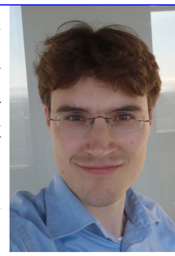
Knot Invariants from Finite Dimensional Integration

Abstract. For the purpose of today, an “I-Type Knot Invariant” is a knot invariant computed from a knot diagram by integrating the exponential of a Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.



joint with R. van der Veen

Q. Are there any such things? **A.** Yes.

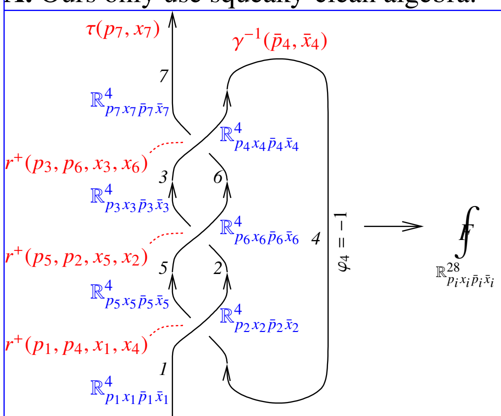
Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.

Q. Didn’t Witten do that back in 1988 with path integrals?

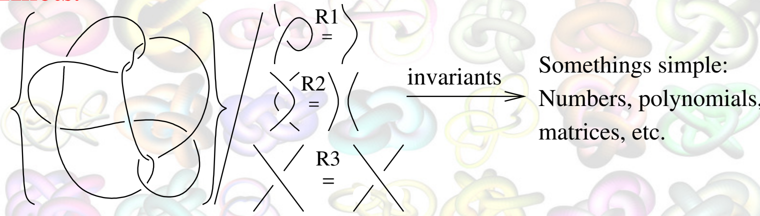
A. No. His constructions are infinite dimensional and far from rigorous.

Q. But integrals belong in analysis!

A. Ours only use squeaky-clean algebra.



Knots.



The Good. 1. At the centre of low dimensional topology.
2. “Invariants” connect to pretty much all of algebra.

The Agony. 1&2 don’t talk to each other.

- Not enough topological applications for all these invariants.
- The fancy algebra doesn’t arise naturally within topology.

⟹ We’re still missing something about the relationship between knots and algebra.

(Alternative) Gausssian Integration.

Goal. Compute
$$I_1(0) := \int d^n x P(x) \exp\left(-\frac{1}{2}a^{ij}x_i x_j + V(x)\right).$$
Solution. Set
$$I_\lambda(x) := \int d^n x' P(x+x') \exp\left(-\frac{1}{2\lambda}a^{ij}x'_i x'_j + V(x+x')\right).$$
Then $I_1(0)$ is what we want, $I_0(x) = (\det A)^{-1/2}P(x)\exp V(x)$, and
$$\partial_\lambda I_\lambda(x) = \frac{1}{2\lambda^2}\int d^n x' a^{ij}x'_i x'_j P(x+x') \exp\left(-\frac{1}{2\lambda}a^{ij}x'_i x'_j + V(x+x')\right)$$
While with g_{ij} the inverse matrix of a^{ij},
$$\frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}I_\lambda(x) = \int d^n x' \frac{1}{2}g_{ij}(\partial_{x_i} - \partial_{x'_i})(\partial_{x_j} - \partial_{x'_j})P(x+x') \exp\left(-\frac{1}{2\lambda}a^{ij}x'_i x'_j + V(x+x')\right)$$
$$= \frac{1}{2\lambda^2}\int d^n x' a^{ij}x'_i x'_j P(x+x') \exp\left(-\frac{1}{2\lambda}a^{ij}x'_i x'_j + V(x+x')\right).$$
Hence
$$\partial_\lambda I_\lambda(x) = \frac{1}{2}g_{ij}\partial_{x_i}\partial_{x_j}I_\lambda(x),$$
and therefore
$$I_\lambda(x) = (\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)P(x)\exp V(x).$$

References.

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