Pensieve header: The rank 2 mod \$\epsilon^2\$ invariant using integration techniques; continues Rank2.nb at pensieve://Talks/Beijing-2407/ and UC4A2.nb and Theta.nb at pensieve://Projects/HigherRank/.

Initialization

```
In[◦]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Geneva-2408"];
        Once[<< IType.m];
```

pdf

```
In[◦]:= T3 = T1 T2;  i_+ := i + 1;
        $π = (CF@Normal[# + O[ϵ]^2] /. {π_is__ :→ B^-1 π_is, x_is__ :→ B^-1 x_is, p_is__ :→ B p_is} /.
              ϵ B^b_ /; b < 0 → 0 /. B → 1) &;
```

pdf

```
In[◦]:= vs_i_ := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
        ℱ[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
        ℒ[K_] := CF[ℒ /@ Features[K]〚2〛];
        vs[K_] := Union @@ Table[{vs_i}, {i, Features[K]〚1〛}]
```

The Lagrangian

tex

\needspace{30mm}
{\bf\red The Lagrangian.}

exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

```
In[◦]:= ℒ[X_{i_,j_}[s_]] := T3^s 𝔼[CF@Plus[
          Sum_{ν=1}^{3} (x_{νi} (p_{νi+} - p_{νi}) + x_{νj} (p_{νj+} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi+} - p_{νj+})),
          (T1^s - 1) p_{3j} x_{1i} (T2^s x_{2i} - x_{2j}),
          ϵ s (T3^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T2^s - 1),
          ϵ s (1/2 + T2^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} - (T2^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} +
             (T3^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} + 2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} - T2^s p_{2j} p_{3j} x_{2i} x_{3j} +
             ((T1^s - 1) p_{1j} x_{1i} (T2^{2s} p_{2j} x_{2i} - T2^s p_{2j} x_{2j} - (T2^s + 1) (T3^s - 1) p_{3j} x_{3i} + T2^s p_{3j} x_{3j}) +
               (T3^s - 1) p_{3j} x_{3i} (1 - T2^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T2^s - 2) p_{2j} x_{2j})) / (T2^s - 1))]]
```

In[∘]:=

$$\text{CF}\Big[\text{Plus}\Big[\sum_{v=1}^{3}\left(x_{vi}\,(p_{vi^+}-p_{vi})+x_{vj}\,(p_{vj^+}-p_{vj})+\left(T_v^s-1\right)x_{vi}\,(p_{vi^+}-p_{vj^+})\right),$$
$$\left(-1+T_1^s\right)p_{3,j}\,x_{1,i}\left(T_2^s\,x_{2,i}-x_{2,j}\right),$$
$$\epsilon\left(\frac{1}{-1+T_2^s}\,s\left(-1+(T_1\,T_2)^s\right)p_{1,j}\,(p_{2,i}-p_{2,j})\,x_{3,i}\right),$$
$$\epsilon\Bigg(\frac{s}{2}+s\,T_2^s\,p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,i}+\frac{s\left(-1+T_1^s\right)T_2^{2s}\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{-1+T_2^s}-s\,p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,j}-$$
$$\frac{s\left(-1+T_1^s\right)T_2^s\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{-1+T_2^s}-s\,p_{3,i}\,x_{3,i}+\frac{s\left(-1+T_3^s\right)p_{3,j}\,x_{3,i}}{-1+T_2^s}-\frac{s\,T_2^s\left(-1+T_3^s\right)p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s}-$$
$$\frac{s\left(-1+T_1^s\right)\left(1+T_2^s\right)\left(-1+T_3^s\right)p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s}-s\left(-1+T_2^s\right)p_{2,j}\,p_{3,i}\,x_{2,i}\,x_{3,i}+s\left(-1+T_3^s\right)p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,i}+$$
$$2\,s\,p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i}+\frac{s\left(-1+T_3^s\right)p_{2,i}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s}+\frac{s\left(-2+T_2^s\right)\left(-1+T_3^s\right)p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s}+s\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}+$$
$$\frac{s\left(-1+T_1^s\right)T_2^s\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{-1+T_2^s}-s\,p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,j}-s\,T_2^s\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}\Bigg)\Big]-\mathcal{L}[X_{i,j}[s]][[2,1]]\Big]$$

Out[∘]=

0

pdf

In[∘]:=

$$\mathcal{L}[C_{i\_}[\varphi\_]] := T_3^{\varphi}\,\mathbb{E}\left[\sum_{v=1}^{3}x_{vi}\,(p_{vi^+}-p_{vi})+\epsilon\,\varphi\,(p_{3i}\,x_{3i}-1/2)\right]$$

exec

```
nb2tex$PDFWidth /= 1.25;
```

Reidemeister 3

tex

{\bf\red Reidemeister 3.}

pdf

In[∘]:=

$$\text{Short}\left[\text{lhs} = \int \mathcal{F}[i,j,k]\;\mathcal{L}\,/@\,(X_{i,j}[1]\,X_{i^+,k}[1]\,X_{j^+,k^+}[1])\,\mathbb{d}\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\}\right]$$

Out[∘]//Short=

pdf

$$T_1^3\,T_2^3\,\mathbb{E}\left[\frac{3\,\epsilon}{2}+T_1^2\,p_{1,2+i}\,\pi_{1,i}-(-1+T_1)\,T_1\,p_{1,2+j}\,\pi_{1,i}+\ll 150 \gg\right]$$

pdf

In[∘]:=

$$\text{rhs} = \int \mathcal{F}[i,j,k]\;\mathcal{L}\,/@\,(X_{j,k}[1]\,X_{i,k^+}[1]\,X_{i^+,j^+}[1])\,\mathbb{d}\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\};\ \text{lhs} == \text{rhs}$$

Out[∘]=

pdf

True

The Trefoil

tex

\needspace{25mm}

\parpic[r]{\includegraphics[width=0.6in]{../Beijing-2407/Trefoil.jpg}}

{\bf\red The Trefoil.}

pdf

In[∘]:=
```
K = Knot[3, 1]; ∫ℒ[K] ⅆ vs[K]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$$-\frac{\mathbb{i}\, T_1^2\, T_2^2\, \mathbb{E}\left[-\frac{\epsilon\left(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}$$

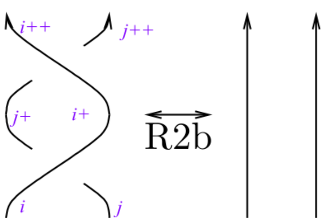
In[∘]:=

$$-\frac{\mathbb{i}\, T_1^2\, T_2^2\, \mathbb{E}\left[-\frac{\epsilon\left(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}$$

Out[∘]=

$$-\frac{\mathbb{i}\, T_1^2\, T_2^2\, \mathbb{E}\left[-\frac{\epsilon\left(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}$$

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+},\ x_{i|j}$.

In[∘]:=
```
lhs = ∫𝓕[i, j] ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}}
rhs = ∫𝓕[i, j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}};
lhs == rhs
```

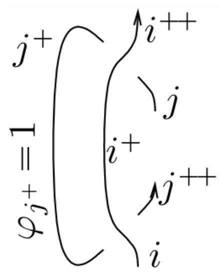
Out[∘]=

$$\mathbb{E}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}]$$

Out[∘]=

True

Invariance Under R2c

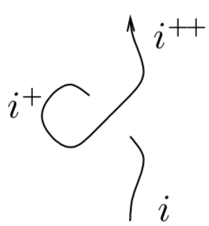


In[∘]:=
$$\text{lhs} = \int \mathcal{F}[i, j]\ \mathcal{L} /@ (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\}$$
$$\text{rhs} = \int \mathcal{F}[i, j]\ \mathcal{L} /@ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\};$$
$$\text{lhs} == \text{rhs}$$

Out[∘]=
$$-\mathbb{i}\ T_1\ T_2\ \mathbb{E}\left[\frac{\epsilon}{2} + p_{1,2+i}\ \pi_{1,i} + p_{1,3+j}\ \pi_{1,j} + p_{2,2+i}\ \pi_{2,i} + p_{2,3+j}\ \pi_{2,j} + p_{3,2+i}\ \pi_{3,i} + p_{3,3+j}\ \pi_{3,j} + \epsilon\ p_{3,3+j}\ \pi_{3,j}\right]$$

Out[∘]=
True

Invariance Under R1l

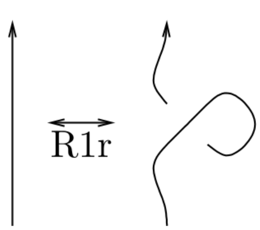


In[∘]:=
$$\text{lhs} = \int \mathcal{F}[i]\ \mathcal{L} /@ (X_{i+2,i}[1]\ C_{i+1}[1])\ \mathbb{d}\{vs_i, vs_{i^+}, vs_{i+2}\}$$
$$\text{rhs} = \int \mathcal{F}[i]\ \mathcal{L} /@ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ \mathbb{d}\{vs_i, vs_{i^+}, vs_{i+2}\};$$
$$\text{lhs} == \text{rhs}$$

Out[∘]=
$$-\mathbb{i}\ \mathbb{E}[p_{1,3+i}\ \pi_{1,i} + p_{2,3+i}\ \pi_{2,i} + p_{3,3+i}\ \pi_{3,i}]$$

Out[∘]=
True

Invariance Under R1r



In[∘]:= **lhs** = $\int \mathcal{F}[i]\ \mathcal{L}$ /@ $(X_{i,i+2}[1]\ C_{i+1}[-1])\ \mathbb{d}\{vs_i, vs_{i^+}, vs_{i+2}\}$

rhs = $\int \mathcal{F}[i]\ \mathcal{L}$ /@ $(C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ \mathbb{d}\{vs_i, vs_{i^+}, vs_{i+2}\}$;

lhs == rhs

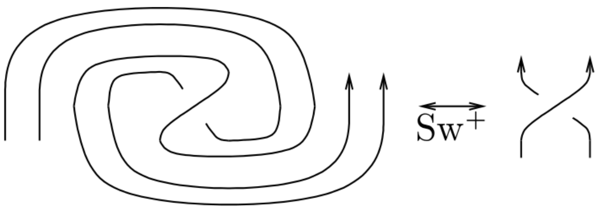
Out[∘]=

$-\mathbb{i}\ \mathbb{E}[p_{1,3+i}\ \pi_{1,i} + p_{2,3+i}\ \pi_{2,i} + p_{3,3+i}\ \pi_{3,i}]$

Out[∘]=

True

Invariance Under Sw



In[∘]:=
```
lhs =
  ∫𝓕[i, j] ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1]) d{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{i+2}, vs_{j+2}}
rhs =
  ∫𝓕[i, j] ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0]) d{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{i+2}, vs_{j+2}};
lhs == rhs
```

Out[∘]=

$$T_1 T_2 \mathbb{E}\Big[\frac{\epsilon}{2} + T_1 p_{1,3+i} \pi_{1,i} + (1 - T_1) p_{1,3+j} \pi_{1,i} + p_{1,3+j} \pi_{1,j} + T_2 p_{2,3+i} \pi_{2,i} - \epsilon T_2 p_{2,3+i} \pi_{2,i} + (1 - T_2) p_{2,3+j} \pi_{2,i} + \epsilon T_1 T_2 p_{1,3+i} p_{2,3+j} \pi_{1,i} \pi_{2,i} + \frac{\epsilon (-1 + T_1) T_2 p_{1,3+j} p_{2,3+j} \pi_{1,i} \pi_{2,i}}{-1 + T_2} + (-1 + T_1) T_2 p_{3,3+j} \pi_{1,i} \pi_{2,i} + p_{2,3+j} \pi_{2,j} + \epsilon p_{2,3+j} \pi_{2,j} - \epsilon T_1 p_{1,3+i} p_{2,3+j} \pi_{1,i} \pi_{2,j} - \frac{\epsilon (-1 + T_1) p_{1,3+j} p_{2,3+j} \pi_{1,i} \pi_{2,j}}{-1 + T_2} + (1 - T_1) p_{3,3+j} \pi_{1,i} \pi_{2,j} + \frac{\epsilon T_2 (-1 + T_1 T_2) p_{1,3+j} p_{2,3+i} \pi_{3,i}}{-1 + T_2} - \frac{\epsilon T_2 (-1 + T_1 T_2) p_{1,3+j} p_{2,3+j} \pi_{3,i}}{-1 + T_2} + T_1 T_2 p_{3,3+i} \pi_{3,i} + \epsilon T_1 T_2 p_{3,3+i} \pi_{3,i} + (1 - T_1 T_2) p_{3,3+j} \pi_{3,i} - \frac{\epsilon T_2 (-1 + T_1 T_2) p_{3,3+j} \pi_{3,i}}{-1 + T_2} - \frac{\epsilon T_1 T_2 (-1 + T_1 T_2) p_{1,3+i} p_{3,3+j} \pi_{1,i} \pi_{3,i}}{-1 + T_2} + \frac{\epsilon (-1 + T_1) T_2 (-1 + T_1 T_2) p_{1,3+j} p_{3,3+j} \pi_{1,i} \pi_{3,i}}{-1 + T_2} - \epsilon T_1 (-1 + T_2) T_2 p_{2,3+j} p_{3,3+i} \pi_{2,i} \pi_{3,i} + \epsilon T_2 (-1 + T_1 T_2) p_{2,3+j} p_{3,3+j} \pi_{2,i} \pi_{3,i} + 2 \epsilon T_1 T_2 p_{2,3+j} p_{3,3+i} \pi_{2,j} \pi_{3,i} + \frac{\epsilon T_2 (-1 + T_1 T_2) p_{2,3+i} p_{3,3+j} \pi_{2,j} \pi_{3,i}}{-1 + T_2} - \frac{\epsilon (-1 + 2 T_2) (-1 + T_1 T_2) p_{2,3+j} p_{3,3+j} \pi_{2,j} \pi_{3,i}}{-1 + T_2} + p_{3,3+j} \pi_{3,j} + \epsilon T_1 p_{1,3+i} p_{3,3+j} \pi_{1,i} \pi_{3,j} + \frac{\epsilon (-1 + T_1) p_{1,3+j} p_{3,3+j} \pi_{1,i} \pi_{3,j}}{-1 + T_2} - \epsilon T_2 p_{2,3+i} p_{3,3+j} \pi_{2,i} \pi_{3,j} - \epsilon p_{2,3+j} p_{3,3+j} \pi_{2,i} \pi_{3,j}\Big]$$

Out[∘]=

True