) (1+5T) p_3^2 p_7 x_7^3}{6T^2} - \frac{(-1+T) p_3 p_7^2 x_7^3}{3T} \right] \right]$$

In[*]:= $\text{vs}[\text{Knot}[3, 1]]$

Out[*]=

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

In[*]:= $\mathbf{K} = \text{Knot}[3, 1]; \int \mathcal{L}[\mathbf{K}] \, d(\text{vs} \circ \mathbf{K})$

Out[*]=

$$\frac{i T \mathbb{E} \left[\epsilon \text{Series} \left[0, \frac{(-1+T)^2 (1+T^2)}{(1-T+T^2)^2}, -\frac{T^2 (1-4T^2+T^4)}{2(1-T+T^2)^4} \right] \right]}{128 \pi^7 (1-T+T^2)}$$

Invariance Under Reidemeister 3

$$\text{In[*]:= lhs} = \int (\mathbb{E}[\pi_i p_i + \pi_j p_j + \pi_k p_k] \mathcal{L} / @ (X_{i,j} [1] X_{i+1,k} [1] X_{j+1,k+1} [1]))$$

$$\mathbb{d}\{p_i, p_j, p_k, p_{i+1}, p_{j+1}, p_{k+1}, x_i, x_j, x_k, x_{i+1}, x_{j+1}, x_{k+1}\}$$

$$\text{rhs} = \int (\mathbb{E}[\pi_i p_i + \pi_j p_j + \pi_k p_k] \mathcal{L} / @ (X_{j,k} [1] X_{i,k+1} [1] X_{i+1,j+1} [1]))$$

$$\mathbb{d}\{p_i, p_j, p_k, p_{i+1}, p_{j+1}, p_{k+1}, x_i, x_j, x_k, x_{i+1}, x_{j+1}, x_{k+1}\};$$

$$\text{lhs} == \text{rhs}$$

Out[*]=

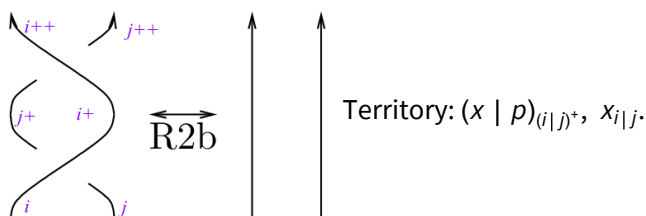
$$\begin{aligned} & T^{3/2} \mathbb{E} \left[\text{Series} \left[T^2 p_{2+i} \pi_i - (-1 + T) T p_{2+j} \pi_i + (1 - T) p_{2+k} \pi_i + T p_{2+j} \pi_j + (1 - T) p_{2+k} \pi_j + p_{2+k} \pi_k, \right. \right. \\ & - \frac{3}{2} + T^2 p_{2+j} \pi_i + T p_{2+k} \pi_i + \frac{1}{2} (-1 + T) T^3 p_{2+i} p_{2+j} \pi_i^2 - \frac{1}{2} (-1 + T) T^3 p_{2+j}^2 \pi_i^2 + \\ & \frac{1}{2} (-1 + T) T^2 p_{2+i} p_{2+k} \pi_i^2 - \frac{1}{2} (-1 + T)^2 T p_{2+j} p_{2+k} \pi_i^2 - \frac{1}{2} (-1 + T) T p_{2+k}^2 \pi_i^2 - T p_{2+j} \pi_j + \\ & (-1 + 2 T) p_{2+k} \pi_j - T^3 p_{2+i} p_{2+j} \pi_i \pi_j + T^3 p_{2+j}^2 \pi_i \pi_j + (-1 + T) T^2 p_{2+i} p_{2+k} \pi_i \pi_j - \\ & (-1 + T)^2 T p_{2+j} p_{2+k} \pi_i \pi_j - (-1 + T) T p_{2+k}^2 \pi_i \pi_j + \frac{1}{2} (-1 + T) T p_{2+j} p_{2+k} \pi_j^2 - \frac{1}{2} (-1 + T) T p_{2+k}^2 \pi_j^2 - \\ & 2 p_{2+k} \pi_k - T^2 p_{2+i} p_{2+k} \pi_i \pi_k + (-1 + T) T p_{2+j} p_{2+k} \pi_i \pi_k + T p_{2+k}^2 \pi_i \pi_k - T p_{2+j} p_{2+k} \pi_j \pi_k + T p_{2+k}^2 \pi_j \pi_k, \\ & - \frac{1}{2} T^2 p_{2+j} \pi_i - \frac{1}{2} T p_{2+k} \pi_i - \frac{1}{4} T^3 (-1 + 3 T) p_{2+i} p_{2+j} \pi_i^2 + \frac{1}{4} T^3 (-3 + 5 T) p_{2+j}^2 \pi_i^2 - \\ & \frac{1}{4} T^2 (-1 + 3 T) p_{2+i} p_{2+k} \pi_i^2 + \frac{1}{4} (-1 + T) T (-1 + 5 T) p_{2+j} p_{2+k} \pi_i^2 + \\ & \frac{1}{4} T (-3 + 5 T) p_{2+k}^2 \pi_i^2 - \frac{1}{6} (-1 + T) T^5 p_{2+i} p_{2+j} \pi_i^3 + \frac{1}{6} (-1 + T) T^4 (-1 + 4 T) p_{2+i} p_{2+j}^2 \pi_i^3 - \\ & \frac{1}{6} (-1 + T) T^4 (-1 + 3 T) p_{2+j}^3 \pi_i^3 - \frac{1}{6} (-1 + T) T^4 p_{2+i} p_{2+k} \pi_i^3 + \frac{5}{6} (-1 + T)^2 T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i^3 - \\ & \frac{1}{6} (-1 + T)^2 T^2 (-1 + 4 T) p_{2+j}^2 p_{2+k} \pi_i^3 + \frac{1}{6} (-1 + T) T^2 (-1 + 4 T) p_{2+i} p_{2+k}^2 \pi_i^3 - \\ & \frac{1}{6} (-1 + T)^2 T (-1 + 4 T) p_{2+j} p_{2+k}^2 \pi_i^3 - \frac{1}{6} (-1 + T) T (-1 + 3 T) p_{2+k}^3 \pi_i^3 + \frac{1}{2} T p_{2+j} \pi_j + \\ & \frac{1}{2} (1 - 4 T) p_{2+k} \pi_j + \frac{3}{2} T^3 p_{2+i} p_{2+j} \pi_i \pi_j - \frac{5}{2} T^3 p_{2+j}^2 \pi_i \pi_j - \frac{1}{2} T^2 (-3 + 5 T) p_{2+i} p_{2+k} \pi_i \pi_j + \\ & \frac{1}{2} (-1 + T) T (-3 + 7 T) p_{2+j} p_{2+k} \pi_i \pi_j + \frac{1}{2} T (-5 + 7 T) p_{2+k}^2 \pi_i \pi_j + \frac{1}{2} T^5 p_{2+i} p_{2+j} \pi_i^2 \pi_j - \\ & \frac{1}{2} T^4 (-1 + 4 T) p_{2+i} p_{2+j}^2 \pi_i^2 \pi_j + \frac{1}{2} T^4 (-1 + 3 T) p_{2+j}^3 \pi_i^2 \pi_j - \frac{1}{2} (-1 + T) T^4 p_{2+i} p_{2+k} \pi_i^2 \pi_j + \\ & \frac{1}{2} (-1 + T) T^3 (-5 + 4 T) p_{2+i} p_{2+j} p_{2+k} \pi_i^2 \pi_j - \frac{1}{2} (-1 + T) T^2 (1 - 5 T + 3 T^2) p_{2+j}^2 p_{2+k} \pi_i^2 \pi_j + \\ & \frac{1}{2} (-1 + T) T^2 (-1 + 4 T) p_{2+i} p_{2+k}^2 \pi_i^2 \pi_j - \frac{1}{2} (-1 + T)^2 T (-1 + 4 T) p_{2+j} p_{2+k}^2 \pi_i^2 \pi_j - \\ & \frac{1}{2} (-1 + T) T (-1 + 3 T) p_{2+k}^3 \pi_i^2 \pi_j - \frac{1}{4} T (-5 + 7 T) p_{2+j} p_{2+k} \pi_j^2 + \frac{1}{4} T (-7 + 9 T) p_{2+k}^2 \pi_j^2 + \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{2} T^4 p_{2+i} p_{2+j}^2 \pi_i \pi_j^2 - \frac{1}{2} T^4 p_{2+j}^3 \pi_i \pi_j^2 - 2(-1+T) T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i \pi_j^2 + \\
 & \frac{1}{2} (-1+T) T^2 (-1+4T) p_{2+j}^2 p_{2+k} \pi_i \pi_j^2 + \frac{1}{2} (-1+T) T^2 (-1+3T) p_{2+i} p_{2+k}^2 \pi_i \pi_j^2 - \\
 & \frac{1}{2} (-1+T) T (1-5T+3T^2) p_{2+j} p_{2+k}^2 \pi_i \pi_j^2 - \frac{1}{2} (-1+T) T (-1+3T) p_{2+k}^3 \pi_i \pi_j^2 - \\
 & \frac{1}{6} (-1+T) T^2 p_{2+j}^2 p_{2+k} \pi_j^3 + \frac{1}{6} (-1+T) T (-1+4T) p_{2+j} p_{2+k}^2 \pi_j^3 - \frac{1}{6} (-1+T) T (-1+3T) p_{2+k}^3 \pi_j^3 + \\
 & 2 p_{2+k} \pi_k + \frac{5}{2} T^2 p_{2+i} p_{2+k} \pi_i \pi_k - \frac{1}{2} T (-5+7T) p_{2+j} p_{2+k} \pi_i \pi_k - \frac{7}{2} T p_{2+k}^2 \pi_i \pi_k + \\
 & \frac{1}{2} T^4 p_{2+i}^2 p_{2+k} \pi_i^2 \pi_k - 2(-1+T) T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i^2 \pi_k + \frac{1}{2} (-1+T) T^2 (-1+3T) p_{2+j}^2 p_{2+k} \pi_i^2 \pi_k - \\
 & \frac{1}{2} T^2 (-1+4T) p_{2+i} p_{2+k}^2 \pi_i^2 \pi_k + \frac{1}{2} (-1+T) T (-1+4T) p_{2+j} p_{2+k}^2 \pi_i^2 \pi_k + \frac{1}{2} T (-1+3T) p_{2+k}^3 \pi_i^2 \pi_k + \\
 & \frac{7}{2} T p_{2+j} p_{2+k} \pi_j \pi_k - \frac{9}{2} T p_{2+k}^2 \pi_j \pi_k + 3 T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i \pi_j \pi_k - T^2 (-1+3T) p_{2+j}^2 p_{2+k} \pi_i \pi_j \pi_k - \\
 & T^2 (-1+3T) p_{2+i} p_{2+k} \pi_i \pi_j \pi_k + T (1-5T+3T^2) p_{2+j} p_{2+k}^2 \pi_i \pi_j \pi_k + T (-1+3T) p_{2+k}^3 \pi_i \pi_j \pi_k + \\
 & \frac{1}{2} T^2 p_{2+j}^2 p_{2+k} \pi_j^2 \pi_k - \frac{1}{2} T (-1+4T) p_{2+j} p_{2+k}^2 \pi_j^2 \pi_k + \frac{1}{2} T (-1+3T) p_{2+k}^3 \pi_j^2 \pi_k + \frac{1}{2} T^2 p_{2+i} p_{2+k}^2 \pi_i \pi_k^2 - \\
 & \frac{1}{2} (-1+T) T p_{2+j} p_{2+k} \pi_i \pi_k^2 - \frac{1}{2} T p_{2+k}^3 \pi_i \pi_k^2 + \frac{1}{2} T p_{2+j} p_{2+k}^2 \pi_j \pi_k^2 - \frac{1}{2} T p_{2+k}^3 \pi_j \pi_k^2 \Big]
 \end{aligned}$$

Out[]=

True

Invariance Under Reidemeister 2b



$$\begin{aligned}
 \text{lhs} &= \int \mathbb{E} [\pi_i p_i + \pi_j p_j] \mathcal{L} / @ (X_{i,j} [1] X_{i+1,j+1} [-1]) \mathbb{d} \{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\} \\
 \text{rhs} &= \int \mathbb{E} [\pi_i p_i + \pi_j p_j] \mathcal{L} / @ (C_i [0] C_{i+1} [0] C_j [0] C_{j+1} [0]) \mathbb{d} \{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}; \\
 \text{lhs} &= \text{rhs}
 \end{aligned}$$

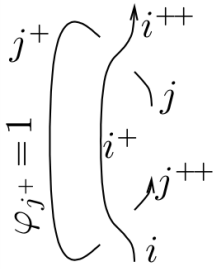
Out[]=

$$\mathbb{E} [\epsilon \text{Series} [p_{2+i} \pi_i + p_{2+j} \pi_j, 0, 0]]$$

Out[]=

True

Invariance Under R2c

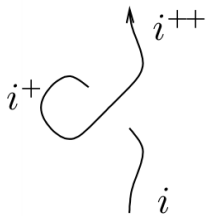


```
In[*]:= lhs = Integrate[E[pi p_i + pj p_j] L /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
  d{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
rhs = Integrate[E[pi p_i + pj p_j] L /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
  d{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
lhs == rhs
```

```
Out[*]= -i sqrt(T) E[Series[p_{2+i} pi + p_{3+j} pj, -1/2 - p_{3+j} pj, 1/2 p_{3+j} pj]]
```

```
Out[*]= True
```

Invariance Under R1l

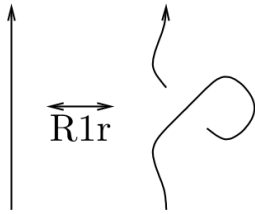


```
In[*]:= lhs = Integrate[E[pi p_i] L /@ (X_{i+2,i}[1] C_{i+1}[1]) d{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = Integrate[E[pi p_i] L /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) d{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs
```

```
Out[*]= -i E[Series[p_{3+i} pi, 0, 0]]
```

```
Out[*]= True
```

Invariance Under R1r

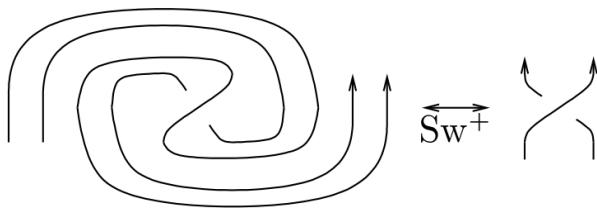


```
In[ ]:= lhs = ∫ E[πi pi] L /@ (Xi,i+2[1] Ci+1[-1]) d{xi, pi, xi+1, pi+1, xi+2, pi+2}
rhs = ∫ E[πi pi] L /@ (Ci[0] Ci+1[0] Ci+2[0]) d{xi, pi, xi+1, pi+1, xi+2, pi+2};
lhs == rhs
```

```
Out[ ]:= - i E[εSeries[p3+i πi, 0, 0]]
```

```
Out[ ]:= True
```

Invariance Under Sw



```
In[*]:= lhs = Integrate[E[p_i p_i + p_j p_j] L / @ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1])
  d[{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}}]
rhs = Integrate[E[p_i p_i + p_j p_j] L / @ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0])
  d[{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}}];
lhs == rhs
```

Out[*]=

$$\sqrt{T} \mathbb{E} \left[\epsilon \text{Series} \left[T p_{3+i} \pi_i + (1 - T) p_{3+j} \pi_i + p_{3+j} \pi_j, -\frac{1}{2} + T p_{3+j} \pi_i + \frac{1}{2} (-1 + T) T p_{3+i} p_{3+j} \pi_i^2 - \frac{1}{2} (-1 + T) T p_{3+j}^2 \pi_i^2 - p_{3+j} \pi_j - T p_{3+i} p_{3+j} \pi_i \pi_j + T p_{3+j}^2 \pi_i \pi_j, -\frac{1}{2} T p_{3+j} \pi_i - \frac{1}{4} T (-1 + 3 T) p_{3+i} p_{3+j} \pi_i^2 + \frac{1}{4} T (-3 + 5 T) p_{3+j}^2 \pi_i^2 - \frac{1}{6} (-1 + T) T^2 p_{3+i}^2 p_{3+j} \pi_i^3 + \frac{1}{6} (-1 + T) T (-1 + 4 T) p_{3+i} p_{3+j}^2 \pi_i^3 - \frac{1}{6} (-1 + T) T (-1 + 3 T) p_{3+j}^3 \pi_i^3 + \frac{1}{2} p_{3+j} \pi_j + \frac{3}{2} T p_{3+i} p_{3+j} \pi_i \pi_j - \frac{5}{2} T p_{3+j}^2 \pi_i \pi_j + \frac{1}{2} T^2 p_{3+i}^2 p_{3+j} \pi_i^2 \pi_j - \frac{1}{2} T (-1 + 4 T) p_{3+i} p_{3+j}^2 \pi_i^2 \pi_j + \frac{1}{2} T (-1 + 3 T) p_{3+j}^3 \pi_i^2 \pi_j + \frac{1}{2} T p_{3+i} p_{3+j}^2 \pi_i \pi_j^2 - \frac{1}{2} T p_{3+j}^3 \pi_i \pi_j^2 \right] \right]$$

Out[*]=

True