Pensieve header: Experiments in 2026: Alternative integrands for θ.

Initialization

```
In[∘]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< "IType (SimplifiedFormatting).m"];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

Minor utilities

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CCF[ℰ_] := Factor[ℰ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, ϵ | (p | x | π | ξ | g)__, F] /. F → CCF];
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼sp___[ℰs___]] := CF /@ 𝔼sp[ℰs];
```

```
In[∘]:= ℒi_[K_] := CF[ℒi /@ Features[K][[2]]]
vs[K_] := Flatten@Table[{pν,i, xν,i}, {ν, 3}, {i, Features[K][[1]]}]
```

```
In[∘]:= {p*, x*, π*, ξ*} = {π, ξ, p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_νi__)* := (u*)νi;
```

```
In[∘]:= HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

g2px and px2g

Modified from pensieve://Talks/Geneva-2408/DataConversions.nb

```
In[∘]:= g2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {gi_,j_ :→ λ pi xj}] /. {λ^k_. :→ 1/k!}]
gν2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {gν_,i_,j_ :→ λν pν,i xν,j}] /. {λ^k_. :→ 1/k!}]
```

```
In[∘]:= Zip{}[ℰ_] := ℰ;
Zip{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip{ξs}, ξ] /. f_. ξ^d_. :→ (D[f, {ξ*, d}])) /. ξ* → 0
```

```
In[•]:= px2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
    ps = Union[Cases[ℰ, p_, ∞]]; xs = Union[Cases[ℰ, x_, ∞]];
    Q = Sum[p0* x0* g_{p0[[2]],x0[[2]]}, {p0, ps}, {x0, xs}];
    Expand[Zip_{ps⋃xs}[ℰ 𝕖^Q]]
   ]
  pxv2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
    ps = Union[Cases[ℰ, p__, ∞]]; xs = Union[Cases[ℰ, x__, ∞]];
    Q = Sum[p0* x0* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
    Expand[Zip_{ps⋃xs}[ℰ 𝕖^Q] /. g_{α_,β_,i_,j_} :→ If[α === β, g_{α,i,j}, 0]]
   ]
```

gRules

From pensieve://Talks/MonteVerita-2604/Theta.nb

```
In[•]:= gR_{s_,i_,j_} := {
    g_{jβ_} :→ g_{j⁺β} + δ_{jβ}, g_{iβ_} :→ T^s g_{i⁺β} + (1 - T^s) g_{j⁺β} + δ_{iβ},
    g_{α_i⁺} :→ T^s g_{αi} + δ_{αi⁺}, g_{α_j⁺} :→ g_{αj} + (1 - T^s) g_{αi} + δ_{αj⁺},
    g_{ν_jβ_} :→ g_{νj⁺β} + δ_{jβ}, g_{ν_iβ_} :→ T_ν^s g_{νi⁺β} + (1 - T_ν^s) g_{νj⁺β} + δ_{iβ},
    g_{ν_α_i⁺} :→ T_ν^s g_{ναi} + δ_{αi⁺}, g_{ν_α_j⁺} :→ g_{ναj} + (1 - T_ν^s) g_{ναi} + δ_{αj⁺}
   }
  gR_{i_} := {g_{i,β_} :→ g_{i⁺,β} + δ_{i,β}, g_{α_,i⁺} :→ g_{α,i} + δ_{α,i⁺}}
```

Inverse gRules written locally:

```
In[•]:= igR_{s_,i_,j_} := {
    g_{j⁺,β_} :→ g_{j,β} - δ_{j,β}, g_{i⁺,β_} :→ T^-s g_{iβ} - (T^-s - 1) g_{j⁺β} - T^-s δ_{iβ},
    g_{α_,i} :→ T^-s (g_{α,i⁺} - δ_{α,i⁺}), g_{α_,j} :→ g_{α,j⁺} - (1 - T^s) g_{α,i} - δ_{α,j⁺},
    g_{ν_,j⁺,β_} :→ g_{ν,j,β} - δ_{j,β}, g_{ν_,i⁺,β_} :→ T_ν^-s g_{νiβ} - (T_ν^-s - 1) g_{νj⁺β} - T_ν^-s δ_{iβ},
    g_{ν_,α_,i} :→ T_ν^-s (g_{ν,α,i⁺} - δ_{α,i⁺}), g_{ν_,α_,j} :→ g_{ν,α,j⁺} - (1 - T_ν^s) g_{ν,α,i} - δ_{α,j⁺}
   }
```

The Base Lagrangian

From pensieve://Talks/Rank2.nb.

pdf

```
In[•]:= T_3 = T_1 T_2;  i_⁺ := i + 1;
  $π =
    (CF@Normal[# + O[ϵ]^2] /. {π_{is__} :→ B^-1 π_{is}, x_{is__} :→ B^-1 x_{is}, p_{is__} :→ B p_{is}} /. ϵ B^{b_} /; b < 0 → 0 /.
      B → 1) &;
```

In[∘]:=

$$\mathcal{L}_1[X_{i\_,j\_}[s\_]] := T_3^s\, \mathbb{E}\Big[\text{CF@Plus}\Big[$$
$$\sum_{\nu=1}^{3} \left(x_{\nu i}\, (p_{\nu i^+} - p_{\nu i}) + x_{\nu j}\, (p_{\nu j^+} - p_{\nu j}) + \left(T_\nu^s - 1\right) x_{\nu i}\, (p_{\nu i^+} - p_{\nu j^+})\right),$$
$$\left(T_1^s - 1\right) p_{3j}\, x_{1i} \left(T_2^s\, x_{2i} - x_{2j}\right),$$
$$\epsilon\, s \left(T_3^s - 1\right) p_{1j}\, (p_{2i} - p_{2j})\, x_{3i} \big/ \left(T_2^s - 1\right),$$
$$\epsilon\, s \Big(1/2 + T_2^s\, p_{1i}\, p_{2j}\, x_{1i}\, x_{2i} - p_{1i}\, p_{2j}\, x_{1i}\, x_{2j} - p_{3i}\, x_{3i} - \left(T_2^s - 1\right) p_{2j}\, p_{3i}\, x_{2i}\, x_{3i} +$$
$$\left(T_3^s - 1\right) p_{2j}\, p_{3j}\, x_{2i}\, x_{3i} + 2\, p_{2j}\, p_{3i}\, x_{2j}\, x_{3i} + p_{1i}\, p_{3j}\, x_{1i}\, x_{3j} - p_{2i}\, p_{3j}\, x_{2i}\, x_{3j} - T_2^s\, p_{2j}\, p_{3j}\, x_{2i}\, x_{3j} +$$
$$\Big(\left(T_1^s - 1\right) p_{1j}\, x_{1i} \left(T_2^{2s}\, p_{2j}\, x_{2i} - T_2^s\, p_{2j}\, x_{2j} - \left(T_2^s + 1\right) \left(T_3^s - 1\right) p_{3j}\, x_{3i} + T_2^s\, p_{3j}\, x_{3j}\right) +$$
$$\left(T_3^s - 1\right) p_{3j}\, x_{3i} \left(1 - T_2^s\, p_{1i}\, x_{1i} + p_{2i}\, x_{2j} + \left(T_2^s - 2\right) p_{2j}\, x_{2j}\right)\Big) \big/ \left(T_2^s - 1\right)\Big)\Big]\Big]$$

$$\mathcal{L}_1[C_{i\_}[\varphi\_]] := T_3^\varphi\, \mathbb{E}\left[\sum_{\nu=1}^{3} x_{\nu i}\, (p_{\nu i^+} - p_{\nu i}) + \epsilon\, \varphi\, (p_{3i}\, x_{3i} - 1/2)\right]$$

In[∘]:= $\$\mathcal{L} = \mathcal{L}_1$;

The Trefoil

In[∘]:= K = Mirror@Knot[3, 1]; Features[K]

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

Features[7, $C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]$]

In[∘]:= {vs[K], $\$\mathcal{L}$[K]}

Out[∘]=

$$\big\{\{p_{1,1}, x_{1,1}, p_{1,2}, x_{1,2}, p_{1,3}, x_{1,3}, p_{1,4}, x_{1,4}, p_{1,5}, x_{1,5}, p_{1,6}, x_{1,6}, p_{1,7},$$
$$x_{1,7}, p_{2,1}, x_{2,1}, p_{2,2}, x_{2,2}, p_{2,3}, x_{2,3}, p_{2,4}, x_{2,4}, p_{2,5}, x_{2,5}, p_{2,6}, x_{2,6}, p_{2,7}, x_{2,7},$$
$$p_{3,1}, x_{3,1}, p_{3,2}, x_{3,2}, p_{3,3}, x_{3,3}, p_{3,4}, x_{3,4}, p_{3,5}, x_{3,5}, p_{3,6}, x_{3,6}, p_{3,7}, x_{3,7}\},$$
$$T_3^2\, \mathbb{E}\Big[2\epsilon - p_{1,1}\, x_{1,1} + T_1\, p_{1,1^+}\, x_{1,1} + (1 - T_1)\, p_{1,5^+}\, x_{1,1} - p_{1,2}\, x_{1,2} + p_{1,2^+}\, x_{1,2} - p_{1,3}\, x_{1,3} + T_1\, p_{1,3^+}\, x_{1,3} +$$
$$(1 - T_1)\, p_{1,7^+}\, x_{1,3} - p_{1,4}\, x_{1,4} + p_{1,4^+}\, x_{1,4} - p_{1,5}\, x_{1,5} + p_{1,5^+}\, x_{1,5} - p_{1,6}\, x_{1,6} + (1 - T_1)\, p_{1,2^+}\, x_{1,6} +$$
$$T_1\, p_{1,6^+}\, x_{1,6} - p_{1,7}\, x_{1,7} + p_{1,7^+}\, x_{1,7} - p_{2,1}\, x_{2,1} + T_2\, p_{2,1^+}\, x_{2,1} + (1 - T_2)\, p_{2,5^+}\, x_{2,1} + \epsilon\, T_2\, p_{1,1}\, p_{2,5}\, x_{1,1}\, x_{2,1} +$$
$$\frac{\epsilon\, (-1 + T_1)\, T_2^2\, p_{1,5}\, p_{2,5}\, x_{1,1}\, x_{2,1}}{-1 + T_2} + (-1 + T_1)\, T_2\, p_{3,5}\, x_{1,1}\, x_{2,1} - p_{2,2}\, x_{2,2} + p_{2,2^+}\, x_{2,2} -$$
$$\frac{\epsilon\, (-1 + T_1)\, T_2\, p_{1,2}\, p_{2,2}\, x_{1,6}\, x_{2,2}}{-1 + T_2} - \epsilon\, p_{1,6}\, p_{2,2}\, x_{1,6}\, x_{2,2} + (1 - T_1)\, p_{3,2}\, x_{1,6}\, x_{2,2} - p_{2,3}\, x_{2,3} +$$
$$T_2\, p_{2,3^+}\, x_{2,3} + (1 - T_2)\, p_{2,7^+}\, x_{2,3} + \epsilon\, T_2\, p_{1,3}\, p_{2,7}\, x_{1,3}\, x_{2,3} + \frac{\epsilon\, (-1 + T_1)\, T_2^2\, p_{1,7}\, p_{2,7}\, x_{1,3}\, x_{2,3}}{-1 + T_2} +$$
$$(-1 + T_1)\, T_2\, p_{3,7}\, x_{1,3}\, x_{2,3} - p_{2,4}\, x_{2,4} + p_{2,4^+}\, x_{2,4} - p_{2,5}\, x_{2,5} + p_{2,5^+}\, x_{2,5} - \epsilon\, p_{1,1}\, p_{2,5}\, x_{1,1}\, x_{2,5} -$$
$$\frac{\epsilon\, (-1 + T_1)\, T_2\, p_{1,5}\, p_{2,5}\, x_{1,1}\, x_{2,5}}{-1 + T_2} + (1 - T_1)\, p_{3,5}\, x_{1,1}\, x_{2,5} - p_{2,6}\, x_{2,6} + (1 - T_2)\, p_{2,2^+}\, x_{2,6} +$$
$$T_2\, p_{2,6^+}\, x_{2,6} + \frac{\epsilon\, (-1 + T_1)\, T_2^2\, p_{1,2}\, p_{2,2}\, x_{1,6}\, x_{2,6}}{-1 + T_2} + \epsilon\, T_2\, p_{1,6}\, p_{2,2}\, x_{1,6}\, x_{2,6} + (-1 + T_1)\, T_2\, p_{3,2}\, x_{1,6}\, x_{2,6} -$$
$$p_{2,7}\, x_{2,7} + p_{2,7^+}\, x_{2,7} - \epsilon\, p_{1,3}\, p_{2,7}\, x_{1,3}\, x_{2,7} - \frac{\epsilon\, (-1 + T_1)\, T_2\, p_{1,7}\, p_{2,7}\, x_{1,3}\, x_{2,7}}{-1 + T_2} +$$
$$(1 - T_1)\, p_{3,7}\, x_{1,3}\, x_{2,7} + \frac{\epsilon\, (-1 + T_3)\, p_{1,5}\, p_{2,1}\, x_{3,1}}{-1 + T_2} - \frac{\epsilon\, (-1 + T_3)\, p_{1,5}\, p_{2,5}\, x_{3,1}}{-1 + T_2} -$$

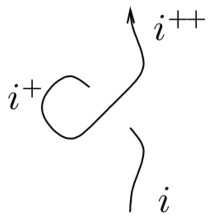
$$p_{3,1}\,x_{3,1}-\in p_{3,1}\,x_{3,1}+\frac{\in(-1+T_3)\,p_{3,5}\,x_{3,1}}{-1+T_2}+T_3\,p_{3,1^+}\,x_{3,1}+(1-T_3)\,p_{3,5^+}\,x_{3,1}-$$
$$\frac{\in T_2(-1+T_3)\,p_{1,1}\,p_{3,5}\,x_{1,1}\,x_{3,1}}{-1+T_2}-\frac{\in(-1+T_1)(1+T_2)(-1+T_3)\,p_{1,5}\,p_{3,5}\,x_{1,1}\,x_{3,1}}{-1+T_2}+$$
$$\in(1-T_2)\,p_{2,5}\,p_{3,1}\,x_{2,1}\,x_{3,1}+\in(-1+T_3)\,p_{2,5}\,p_{3,5}\,x_{2,1}\,x_{3,1}+2\in p_{2,5}\,p_{3,1}\,x_{2,5}\,x_{3,1}+$$
$$\frac{\in(-1+T_3)\,p_{2,1}\,p_{3,5}\,x_{2,5}\,x_{3,1}}{-1+T_2}+\frac{\in(-2+T_2)(-1+T_3)\,p_{2,5}\,p_{3,5}\,x_{2,5}\,x_{3,1}}{-1+T_2}-p_{3,2}\,x_{3,2}+$$
$$p_{3,2^+}\,x_{3,2}+\frac{\in(-1+T_1)\,T_2\,p_{1,2}\,p_{3,2}\,x_{1,6}\,x_{3,2}}{-1+T_2}+\in p_{1,6}\,p_{3,2}\,x_{1,6}\,x_{3,2}-\in T_2\,p_{2,2}\,p_{3,2}\,x_{2,6}\,x_{3,2}-$$
$$\in p_{2,6}\,p_{3,2}\,x_{2,6}\,x_{3,2}+\frac{\in(-1+T_3)\,p_{1,7}\,p_{2,3}\,x_{3,3}}{-1+T_2}-\frac{\in(-1+T_3)\,p_{1,7}\,p_{2,7}\,x_{3,3}}{-1+T_2}-p_{3,3}\,x_{3,3}-\in p_{3,3}\,x_{3,3}+$$
$$\frac{\in(-1+T_3)\,p_{3,7}\,x_{3,3}}{-1+T_2}+T_3\,p_{3,3^+}\,x_{3,3}+(1-T_3)\,p_{3,7^+}\,x_{3,3}-\frac{\in T_2(-1+T_3)\,p_{1,3}\,p_{3,7}\,x_{1,3}\,x_{3,3}}{-1+T_2}-$$
$$\frac{\in(-1+T_1)(1+T_2)(-1+T_3)\,p_{1,7}\,p_{3,7}\,x_{1,3}\,x_{3,3}}{-1+T_2}+\in(1-T_2)\,p_{2,7}\,p_{3,3}\,x_{2,3}\,x_{3,3}+$$
$$\in(-1+T_3)\,p_{2,7}\,p_{3,7}\,x_{2,3}\,x_{3,3}+2\in p_{2,7}\,p_{3,3}\,x_{2,7}\,x_{3,3}+\frac{\in(-1+T_3)\,p_{2,3}\,p_{3,7}\,x_{2,7}\,x_{3,3}}{-1+T_2}+$$
$$\frac{\in(-2+T_2)(-1+T_3)\,p_{2,7}\,p_{3,7}\,x_{2,7}\,x_{3,3}}{-1+T_2}-p_{3,4}\,x_{3,4}-\in p_{3,4}\,x_{3,4}+p_{3,4^+}\,x_{3,4}-p_{3,5}\,x_{3,5}+$$
$$p_{3,5^+}\,x_{3,5}+\in p_{1,1}\,p_{3,5}\,x_{1,1}\,x_{3,5}+\frac{\in(-1+T_1)\,T_2\,p_{1,5}\,p_{3,5}\,x_{1,1}\,x_{3,5}}{-1+T_2}-\in p_{2,1}\,p_{3,5}\,x_{2,1}\,x_{3,5}-$$
$$\in T_2\,p_{2,5}\,p_{3,5}\,x_{2,1}\,x_{3,5}-\frac{\in(-1+T_3)\,p_{1,2}\,p_{2,2}\,x_{3,6}}{-1+T_2}+\frac{\in(-1+T_3)\,p_{1,2}\,p_{2,6}\,x_{3,6}}{-1+T_2}+$$
$$\frac{\in(-1+T_3)\,p_{3,2}\,x_{3,6}}{-1+T_2}-p_{3,6}\,x_{3,6}-\in p_{3,6}\,x_{3,6}+(1-T_3)\,p_{3,2^+}\,x_{3,6}+T_3\,p_{3,6^+}\,x_{3,6}-$$
$$\frac{\in(-1+T_1)(1+T_2)(-1+T_3)\,p_{1,2}\,p_{3,2}\,x_{1,6}\,x_{3,6}}{-1+T_2}-\frac{\in T_2(-1+T_3)\,p_{1,6}\,p_{3,2}\,x_{1,6}\,x_{3,6}}{-1+T_2}+$$
$$\frac{\in(-2+T_2)(-1+T_3)\,p_{2,2}\,p_{3,2}\,x_{2,2}\,x_{3,6}}{-1+T_2}+\frac{\in(-1+T_3)\,p_{2,6}\,p_{3,2}\,x_{2,2}\,x_{3,6}}{-1+T_2}+2\in p_{2,2}\,p_{3,6}\,x_{2,2}\,x_{3,6}+$$
$$\in(-1+T_3)\,p_{2,2}\,p_{3,2}\,x_{2,6}\,x_{3,6}+\in(1-T_2)\,p_{2,2}\,p_{3,6}\,x_{2,6}\,x_{3,6}-p_{3,7}\,x_{3,7}+p_{3,7^+}\,x_{3,7}+$$
$$\in p_{1,3}\,p_{3,7}\,x_{1,3}\,x_{3,7}+\frac{\in(-1+T_1)\,T_2\,p_{1,7}\,p_{3,7}\,x_{1,3}\,x_{3,7}}{-1+T_2}-\in p_{2,3}\,p_{3,7}\,x_{2,3}\,x_{3,7}-\in T_2\,p_{2,7}\,p_{3,7}\,x_{2,3}\,x_{3,7}\Big]\Big\}$$

In[∘]:=
```
∫$ℒ[K] ⅆ vs[K]
```

Out[∘]=

$$-\frac{\mathbb{i}\,T_1^2\,T_2^2\,\mathbb{E}\left[\frac{\in\left(1-T_1+T_1^2-T_2-T_1^3T_2+T_2^2+T_1^4T_2^2-T_1T_2^3-T_1^4T_2^3+T_1^2T_2^4-T_1^3T_2^4+T_1^4T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1T_2+T_1^2T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1T_2+T_1^2T_2^2\right)}$$

In[∘]:=
```
ℱ[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
vs_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
```

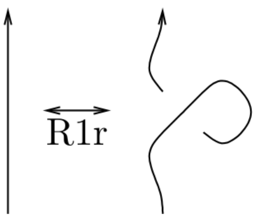


In[∘]:= $\text{lhs} = \int \mathcal{F}[i]\ \$\mathcal{L} \mathbin{/@} (X_{i+2,i}[1]\ C_{i+1}[1])\ d\{vs_i, vs_{i^+}, vs_{i+2}\}$

$\text{rhs} = \int \mathcal{F}[i]\ \$\mathcal{L} \mathbin{/@} (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{vs_i, vs_{i^+}, vs_{i+2}\};$

$\text{lhs} == \text{rhs}\ //\ \text{HL}$

» 1



In[∘]:= $\text{lhs} = \int \mathcal{F}[i]\ \$\mathcal{L} \mathbin{/@} (X_{i,i+2}[1]\ C_{i+1}[-1])\ d\{vs_i, vs_{i^+}, vs_{i+2}\}$

$\text{rhs} = \int \mathcal{F}[i]\ \$\mathcal{L} \mathbin{/@} (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{vs_i, vs_{i^+}, vs_{i+2}\};$

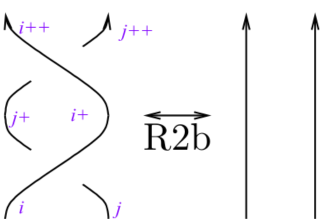
$\text{lhs} == \text{rhs}\ //\ \text{HL}$

Out[∘]=

$-i\,\mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$

Out[∘]=

True



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:= $\text{lhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L} \mathbin{/@} (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\}$

$\text{rhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L} \mathbin{/@} (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\};$

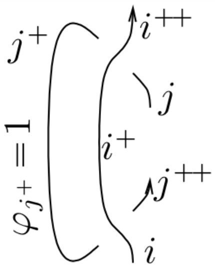
$\text{lhs} == \text{rhs}\ //\ \text{HL}$

Out[∘]=

$\mathbb{E}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}]$

Out[∘]=

True



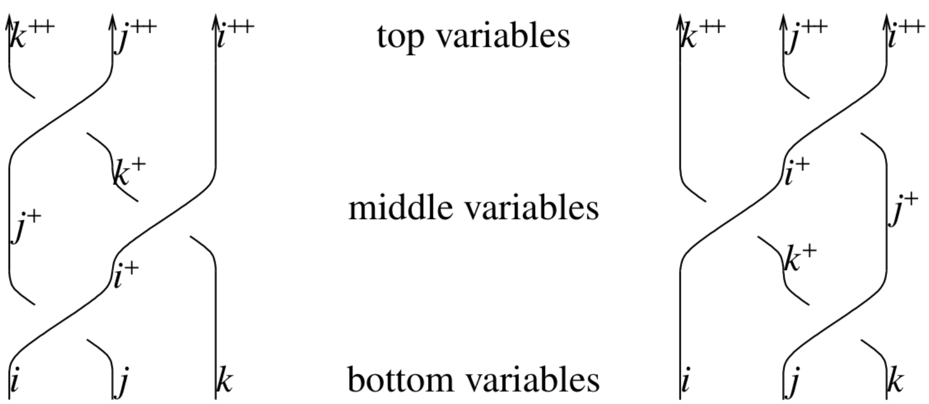
In[ ]:=
```
lhs = ∫ 𝓕[i, j] $ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1]) ⅆ{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}}
rhs = ∫ 𝓕[i, j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0]) ⅆ{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}};
lhs == rhs // HL
```

Out[ ]=

$$-\mathbb{i}\, T_1\, T_2\, \mathbb{E}\left[\frac{\epsilon}{2} + p_{1,2+i}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j} + \epsilon\, p_{3,3+j}\,\pi_{3,j}\right]$$

Out[ ]=

True



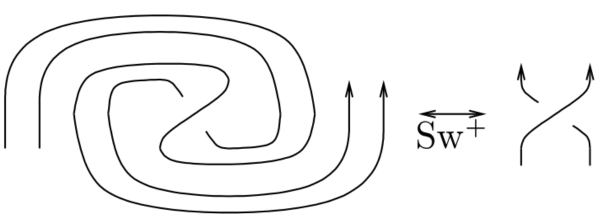
In[ ]:=
```
Short[lhs = ∫ (𝓕[i, j, k] $ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])) ⅆ{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}}]
rhs = ∫ (𝓕[i, j, k] $ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1])) ⅆ{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}};
lhs == rhs // HL
```

Out[ ]//Short=

$$T_1^3\, T_2^3\, \mathbb{E}\left[\frac{3\,\epsilon}{2} + \ll 153 \gg\right]$$

Out[ ]=

True



In[∘]:=
```
Short[lhs =
   ∫𝓕[i, j] $ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{i+2}, vs_{j+2}}]
rhs =
   ∫𝓕[i, j] $ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{i+2}, vs_{j+2}};
lhs == rhs // HL
```

Out[∘]//Short=

$$T_1 \ll 1\gg \ll 1\gg \mathbb{E}\left[\frac{\epsilon}{2} + T_1\, p_{1,3+i}\, \pi_{1,i} + (1 - T_1)\, p_{1,3+j}\, \pi_{1,i} + p_{1,3+j}\, \pi_{1,j} + \ll 33\gg + \epsilon \ll 4\gg \pi_{3\,\ll 1\gg\,\ll 1\gg} + \frac{\epsilon\,(-1 + T_1) \ll 3\gg \pi_{3,j}}{-1 + T_2} - \epsilon\, T_2\, p_{2,3+i}\, p_{3,3+j}\, \pi_{2,i}\, \pi_{3,j} - \epsilon\, p_{2,3+j}\, p_{3,3+j}\, \pi_{2,i}\, \pi_{3,j}\right]$$

Out[∘]=

True

The pPush Lagrangian

In[∘]:=
```
pPush_{loc__}[ℰ_] := g2px[px2g[ℰ] /. gR_{loc} /. {δ_{α_,α_} → 1, δ_{i,j} → 0, δ_{j,i} → 0} /. α_^+ :→ α + 1];
```

In[∘]:=
```
{pPush_{s,i,j}[x_i p_i], pPush_i[x_i p_i]}
```

Out[∘]=

$$\{1 + T^s\, p_{1+i}\, x_i + (1 - T^s)\, p_{1+j}\, x_i,\ 1 + p_{1+i}\, x_i\}$$

In[∘]:=
```
ℒ_pPush[X_{i_,j_}[s_]] := T^{s/2} CF@𝔼[Plus[
     x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}),
     pPush_{s,i,j}[(ϵ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
   ]]
ℒ_pPush[C_{i_}[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + pPush_i[ϵ φ (1/2 - x_i p_i)]]
$ℒ = ℒ_pPush;
```

In[∘]:=
```
$ℒ[X_{i,j}[s]]
$ℒ[C_i[φ]]
```

Out[∘]=

$$T^{s/2}\, \mathbb{E}\left[-\frac{s\,\epsilon}{2} - p_i\, x_i + T^s\, p_{1+i}\, x_i + (1 - T^s)\, p_{1+j}\, x_i + s\, T^s\, \epsilon\, p_{1+j}\, x_i + \frac{1}{2}\, s\, T^s\, (-1 + T^s)\, \epsilon\, p_{1+i}\, p_{1+j}\, x_i^2 - \frac{1}{2}\, s\, T^s\, (-1 + T^s)\, \epsilon\, p_{1+j}^2\, x_i^2 - p_j\, x_j + p_{1+j}\, x_j - s\, \epsilon\, p_{1+j}\, x_j - s\, T^s\, \epsilon\, p_{1+i}\, p_{1+j}\, x_i\, x_j + s\, T^s\, \epsilon\, p_{1+j}^2\, x_i\, x_j\right]$$

Out[∘]=

$$T^{\varphi/2}\, \mathbb{E}\left[-\frac{\epsilon\,\varphi}{2} - \epsilon\,\varphi\, p_{1+i}\, x_i + (-p_i + p_{1+i})\, x_i\right]$$

In[∘]:=
```
Collect[$ℒ[X_{i,j}[s]]〚2, 1〛, ϵ]
Collect[$ℒ[C_i[φ]]〚2, 1〛, ϵ]
```

Out[∘]=

$$-p_i x_i + T^s p_{1+i} x_i + (1 - T^s) p_{1+j} x_i - p_j x_j + p_{1+j} x_j + \in \left(-\frac{s}{2} + s T^s p_{1+j} x_i + \frac{1}{2} s T^s (-1 + T^s) p_{1+i} p_{1+j} x_i^2 - \frac{1}{2} s T^s (-1 + T^s) p_{1+j}^2 x_i^2 - s p_{1+j} x_j - s T^s p_{1+i} p_{1+j} x_i x_j + s T^s p_{1+j}^2 x_i x_j\right)$$

Out[∘]=

$$(-p_i + p_{1+i}) x_i + \in \left(-\frac{\varphi}{2} - \varphi p_{1+i} x_i\right)$$

The Trefoil

In[∘]:=
```
K = Mirror@Knot[3, 1]; {vs[K], $ℒ[K]}
```

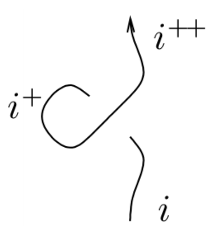
Out[∘]=

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-\in - p_1 x_1 + T p_2 x_1 + (1 - T) p_6 x_1 + T \in p_6 x_1 + \frac{1}{2}(-1 + T) T \in p_2 p_6 x_1^2 - \frac{1}{2}(-1 + T) T \in p_6^2 x_1^2 - p_2 x_2 + p_3 x_2 - \in p_3 x_2 - p_3 x_3 + T p_4 x_3 + (1 - T) p_8 x_3 + T \in p_8 x_3 + \frac{1}{2}(-1 + T) T \in p_4 p_8 x_3^2 - \frac{1}{2}(-1 + T) T \in p_8^2 x_3^2 - p_4 x_4 + p_5 x_4 + \in p_5 x_4 - p_5 x_5 + p_6 x_5 - \in p_6 x_5 - T \in p_2 p_6 x_1 x_5 + T \in p_6^2 x_1 x_5 + (1 - T) p_3 x_6 + T \in p_3 x_6 - p_6 x_6 + T p_7 x_6 + T \in p_3^2 x_2 x_6 - T \in p_3 p_7 x_2 x_6 - \frac{1}{2}(-1 + T) T \in p_3^2 x_6^2 + \frac{1}{2}(-1 + T) T \in p_3 p_7 x_6^2 - p_7 x_7 + p_8 x_7 - \in p_8 x_7 - T \in p_4 p_8 x_3 x_7 + T \in p_8^2 x_3 x_7\Big]\Big\}$$

In[∘]:=
```
$π = Normal[# + O[ϵ]^2] &; ∫$ℒ[K] ⅆ vs[K]
```

Out[∘]=

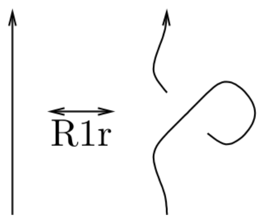
$$-\frac{i\, T\, \mathbb{E}\left[-\frac{(-1+T)^2 (1+T^2) \in}{(1-T+T^2)^2}\right]}{1 - T + T^2}$$



In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i] $ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[ⅈ π_i p_i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs // HL
```

Out[∘]=
$-i\,\mathbb{E}[i\, p_{3+i}\, \pi_i]$

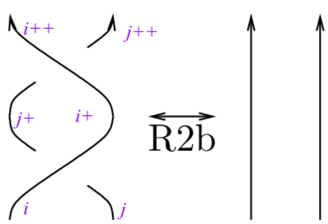
Out[∘]=
True



In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i] $ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[ⅈ π_i p_i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs // HL
```

Out[∘]=
$-i\,\mathbb{E}[i\, p_{3+i}\, \pi_i]$

Out[∘]=
True

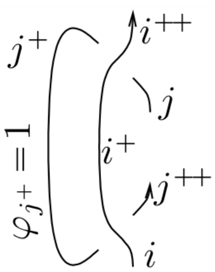


Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] $ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
rhs =
  ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
lhs == rhs // HL
```

Out[∘]=
$\mathbb{E}[i\, p_{2+i}\, \pi_i + i\, p_{2+j}\, \pi_j]$

Out[∘]=
True



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \$\mathcal{L}\ /@\ (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ \mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\}$$

$$\text{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ \mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}\};$$

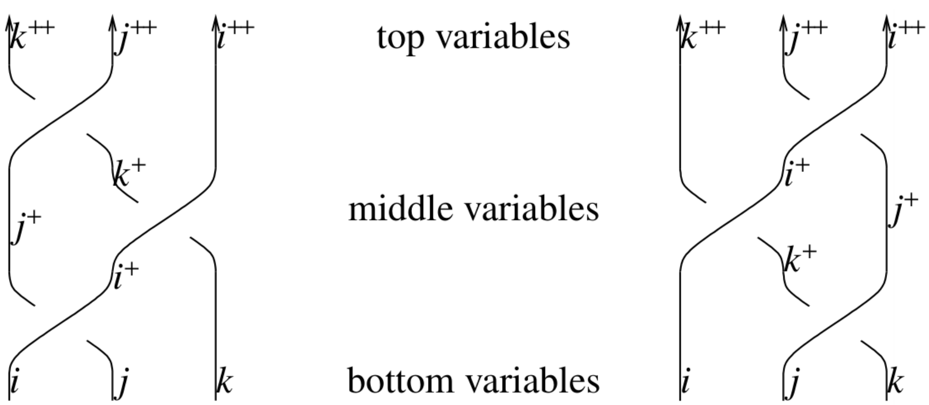
```
lhs == rhs // HL
```

Out[∘]=

$$-\mathbb{i}\,\sqrt{T}\,\mathbb{E}\left[-\frac{\epsilon}{2} + \mathbb{i}\,p_{2+i}\,\pi_i + \mathbb{i}\,p_{3+j}\,\pi_j - \mathbb{i}\,\epsilon\,p_{3+j}\,\pi_j\right]$$

Out[∘]=

True



In[∘]:=

$$\$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1])$$

Out[∘]=

$$\begin{aligned}
T^{3/2}\,\mathbb{E}\Big[&-\frac{3\,\epsilon}{2} - p_i\,x_i + T\,p_{1+i}\,x_i + (1-T)\,p_{1+j}\,x_i + T\,\epsilon\,p_{1+j}\,x_i + \frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{1+i}\,p_{1+j}\,x_i^2 - \\
&\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{1+j}^2\,x_i^2 - p_{1+i}\,x_{1+i} + T\,p_{2+i}\,x_{1+i} + (1-T)\,p_{1+k}\,x_{1+i} + T\,\epsilon\,p_{1+k}\,x_{1+i} + \\
&\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+i}\,p_{1+k}\,x_{1+i}^2 - \frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{1+k}^2\,x_{1+i}^2 - p_j\,x_j + p_{1+j}\,x_j - \epsilon\,p_{1+j}\,x_j - \\
&T\,\epsilon\,p_{1+i}\,p_{1+j}\,x_i\,x_j + T\,\epsilon\,p_{1+j}^2\,x_i\,x_j - p_{1+j}\,x_{1+j} + T\,p_{2+j}\,x_{1+j} + (1-T)\,p_{2+k}\,x_{1+j} + T\,\epsilon\,p_{2+k}\,x_{1+j} + \\
&\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+j}\,p_{2+k}\,x_{1+j}^2 - \frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{2+k}^2\,x_{1+j}^2 - p_k\,x_k + p_{1+k}\,x_k - \epsilon\,p_{1+k}\,x_k - T\,\epsilon\,p_{2+i}\,p_{1+k}\,x_{1+i}\,x_k + \\
&T\,\epsilon\,p_{1+k}^2\,x_{1+i}\,x_k - p_{1+k}\,x_{1+k} + p_{2+k}\,x_{1+k} - \epsilon\,p_{2+k}\,x_{1+k} - T\,\epsilon\,p_{2+j}\,p_{2+k}\,x_{1+j}\,x_{1+k} + T\,\epsilon\,p_{2+k}^2\,x_{1+j}\,x_{1+k}\Big]
\end{aligned}$$

pdf

In[∘]:=

$$\text{lhs} = \int \left(\mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j + \mathbb{i}\,\pi_k\,p_k]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1])\right)$$
$$\mathbb{d}\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\}$$
$$\text{rhs} = \int \left(\mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j + \mathbb{i}\,\pi_k\,p_k]\ \$\mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1])\right)$$
$$\mathbb{d}\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$

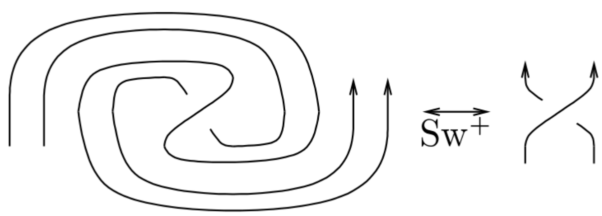
```
lhs == rhs // HL
```

Out[∘]=
pdf

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\in}{2} + \mathbb{i}\,T^2 p_{2+i}\pi_i - \mathbb{i}\,(-1+T)\,T\,p_{2+j}\pi_i + \mathbb{i}\,T^2\in p_{2+j}\pi_i - \mathbb{i}\,(-1+T)\,p_{2+k}\pi_i + \mathbb{i}\,T\in p_{2+k}\pi_i - \frac{1}{2}(-1+T)\,T^3\in p_{2+i}p_{2+j}\pi_i^2 + \frac{1}{2}(-1+T)\,T^3\in p_{2+j}^2\pi_i^2 - \frac{1}{2}(-1+T)\,T^2\in p_{2+i}p_{2+k}\pi_i^2 + \frac{1}{2}(-1+T)^2\,T\in p_{2+j}p_{2+k}\pi_i^2 + \frac{1}{2}(-1+T)\,T\in p_{2+k}^2\pi_i^2 + \mathbb{i}\,T\,p_{2+j}\pi_j - \mathbb{i}\,T\in p_{2+j}\pi_j - \mathbb{i}\,(-1+T)\,p_{2+k}\pi_j + \mathbb{i}\,(-1+2T)\in p_{2+k}\pi_j + T^3\in p_{2+i}p_{2+j}\pi_i\pi_j - T^3\in p_{2+j}^2\pi_i\pi_j - (-1+T)\,T^2\in p_{2+i}p_{2+k}\pi_i\pi_j + (-1+T)^2\,T\in p_{2+j}p_{2+k}\pi_i\pi_j + (-1+T)\,T\in p_{2+k}^2\pi_i\pi_j - \frac{1}{2}(-1+T)\,T\in p_{2+j}p_{2+k}\pi_j^2 + \frac{1}{2}(-1+T)\,T\in p_{2+k}^2\pi_j^2 + \mathbb{i}\,p_{2+k}\pi_k - 2\,\mathbb{i}\in p_{2+k}\pi_k + T^2\in p_{2+i}p_{2+k}\pi_i\pi_k - (-1+T)\,T\in p_{2+j}p_{2+k}\pi_i\pi_k - T\in p_{2+k}^2\pi_i\pi_k + T\in p_{2+j}p_{2+k}\pi_j\pi_k - T\in p_{2+k}^2\pi_j\pi_k\Big]$$

Out[∘]=
pdf

True



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \$\mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])$$
$$\mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\}$$
$$\text{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \$\mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])$$
$$\mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\};$$

```
lhs == rhs // HL
```

Out[∘]=

$$\sqrt{T}\,\mathbb{E}\Big[-\frac{\in}{2} + \mathbb{i}\,T\,p_{3+i}\pi_i - \mathbb{i}\,(-1+T)\,p_{3+j}\pi_i + \mathbb{i}\,T\in p_{3+j}\pi_i - \frac{1}{2}(-1+T)\,T\in p_{3+i}p_{3+j}\pi_i^2 + \frac{1}{2}(-1+T)\,T\in p_{3+j}^2\pi_i^2 + \mathbb{i}\,p_{3+j}\pi_j - \mathbb{i}\in p_{3+j}\pi_j + T\in p_{3+i}p_{3+j}\pi_i\pi_j - T\in p_{3+j}^2\pi_i\pi_j\Big]$$

Out[∘]=

True