Pensieve header: Mathematica notebook for Talks: Geneva-2408.

Ancestors in Talks/Beijing-2407 and in Projects/HigherRank.

exec

```
nb2tex$TeXFileName = "IType1.tex";
```

In[ ]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Geneva-2408"];
```

Preliminaries

tex

{\bf\red Implementation} (see IType.nb of \web{ap}).

pdf

In[ ]:=

```
Once[<< KnotTheory`; << Rot.m];
```

pdf

```
C:\drorbn\AcademicPensieve\Projects\KnotTheory\KnotTheory
```

pdf

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/AP/Talks/Geneva-2408 to compute rotation numbers.
```

pdf

In[ ]:=

```
CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
   vs = Cases[ℰ, (x | p | ξ | π | g)__, ∞] ⋃ {ϵ};
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ Factor[c] (Times @@ vs^ps)] ];
```

tex

\vskip 1mm\rule{\linewidth}{1pt}\vspace{-2mm}

Integration

tex

{\bf\red Integration} using Picard iteration. The \myellow{core is in yellow} and \mpink{hacks are in pink}.

pdf

In[ ]:=

```
𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B];
```

pdf

In[ ]:=

```
$π = Identity; (* The Wisdom Projection *)
```

pdf

In[∘]:=
```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, DDZ, FZ, a, b},
    n = Length@vs; L0 = L /. ϵ → 0;
    Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /. (p | x)__ → 0, {a, n}, {b, n}];
    If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
    Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
    FixedPoint[(DZ = Table[∂_v Z, {v, vs}];
        DDZ = Table[∂_u DZ, {u, vs}];
        FZ = Sum[G⟦a, b⟧ (DDZ⟦a, b⟧ + DZ⟦a⟧ DZ⟦b⟧), {a, n}, {b, n}] / 2;
        Z = CF[Z0 + ∫_0^λ $π[FZ] ⅆλ]) &, Z];
    PowerExpand@Factor[ω Δ^(-1/2)] 𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

tex

\parpic[r]{\parbox{0.75in}{
\includegraphics[width=0.75in]{../../Projects/Gallery/Fourier.jpg}
\footnotesize Joseph Fourier
}}
\picskip{2}

pdf

In[∘]:= $\int \mathbb{E}\left[-\mu x^2/2 + i\,\xi\,x\right] ⅆ\{x\}$

Out[∘]=
pdf

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

tex

\needspace{12mm}

pdf

In[∘]:= FofG = $\int \mathbb{E}\left[-\mu\,(x-a)^2/2 + i\,\xi\,x\right] ⅆ\{x\}$

Out[∘]=
pdf

$$\frac{\mathbb{E}\left[\frac{i\,(2\,a\,\mu + i\,\xi)\,\xi}{2\,\mu}\right]}{\sqrt{\mu}}$$

tex

\needspace{12mm}

pdf

In[∘]:= $\int \text{FofG}\, \mathbb{E}[-i\, \xi\, x]\, d\{\xi\}$

Out[∘]=
pdf

$$\mathbb{E}\left[-\frac{1}{2}(a-x)^2\mu\right]$$

tex

So we've tested and nearly proven the Fourier inversion formula!

pdf

In[∘]:= $L = -\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$

$Z12 = \int \mathbb{E}[L]\, d\{x_1, x_2\}$

Out[∘]=
pdf

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\,(-b^2+a\,c)} + \frac{b\,\xi_1\,\xi_2}{b^2-a\,c} + \frac{a\,\xi_2^2}{2\,(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$$

tex

\parpic[r]{\parbox{0.65in}{
\includegraphics[width=0.65in]{../../Projects/Gallery/Fubini.jpg}
\scriptsize Guido Fubini
}}
\picskip{2}

pdf

In[∘]:= $\left\{Z1 = \int \mathbb{E}[L]\, d\{x_1\},\ Z12 == \int Z1\, d\{x_2\}\right\}$

Out[∘]=
pdf

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\,x_2^2}{2\,a} - \frac{b\,x_2\,\xi_1}{a} + \frac{\xi_1^2}{2\,a} + x_2\,\xi_2\right]}{\sqrt{a}}, \text{True}\right\}$$

pdf

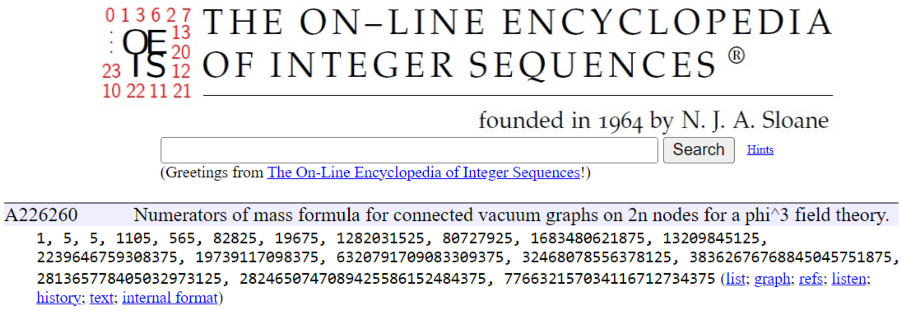
In[∘]:= $\$\pi = \text{Normal}[\# + O[\epsilon]^{13}]\ \&;\ \int \mathbb{E}[-\phi^2/2 + \epsilon\,\phi^3/6]\, d\{\phi\}$

Out[∘]=
pdf

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24} + \frac{5\,\epsilon^4}{16} + \frac{1105\,\epsilon^6}{1152} + \frac{565\,\epsilon^8}{128} + \frac{82\,825\,\epsilon^{10}}{3072} + \frac{19\,675\,\epsilon^{12}}{96}\right]$$

tex

\vskip 1mm
From \surl{oeis.org/A226260}:
\vskip 1mm
\includegraphics[width=\linewidth]{../Groningen-240530/OEIS.png}



THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.

1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 38362676768845045751875, 281365778405032973125, 2824650747089425586152484375, 776632157034116712734375 (list; graph; refs; listen; history; text; internal format)

tex

\vskip -3mm\rule{\linewidth}{1pt}\vspace{-2mm}

The Right-Handed Trefoil

tex

{\bf\red The Right-Handed Trefoil.}

pdf

```
In[∘]:= K = Mirror@Knot[3, 1]; Features[K]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$\text{Features}[7, C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]]$

pdf

```
In[∘]:= ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[
          x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
           (ϵ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
        ℒ[C_{i_}[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
        ℒ[K_] := CF[ℒ /@ Features[K][[2]]]
        vs[K_] := Join @@ Table[{p_i, x_i}, {i, Features[K][[1]]}]
```

exec

```
In[∘]:= nb2tex$PDFWidth *= 1.25;
```

tex

\needspace{5cm}

pdf

In[•]:= {vs[K], ℒ[K]}

Out[•]=
pdf

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},\ T\,\mathbb{E}\Big[-2\in - p_1 x_1 + \in p_1 x_1 + T p_2 x_1 - \in p_5 x_1 + (1-T)\, p_6 x_1 + \frac{1}{2}(-1+T)\in p_1 p_5 x_1^2 + \frac{1}{2}(1-T)\in p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 - p_3 x_3 + \in p_3 x_3 + T p_4 x_3 - \in p_7 x_3 + (1-T)\, p_8 x_3 + \frac{1}{2}(-1+T)\in p_3 p_7 x_3^2 + \frac{1}{2}(1-T)\in p_7^2 x_3^2 - p_4 x_4 + \in p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \in p_1 p_5 x_1 x_5 + \in p_5^2 x_1 x_5 - \in p_2 x_6 + (1-T)\, p_3 x_6 - p_6 x_6 + \in p_6 x_6 + T p_7 x_6 + \in p_2^2 x_2 x_6 - \in p_2 p_6 x_2 x_6 + \frac{1}{2}(1-T)\in p_2^2 x_6^2 + \frac{1}{2}(-1+T)\in p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \in p_3 p_7 x_3 x_7 + \in p_7^2 x_3 x_7\Big]\Big\}$$

exec

In[•]:= nb2tex$PDFWidth /= 1.25;

tex

\needspace{10mm}

pdf

In[•]:= $\$\pi$ = Normal[# + O[∈]^2] &; $\int$ ℒ[K] ⅆ vs[K]

Out[•]=
pdf

$$-\frac{i\,T\,\mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^2)\in}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

In[•]:= $\int$(ℒ[K] /. $x_{i\_}$:→ ⅈ x_i) ⅆ (vs@K)

Out[•]=

$$\frac{T\,\mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^2)\in}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

tex

\vskip 1mm
A faster program to compute \$\rho_1\$, and more stories about it, are at~\cite{APAI}.

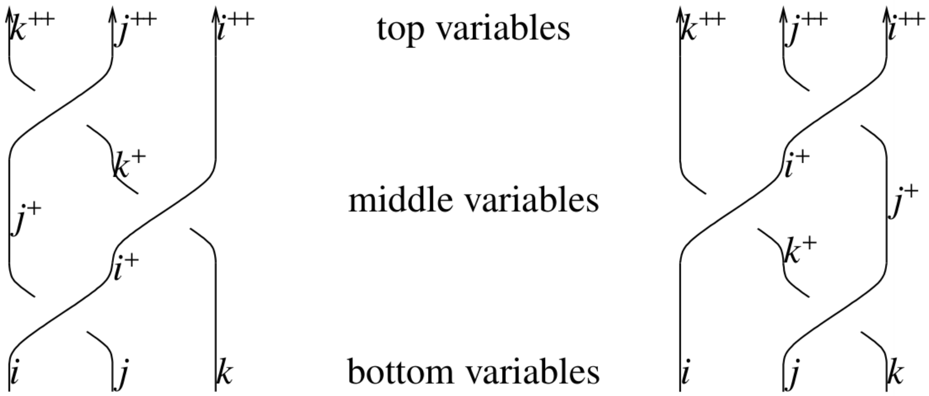
\rule{\linewidth}{1pt}\vspace{2mm}\vskip -2mm
%\newcolumn

Invariance Under Reidemeister 3

tex

{\bf\red Invariance Under Reidemeister 3.}
\vskip 2mm
\def\ip{{i^+}} \def\jp{{j^+}} \def\kp{{k^+}}

\def\ipp{{i^{+\!+}}} \def\jpp{{j^{+\!+}}} \def\kpp{{k^{+\!+}}}
\import{../Beijing-2407/figs}{R3.pdf_t}



pdf

In[∘]:=
```
lhs = ∫(ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])) ⅆ{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
rhs = ∫(ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1])) ⅆ{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}};
lhs === rhs
```

Out[∘]=
pdf

```
False
```

tex

\vskip 1mm\rule{\linewidth}{1pt}\vspace{2mm}

Invariance Under Reidemeister 3, Take 2

tex

{\bf\red Invariance Under Reidemeister 3, Take 2.}

pdf

In[∘]:=
```
lhs = ∫(ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])) ⅆ{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
rhs = ∫(ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1])) ⅆ{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}};
lhs === rhs
```

Out[∘]=
pdf

```
True
```

pdf

In[∘]:=
```
lhs
```

Out[∘]=
pdf

```
Degenerate Q!
```

tex

\newcolumn

Invariance Under Reidemeister 3, Take 3

tex

{\bf\red Invariance Under Reidemeister 3, Take 3.}

exec

In[∘]:= `nb2tex$PDFWidth *= 1.25;`

pdf

In[∘]:=
$$\text{lhs} = \int \left(\mathbb{E}[i\,\pi_i\,p_i + i\,\pi_j\,p_j + i\,\pi_k\,p_k]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1])\right)$$
$$d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$
$$\text{rhs} = \int \left(\mathbb{E}[i\,\pi_i\,p_i + i\,\pi_j\,p_j + i\,\pi_k\,p_k]\ \mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1])\right)$$
$$d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$
`lhs == rhs`

Out[∘]=
pdf

True

tex

\needspace{20mm}

pdf

In[∘]:= `lhs`

Out[∘]=
pdf

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\in}{2} + i\,T^2\,p_{2+i}\,\pi_i - i\,(-1+T)\,T\,p_{2+j}\,\pi_i + i\,T^2\in p_{2+j}\,\pi_i - i\,(-1+T)\,p_{2+k}\,\pi_i + i\,T\in p_{2+k}\,\pi_i -$$
$$\frac{1}{2}(-1+T)\,T^3\in p_{2+i}\,p_{2+j}\,\pi_i^2 + \frac{1}{2}(-1+T)\,T^3\in p_{2+j}^2\,\pi_i^2 - \frac{1}{2}(-1+T)\,T^2\in p_{2+i}\,p_{2+k}\,\pi_i^2 +$$
$$\frac{1}{2}(-1+T)^2\,T\in p_{2+j}\,p_{2+k}\,\pi_i^2 + \frac{1}{2}(-1+T)\,T\in p_{2+k}^2\,\pi_i^2 + i\,T\,p_{2+j}\,\pi_j - i\,T\in p_{2+j}\,\pi_j -$$
$$i\,(-1+T)\,p_{2+k}\,\pi_j + i\,(-1+2T)\in p_{2+k}\,\pi_j + T^3\in p_{2+i}\,p_{2+j}\,\pi_i\,\pi_j - T^3\in p_{2+j}^2\,\pi_i\,\pi_j -$$
$$(-1+T)\,T^2\in p_{2+i}\,p_{2+k}\,\pi_i\,\pi_j + (-1+T)^2\,T\in p_{2+j}\,p_{2+k}\,\pi_i\,\pi_j + (-1+T)\,T\in p_{2+k}^2\,\pi_i\,\pi_j -$$
$$\frac{1}{2}(-1+T)\,T\in p_{2+j}\,p_{2+k}\,\pi_j^2 + \frac{1}{2}(-1+T)\,T\in p_{2+k}^2\,\pi_j^2 + i\,p_{2+k}\,\pi_k - 2\,i\in p_{2+k}\,\pi_k + T^2\in p_{2+i}\,p_{2+k}\,\pi_i\,\pi_k -$$
$$(-1+T)\,T\in p_{2+j}\,p_{2+k}\,\pi_i\,\pi_k - T\in p_{2+k}^2\,\pi_i\,\pi_k + T\in p_{2+j}\,p_{2+k}\,\pi_j\,\pi_k - T\in p_{2+k}^2\,\pi_j\,\pi_k\Big]$$

exec

In[∘]:= `nb2tex$PDFWidth /= 1.25;`

tex

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at \web{ap}.

Invariance Under Reidemeister 3, Take 4 (just for fun)

In[•]:= lhs = $\int$ ($\mathbb{E}$[ⅈ π_i p_i + ⅈ π_j p_j + ⅈ π_k p_k + ⅈ π_{i+2} p_{i+2} + ⅈ π_{j+2} p_{j+2} + ⅈ π_{k+2} p_{k+2} + ⅈ ξ_{i+2} x_{i+2} + ⅈ ξ_{j+2} x_{j+2} + ⅈ ξ_{k+2} x_{k+2}] $\mathcal{L}$ /@ ($X_{i,j}$[1] $X_{i+1,k}$[1] $X_{j+1,k+1}$[1])) ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+2}, p_{j+2}, p_{k+2}, x_{i+2}, x_{j+2}, x_{k+2}};

rhs = $\int$ ($\mathbb{E}$[ⅈ π_i p_i + ⅈ π_j p_j + ⅈ π_k p_k + ⅈ π_{i+2} p_{i+2} + ⅈ π_{j+2} p_{j+2} + ⅈ π_{k+2} p_{k+2} + ⅈ ξ_{i+2} x_{i+2} + ⅈ ξ_{j+2} x_{j+2} + ⅈ ξ_{k+2} x_{k+2}] $\mathcal{L}$ /@ ($X_{j,k}$[1] $X_{i,k+1}$[1] $X_{i+1,j+1}$[1])) ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+2}, p_{j+2}, p_{k+2}, x_{i+2}, x_{j+2}, x_{k+2}};

lhs == rhs

Out[•]=

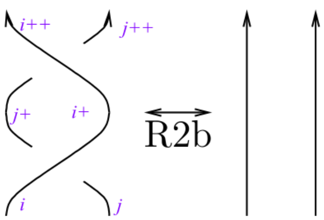
True

In[•]:= lhs

Out[•]=

Degenerate Q!

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[•]:= lhs = $\int$ $\mathbb{E}$[ⅈ π_i p_i + ⅈ π_j p_j] $\mathcal{L}$ /@ ($X_{i,j}$[1] $X_{i+1,j+1}$[-1]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}

rhs = $\int$ $\mathbb{E}$[ⅈ π_i p_i + ⅈ π_j p_j] $\mathcal{L}$ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};

lhs == rhs

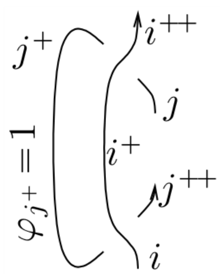
Out[•]=

$\mathbb{E}$[ⅈ p_{2+i} π_i + ⅈ p_{2+j} π_j]

Out[•]=

True

Invariance Under R2c

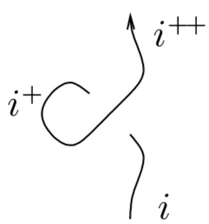


In[∘]:= lhs = $\int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}$ /@ ($X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1]$)
$d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\}$
rhs = $\int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}$ /@ ($C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0]$)
$d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\}$;
lhs == rhs

Out[∘]=
$$-\mathbb{i}\,\sqrt{T}\,\mathbb{E}\left[-\frac{\epsilon}{2} + \mathbb{i}\,p_{2+i}\,\pi_i + \mathbb{i}\,p_{3+j}\,\pi_j - \mathbb{i}\,\epsilon\,p_{3+j}\,\pi_j\right]$$

Out[∘]=
True

Invariance Under R1l

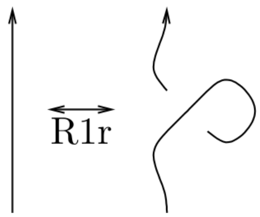


In[∘]:= lhs = $\int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}$ /@ ($X_{i+2,i}[1]\ C_{i+1}[1]$) $d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$
rhs = $\int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}$ /@ ($C_i[0]\ C_{i+1}[0]\ C_{i+2}[0]$) $d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$;
lhs == rhs

Out[∘]=
$$-\mathbb{i}\,\mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Out[∘]=
True

Invariance Under R1r



In[∘]:= **lhs** = $\int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}$ /@ $(X_{i,i+2}[1]\ C_{i+1}[-1])$ $\mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$

rhs = $\int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \mathcal{L}$ /@ $(C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])$ $\mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$;

lhs == rhs

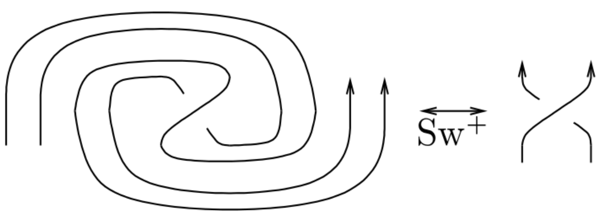
Out[∘]=

$-\mathbb{i}\ \mathbb{E}[\mathbb{i}\ p_{3+i}\ \pi_i]$

Out[∘]=

True

Invariance Under Sw



In[∘]:= **lhs** = $\int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}$ /@ $(X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])$

$\mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\}$

rhs = $\int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \mathcal{L}$ /@ $(X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])$

$\mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\}$;

lhs == rhs

Out[∘]=

$$\sqrt{T}\ \mathbb{E}\Big[-\frac{\epsilon}{2} + \mathbb{i}\,T\,p_{3+i}\,\pi_i - \mathbb{i}\,(-1+T)\,p_{3+j}\,\pi_i + \mathbb{i}\,T\,\epsilon\,p_{3+j}\,\pi_i - \frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i^2 + \frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+j}^2\,\pi_i^2 + \mathbb{i}\,p_{3+j}\,\pi_j - \mathbb{i}\,\epsilon\,p_{3+j}\,\pi_j + T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j - T\,\epsilon\,p_{3+j}^2\,\pi_i\,\pi_j\Big]$$

Out[∘]=

True