Pensieve header: Experiments in 2026: Alternative integrands for θ.

Initialization

(Alt) In[ ]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< "IType(SimplifiedFormatting).m"];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

Minor utilities

(Alt) In[ ]:=

```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CCF[ℰ_] := Factor[ℰ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, ϵ | (p | x | π | ξ | g)__, F] /. F → CCF];
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

(Alt) In[ ]:=

```
ℒ_i_[K_] := CF[ℒ_i /@ Features[K]⟦2⟧]
vs[K_] := Flatten@Table[{p_{ν,i}, x_{ν,i}}, {ν, 3}, {i, Features[K]⟦1⟧}]
```

(Alt) In[ ]:=

```
{p*, x*, π*, ξ*} = {π, ξ, p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_vi__)* := (u*)_vi;
```

(Alt) In[ ]:=

```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

g2px and px2g

Modified from pensieve://Talks/Geneva-2408/DataConversions.nb

(Alt) In[ ]:=

```
g2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {g_{i_,j_} :> λ p_i x_j}] /. {λ^k_. :> 1/k!}]
gν2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {g_{ν_,i_,j_} :> λ_ν p_{ν,i} x_{ν,j}}] /. {λ_^k_. :> 1/k!}]
```

(Alt) In[ ]:=

```
Zip_{}[ℰ_] := ℰ;
Zip_{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip_{ξs}, ξ] /. f_. ξ^d_. :> (D[f, {ξ*, d}])) /. ξ* → 0
```

(Alt) In[ ]:=

```
px2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
  ps = Union[Cases[ℰ, p_, ∞]]; xs = Union[Cases[ℰ, x_, ∞]];
  Q = Sum[p0* x0* g_{p0⟦2⟧,x0⟦2⟧}, {p0, ps}, {x0, xs}];
  Expand[Zip_{ps⋃xs}[ℰ 𝕖^Q]]
 ]
pxv2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
  ps = Union[Cases[ℰ, p__, ∞]]; xs = Union[Cases[ℰ, x__, ∞]];
  Q = Sum[p0* x0* g_{p0⟦2⟧,x0⟦2⟧,p0⟦3⟧,x0⟦3⟧}, {p0, ps}, {x0, xs}];
  Expand[Zip_{ps⋃xs}[ℰ 𝕖^Q] /. g_{α_,β_,i_,j_} :> If[α === β, g_{α,i,j}, 0]]
 ]
```

gRules

From pensieve://Talks/MonteVerita-2604/Theta.nb

(Alt) In[ ]:=

```
gR_{s_,i_,j_} := {
  g_{jβ_} :> g_{j⁺β} + δ_{jβ}, g_{iβ_} :> T^s g_{i⁺β} + (1 - T^s) g_{j⁺β} + δ_{iβ},
  g_{α_i⁺} :> T^s g_{αi} + δ_{αi⁺}, g_{α_j⁺} :> g_{αj} + (1 - T^s) g_{αi} + δ_{αj⁺},
  g_{ν_jβ_} :> g_{νj⁺β} + δ_{jβ}, g_{ν_iβ_} :> T_ν^s g_{νi⁺β} + (1 - T_ν^s) g_{νj⁺β} + δ_{iβ},
  g_{ν_α_i⁺} :> T_ν^s g_{ναi} + δ_{αi⁺}, g_{ν_α_j⁺} :> g_{ναj} + (1 - T_ν^s) g_{ναi} + δ_{αj⁺}
 }
gR_{i_} := {g_{i,β_} :> g_{i⁺,β} + δ_{i,β}, g_{α_,i⁺} :> g_{α,i} + δ_{α,i⁺}}
```

Inverse gRules written locally:

(Alt) In[ ]:=

```
igR_{s_,i_,j_} := {
  g_{j⁺,β_} :> g_{j,β} - δ_{j,β}, g_{i⁺,β_} :> T^{-s} g_{iβ} - (T^{-s} - 1) g_{j⁺β} - T^{-s} δ_{iβ},
  g_{α_,i} :> T^{-s} (g_{α,i⁺} - δ_{α,i⁺}), g_{α_,j} :> g_{α,j⁺} - (1 - T^s) g_{α,i} - δ_{α,j⁺},
  g_{ν_,j⁺,β_} :> g_{ν,j,β} - δ_{j,β}, g_{ν_,i⁺,β_} :> T_ν^{-s} g_{νiβ} - (T_ν^{-s} - 1) g_{νj⁺β} - T_ν^{-s} δ_{iβ},
  g_{ν_,α_,i} :> T_ν^{-s} (g_{ν,α,i⁺} - δ_{α,i⁺}), g_{ν_,α_,j} :> g_{ν,α,j⁺} - (1 - T_ν^s) g_{ν,α,i} - δ_{α,j⁺}
 }
```

The Base Lagrangian

From pensieve://Talks/Rank2.nb.

(Alt) In[ ]:=
pdf

```
T_3 = T_1 T_2;  i_⁺ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /. {π_{is__} :> B^-1 π_{is}, x_{is__} :> B^-1 x_{is}, p_{is__} :> B p_{is}} /. ϵ B^{b_} /; b < 0 → 0 /.
     B → 1) &;
```

(Alt) In[ ]:=

$$\mathcal{L}_1[X_{i\_,j\_}[s\_]] := T_3^s\,\mathbb{E}\Big[\text{CF@Plus}\Big[$$
$$\sum_{\nu=1}^{3}\left(x_{\nu i}\,(p_{\nu i^+}-p_{\nu i})+x_{\nu j}\,(p_{\nu j^+}-p_{\nu j})+\left(T_\nu^s-1\right)x_{\nu i}\,(p_{\nu i^+}-p_{\nu j^+})\right),$$
$$\left(T_1^s-1\right)p_{3j}\,x_{1i}\left(T_2^s\,x_{2i}-x_{2j}\right),$$
$$\epsilon\,s\left(T_3^s-1\right)p_{1j}\,(p_{2i}-p_{2j})\,x_{3i}\Big/\left(T_2^s-1\right),$$
$$\epsilon\,s\Big(1/2+T_2^s\,p_{1i}\,p_{2j}\,x_{1i}\,x_{2i}-p_{1i}\,p_{2j}\,x_{1i}\,x_{2j}-p_{3i}\,x_{3i}-\left(T_2^s-1\right)p_{2j}\,p_{3i}\,x_{2i}\,x_{3i}+$$
$$\left(T_3^s-1\right)p_{2j}\,p_{3j}\,x_{2i}\,x_{3i}+2\,p_{2j}\,p_{3i}\,x_{2j}\,x_{3i}+p_{1i}\,p_{3j}\,x_{1i}\,x_{3j}-p_{2i}\,p_{3j}\,x_{2i}\,x_{3j}-T_2^s\,p_{2j}\,p_{3j}\,x_{2i}\,x_{3j}+$$
$$\Big(\left(T_1^s-1\right)p_{1j}\,x_{1i}\left(T_2^{2s}\,p_{2j}\,x_{2i}-T_2^s\,p_{2j}\,x_{2j}-\left(T_2^s+1\right)\left(T_3^s-1\right)p_{3j}\,x_{3i}+T_2^s\,p_{3j}\,x_{3j}\right)+$$
$$\left(T_3^s-1\right)p_{3j}\,x_{3i}\left(1-T_2^s\,p_{1i}\,x_{1i}+p_{2i}\,x_{2j}+\left(T_2^s-2\right)p_{2j}\,x_{2j}\right)\Big)\Big/\left(T_2^s-1\right)\Big)\Big]\Big]$$
$$\mathcal{L}_1[C_{i\_}[\varphi\_]] := T_3^{\varphi}\,\mathbb{E}\Big[\sum_{\nu=1}^{3}x_{\nu i}\,(p_{\nu i^+}-p_{\nu i})+\epsilon\,\varphi\,(p_{3i}\,x_{3i}-1/2)\Big]$$

In[ ]:= $\$\mathcal{L} = \mathcal{L}_1;$

The Trefoil

In[ ]:= `K = Mirror@Knot[3, 1]; Features[K]`

KnotTheory: Loading precomputed data in PD4Knots`.

Out[ ]=

$\text{Features}[7, C_4[-1]\,X_{1,5}[1]\,X_{3,7}[1]\,X_{6,2}[1]]$

In[ ]:= $\{\text{vs}[K], \$\mathcal{L}[K]\}$

Out[ ]=

$$\{\{p_{1,1}, x_{1,1}, p_{1,2}, x_{1,2}, p_{1,3}, x_{1,3}, p_{1,4}, x_{1,4}, p_{1,5}, x_{1,5}, p_{1,6}, x_{1,6}, p_{1,7},$$
$$x_{1,7}, p_{2,1}, x_{2,1}, p_{2,2}, x_{2,2}, p_{2,3}, x_{2,3}, p_{2,4}, x_{2,4}, p_{2,5}, x_{2,5}, p_{2,6}, x_{2,6}, p_{2,7}, x_{2,7},$$
$$p_{3,1}, x_{3,1}, p_{3,2}, x_{3,2}, p_{3,3}, x_{3,3}, p_{3,4}, x_{3,4}, p_{3,5}, x_{3,5}, p_{3,6}, x_{3,6}, p_{3,7}, x_{3,7}\},$$
$$T_3^2\,\mathbb{E}\Big[2\in-p_{1,1}\,x_{1,1}+T_1\,p_{1,1^+}\,x_{1,1}+(1-T_1)\,p_{1,5^+}\,x_{1,1}-p_{1,2}\,x_{1,2}+p_{1,2^+}\,x_{1,2}-p_{1,3}\,x_{1,3}+T_1\,p_{1,3^+}\,x_{1,3}+$$
$$(1-T_1)\,p_{1,7^+}\,x_{1,3}-p_{1,4}\,x_{1,4}+p_{1,4^+}\,x_{1,4}-p_{1,5}\,x_{1,5}+p_{1,5^+}\,x_{1,5}-p_{1,6}\,x_{1,6}+(1-T_1)\,p_{1,2^+}\,x_{1,6}+$$
$$T_1\,p_{1,6^+}\,x_{1,6}-p_{1,7}\,x_{1,7}+p_{1,7^+}\,x_{1,7}-p_{2,1}\,x_{2,1}+T_2\,p_{2,1^+}\,x_{2,1}+(1-T_2)\,p_{2,5^+}\,x_{2,1}+\in T_2\,p_{1,1}\,p_{2,5}\,x_{1,1}\,x_{2,1}+$$
$$\frac{\in(-1+T_1)\,T_2^2\,p_{1,5}\,p_{2,5}\,x_{1,1}\,x_{2,1}}{-1+T_2}+(-1+T_1)\,T_2\,p_{3,5}\,x_{1,1}\,x_{2,1}-p_{2,2}\,x_{2,2}+p_{2,2^+}\,x_{2,2}-$$
$$\frac{\in(-1+T_1)\,T_2\,p_{1,2}\,p_{2,2}\,x_{1,6}\,x_{2,2}}{-1+T_2}-\in p_{1,6}\,p_{2,2}\,x_{1,6}\,x_{2,2}+(1-T_1)\,p_{3,2}\,x_{1,6}\,x_{2,2}-p_{2,3}\,x_{2,3}+$$
$$T_2\,p_{2,3^+}\,x_{2,3}+(1-T_2)\,p_{2,7^+}\,x_{2,3}+\in T_2\,p_{1,3}\,p_{2,7}\,x_{1,3}\,x_{2,3}+\frac{\in(-1+T_1)\,T_2^2\,p_{1,7}\,p_{2,7}\,x_{1,3}\,x_{2,3}}{-1+T_2}+$$
$$(-1+T_1)\,T_2\,p_{3,7}\,x_{1,3}\,x_{2,3}-p_{2,4}\,x_{2,4}+p_{2,4^+}\,x_{2,4}-p_{2,5}\,x_{2,5}+p_{2,5^+}\,x_{2,5}-\in p_{1,1}\,p_{2,5}\,x_{1,1}\,x_{2,5}-$$
$$\frac{\in(-1+T_1)\,T_2\,p_{1,5}\,p_{2,5}\,x_{1,1}\,x_{2,5}}{-1+T_2}+(1-T_1)\,p_{3,5}\,x_{1,1}\,x_{2,5}-p_{2,6}\,x_{2,6}+(1-T_2)\,p_{2,2^+}\,x_{2,6}+$$
$$T_2\,p_{2,6^+}\,x_{2,6}+\frac{\in(-1+T_1)\,T_2^2\,p_{1,2}\,p_{2,2}\,x_{1,6}\,x_{2,6}}{-1+T_2}+\in T_2\,p_{1,6}\,p_{2,2}\,x_{1,6}\,x_{2,6}+(-1+T_1)\,T_2\,p_{3,2}\,x_{1,6}\,x_{2,6}-$$
$$p_{2,7}\,x_{2,7}+p_{2,7^+}\,x_{2,7}-\in p_{1,3}\,p_{2,7}\,x_{1,3}\,x_{2,7}-\frac{\in(-1+T_1)\,T_2\,p_{1,7}\,p_{2,7}\,x_{1,3}\,x_{2,7}}{-1+T_2}+$$

$$(1-T_1)\ p_{3,7}\ x_{1,3}\ x_{2,7}+\frac{\in\ (-1+T_3)\ p_{1,5}\ p_{2,1}\ x_{3,1}}{-1+T_2}-\frac{\in\ (-1+T_3)\ p_{1,5}\ p_{2,5}\ x_{3,1}}{-1+T_2}-$$

$$p_{3,1}\ x_{3,1}-\in p_{3,1}\ x_{3,1}+\frac{\in\ (-1+T_3)\ p_{3,5}\ x_{3,1}}{-1+T_2}+T_3\ p_{3,1^+}\ x_{3,1}+(1-T_3)\ p_{3,5^+}\ x_{3,1}-$$

$$\frac{\in T_2\ (-1+T_3)\ p_{1,1}\ p_{3,5}\ x_{1,1}\ x_{3,1}}{-1+T_2}-\frac{\in\ (-1+T_1)\ (1+T_2)\ (-1+T_3)\ p_{1,5}\ p_{3,5}\ x_{1,1}\ x_{3,1}}{-1+T_2}+$$

$$\in\ (1-T_2)\ p_{2,5}\ p_{3,1}\ x_{2,1}\ x_{3,1}+\in\ (-1+T_3)\ p_{2,5}\ p_{3,5}\ x_{2,1}\ x_{3,1}+2\in p_{2,5}\ p_{3,1}\ x_{2,5}\ x_{3,1}+$$

$$\frac{\in\ (-1+T_3)\ p_{2,1}\ p_{3,5}\ x_{2,5}\ x_{3,1}}{-1+T_2}+\frac{\in\ (-2+T_2)\ (-1+T_3)\ p_{2,5}\ p_{3,5}\ x_{2,5}\ x_{3,1}}{-1+T_2}-p_{3,2}\ x_{3,2}+$$

$$p_{3,2^+}\ x_{3,2}+\frac{\in\ (-1+T_1)\ T_2\ p_{1,2}\ p_{3,2}\ x_{1,6}\ x_{3,2}}{-1+T_2}+\in p_{1,6}\ p_{3,2}\ x_{1,6}\ x_{3,2}-\in T_2\ p_{2,2}\ p_{3,2}\ x_{2,6}\ x_{3,2}-$$

$$\in p_{2,6}\ p_{3,2}\ x_{2,6}\ x_{3,2}+\frac{\in\ (-1+T_3)\ p_{1,7}\ p_{2,3}\ x_{3,3}}{-1+T_2}-\frac{\in\ (-1+T_3)\ p_{1,7}\ p_{2,7}\ x_{3,3}}{-1+T_2}-p_{3,3}\ x_{3,3}-\in p_{3,3}\ x_{3,3}+$$

$$\frac{\in\ (-1+T_3)\ p_{3,7}\ x_{3,3}}{-1+T_2}+T_3\ p_{3,3^+}\ x_{3,3}+(1-T_3)\ p_{3,7^+}\ x_{3,3}-\frac{\in T_2\ (-1+T_3)\ p_{1,3}\ p_{3,7}\ x_{1,3}\ x_{3,3}}{-1+T_2}-$$

$$\frac{\in\ (-1+T_1)\ (1+T_2)\ (-1+T_3)\ p_{1,7}\ p_{3,7}\ x_{1,3}\ x_{3,3}}{-1+T_2}+\in\ (1-T_2)\ p_{2,7}\ p_{3,3}\ x_{2,3}\ x_{3,3}+$$

$$\in\ (-1+T_3)\ p_{2,7}\ p_{3,7}\ x_{2,3}\ x_{3,3}+2\in p_{2,7}\ p_{3,3}\ x_{2,7}\ x_{3,3}+\frac{\in\ (-1+T_3)\ p_{2,3}\ p_{3,7}\ x_{2,7}\ x_{3,3}}{-1+T_2}+$$

$$\frac{\in\ (-2+T_2)\ (-1+T_3)\ p_{2,7}\ p_{3,7}\ x_{2,7}\ x_{3,3}}{-1+T_2}-p_{3,4}\ x_{3,4}-\in p_{3,4}\ x_{3,4}+p_{3,4^+}\ x_{3,4}-p_{3,5}\ x_{3,5}+$$

$$p_{3,5^+}\ x_{3,5}+\in p_{1,1}\ p_{3,5}\ x_{1,1}\ x_{3,5}+\frac{\in\ (-1+T_1)\ T_2\ p_{1,5}\ p_{3,5}\ x_{1,1}\ x_{3,5}}{-1+T_2}-\in p_{2,1}\ p_{3,5}\ x_{2,1}\ x_{3,5}-$$

$$\in T_2\ p_{2,5}\ p_{3,5}\ x_{2,1}\ x_{3,5}-\frac{\in\ (-1+T_3)\ p_{1,2}\ p_{2,2}\ x_{3,6}}{-1+T_2}+\frac{\in\ (-1+T_3)\ p_{1,2}\ p_{2,6}\ x_{3,6}}{-1+T_2}+$$

$$\frac{\in\ (-1+T_3)\ p_{3,2}\ x_{3,6}}{-1+T_2}-p_{3,6}\ x_{3,6}-\in p_{3,6}\ x_{3,6}+(1-T_3)\ p_{3,2^+}\ x_{3,6}+T_3\ p_{3,6^+}\ x_{3,6}-$$

$$\frac{\in\ (-1+T_1)\ (1+T_2)\ (-1+T_3)\ p_{1,2}\ p_{3,2}\ x_{1,6}\ x_{3,6}}{-1+T_2}-\frac{\in T_2\ (-1+T_3)\ p_{1,6}\ p_{3,2}\ x_{1,6}\ x_{3,6}}{-1+T_2}+$$

$$\frac{\in\ (-2+T_2)\ (-1+T_3)\ p_{2,2}\ p_{3,2}\ x_{2,2}\ x_{3,6}}{-1+T_2}+\frac{\in\ (-1+T_3)\ p_{2,6}\ p_{3,2}\ x_{2,2}\ x_{3,6}}{-1+T_2}+2\in p_{2,2}\ p_{3,6}\ x_{2,2}\ x_{3,6}+$$

$$\in\ (-1+T_3)\ p_{2,2}\ p_{3,2}\ x_{2,6}\ x_{3,6}+\in\ (1-T_2)\ p_{2,2}\ p_{3,6}\ x_{2,6}\ x_{3,6}-p_{3,7}\ x_{3,7}+p_{3,7^+}\ x_{3,7}+$$

$$\in p_{1,3}\ p_{3,7}\ x_{1,3}\ x_{3,7}+\frac{\in\ (-1+T_1)\ T_2\ p_{1,7}\ p_{3,7}\ x_{1,3}\ x_{3,7}}{-1+T_2}-\in p_{2,3}\ p_{3,7}\ x_{2,3}\ x_{3,7}-\in T_2\ p_{2,7}\ p_{3,7}\ x_{2,3}\ x_{3,7}\Big]\Big\}$$

In[∘]:=

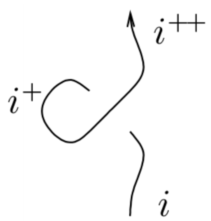
$$\int \$\mathcal{L}[K]\ d\!\!l\ vs[K]$$

Out[∘]=

$$-\frac{i\ T_1^2\ T_2^2\ \mathbb{E}\left[\frac{\in\ \left(1-T_1+T_1^2-T_2-T_1^3\ T_2+T_2^2+T_1^4\ T_2^2-T_1\ T_2^3-T_1^4\ T_2^3+T_1^2\ T_2^4-T_1^3\ T_2^4+T_1^4\ T_2^4\right)}{\left(1-T_1+T_1^2\right)\ \left(1-T_2+T_2^2\right)\ \left(1-T_1\ T_2+T_1^2\ T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\ \left(1-T_2+T_2^2\right)\ \left(1-T_1\ T_2+T_1^2\ T_2^2\right)}$$

(Alt) In[ ]:=

```
𝓕[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
vs_i_ := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
```



In[ ]:=

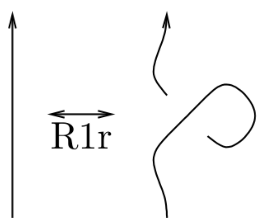
```
lhs = ∫𝓕[i] $ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) d{vs_i, vs_{i+}, vs_{i+2}}
rhs = ∫𝓕[i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) d{vs_i, vs_{i+}, vs_{i+2}};
lhs == rhs // HL
```

Out[ ]=

$-i\,\mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$

Out[ ]=

True



In[ ]:=

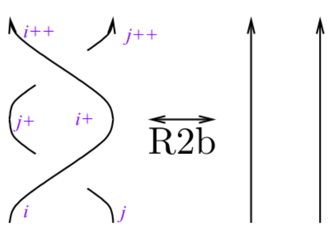
```
lhs = ∫𝓕[i] $ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) d{vs_i, vs_{i+}, vs_{i+2}}
rhs = ∫𝓕[i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) d{vs_i, vs_{i+}, vs_{i+2}};
lhs == rhs // HL
```

Out[ ]=

$-i\,\mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$

Out[ ]=

True



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:= $\texttt{lhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\}$

$\texttt{rhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\};$

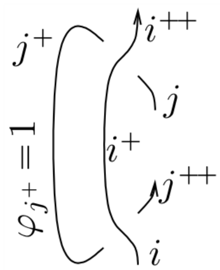
$\texttt{lhs} == \texttt{rhs}\ //\ \texttt{HL}$

Out[∘]=

$\mathbb{E}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}]$

Out[∘]=

True



In[∘]:= $\texttt{lhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L}\ /@\ (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\}$

$\texttt{rhs} = \int \mathcal{F}[i, j]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\};$

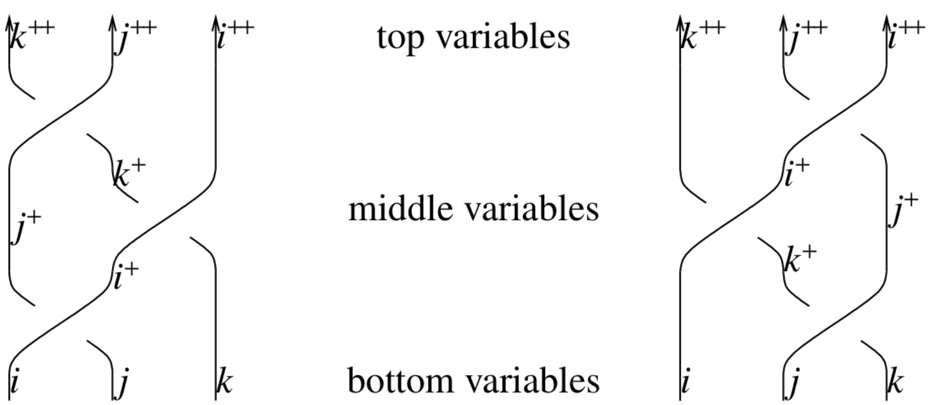
$\texttt{lhs} == \texttt{rhs}\ //\ \texttt{HL}$

Out[∘]=

$-\mathbb{i}\,T_1\,T_2\,\mathbb{E}\left[\frac{\epsilon}{2} + p_{1,2+i}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j} + \epsilon\,p_{3,3+j}\,\pi_{3,j}\right]$

Out[∘]=

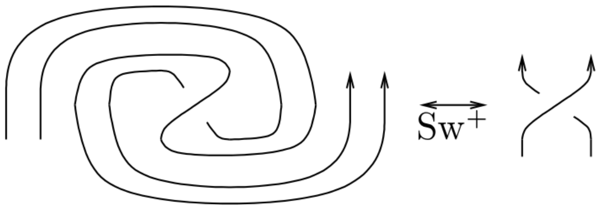
True



In[∘]:= $\texttt{Short}\left[\texttt{lhs} = \int (\mathcal{F}[i, j, k]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1]))\ \mathbb{d}\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\}\right]$

$\texttt{rhs} = \int (\mathcal{F}[i, j, k]\ \$\mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1]))\ \mathbb{d}\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\};$

$\texttt{lhs} == \texttt{rhs}\ //\ \texttt{HL}$




```
Short[lhs =
  ∫ℱ[i, j] $ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1]) d{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{i+2}, vs_{j+2}}]
rhs =
  ∫ℱ[i, j] $ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0]) d{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{i+2}, vs_{j+2}};
lhs == rhs // HL
```

The pPush Lagrangian

(Alt) In[ ]:=

```
pPush[ω_. 𝔼[L_]] := Module[{B, L1, Q, P0, P1a, P1b, θ0},
   L1 = CoefficientRules[B L /. {p_{αβ__} :→ B p_{αβ}, x_{αβ__} :→ B^-1 x_{αβ}}, {ϵ, B}];
   {ω, Q = {0, 1} /. L1, P0 = {0, 0} /. L1, P1a = {1, 2} /. L1, P1b = {1, 1} /. L1};
   θ0 = CF[
     px∨2g[
        CF[(P0 /. Thread[{s, i, j} → {s0, i0, j0}]) (P1a /. Thread[{s, i, j} → {s1, i1, j1}])
        ]] /. gR_{s0,i0,j0} /. gR_{s1,i1,j1} /. {δ_{j1,i0} | δ_{j0,i1} | δ_{i1,j0} → 0, δ_{i1,i0} | δ_{j1,j0} → B}
    ];
   θ1 = List @@ Factor[g∨2px[θ0 /. B → 0]];
   {Y0 = Times @@ Select[θ1, FreeQ[s1 | i1 | j1]],
    Y1 = Times @@ Select[θ1, FreeQ[s0 | i0 | j0]]};
   Times @@ θ1 == Y0 Y1;
   g∨2px[Coefficient[θ0, B] /. {s0 | s1 → s, i0 | i1 → i, j0 | j1 → j}]
  ];
pPush@ℒ_1[X_{i,j}[s]]
```

(Alt) Out[ ]=

$$\frac{s\,(-1+T_1^s)\,(1+T_2^s)\,(-1+(T_1 T_2)^s)\,p_{1,1+j}\,p_{3,1+j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s}$$

(Alt) In[ ]:=

```
pPush_{loc__}[ℰ_] := g2px[px2g[ℰ] /. gR_{loc} /. {δ_{α_,α_} → 1, δ_{i,j} → 0, δ_{j,i} → 0} /. α_^+ :→ α + 1];
```

In[ ]:=

```
{pPush_{s,i,j}[x_i p_i], pPush_i[x_i p_i]}
```

Out[ ]=

$$\{1+T^s\,p_{1+i}\,x_i+(1-T^s)\,p_{1+j}\,x_i,\ 1+p_{1+i}\,x_i\}$$

(Alt) In[∘]:=

```
ℒpPush[X_{i_,j_}[s_]] := T^{s/2} CF@𝔼[Plus[
    x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}),
    pPush_{s,i,j}[(∈ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
  ]]
ℒpPush[C_{i_}[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + pPush_i[∈ φ (1/2 - x_i p_i)]]
$ℒ = ℒpPush;
```

In[∘]:=

```
$ℒ[X_{i,j}[s]]
$ℒ[C_i[φ]]
```

Out[∘]=

$$T^{s/2}\,\mathbb{E}\Big[-\frac{s\,\epsilon}{2}-p_i x_i+T^s p_{1+i} x_i+(1-T^s)\,p_{1+j} x_i+s\,T^s\,\epsilon\,p_{1+j} x_i+\frac{1}{2}\,s\,T^s\,(-1+T^s)\,\epsilon\,p_{1+i}\,p_{1+j}\,x_i^2-\frac{1}{2}\,s\,T^s\,(-1+T^s)\,\epsilon\,p_{1+j}^2\,x_i^2-p_j x_j+p_{1+j} x_j-s\,\epsilon\,p_{1+j} x_j-s\,T^s\,\epsilon\,p_{1+i}\,p_{1+j}\,x_i x_j+s\,T^s\,\epsilon\,p_{1+j}^2\,x_i x_j\Big]$$

Out[∘]=

$$T^{\varphi/2}\,\mathbb{E}\Big[-\frac{\epsilon\,\varphi}{2}-\epsilon\,\varphi\,p_{1+i} x_i+(-p_i+p_{1+i})\,x_i\Big]$$

In[∘]:=

```
Collect[$ℒ[X_{i,j}[s]]⟦2, 1⟧, ∈]
Collect[$ℒ[C_i[φ]]⟦2, 1⟧, ∈]
```

Out[∘]=

$$-p_i x_i+T^s p_{1+i} x_i+(1-T^s)\,p_{1+j} x_i-p_j x_j+p_{1+j} x_j+\epsilon\left(-\frac{s}{2}+s\,T^s p_{1+j} x_i+\frac{1}{2}\,s\,T^s\,(-1+T^s)\,p_{1+i}\,p_{1+j}\,x_i^2-\frac{1}{2}\,s\,T^s\,(-1+T^s)\,p_{1+j}^2\,x_i^2-s\,p_{1+j} x_j-s\,T^s p_{1+i}\,p_{1+j}\,x_i x_j+s\,T^s p_{1+j}^2\,x_i x_j\right)$$

Out[∘]=

$$(-p_i+p_{1+i})\,x_i+\epsilon\left(-\frac{\varphi}{2}-\varphi\,p_{1+i} x_i\right)$$

The Trefoil

In[∘]:=

```
K = Mirror@Knot[3, 1]; {vs[K], $ℒ[K]}
```

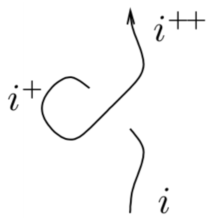
Out[∘]=

$$\Big\{\{p_1,x_1,p_2,x_2,p_3,x_3,p_4,x_4,p_5,x_5,p_6,x_6,p_7,x_7\},\ T\,\mathbb{E}\Big[-\epsilon-p_1 x_1+T p_2 x_1+(1-T)\,p_6 x_1+T\,\epsilon\,p_6 x_1+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_2\,p_6\,x_1^2-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_6^2\,x_1^2-p_2 x_2+p_3 x_2-\epsilon\,p_3 x_2-p_3 x_3+T p_4 x_3+(1-T)\,p_8 x_3+T\,\epsilon\,p_8 x_3+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_4\,p_8\,x_3^2-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_8^2\,x_3^2-p_4 x_4+p_5 x_4+\epsilon\,p_5 x_4-p_5 x_5+p_6 x_5-\epsilon\,p_6 x_5-T\,\epsilon\,p_2\,p_6\,x_1 x_5+T\,\epsilon\,p_6^2\,x_1 x_5+(1-T)\,p_3 x_6+T\,\epsilon\,p_3 x_6-p_6 x_6+T p_7 x_6+T\,\epsilon\,p_3^2\,x_2 x_6-T\,\epsilon\,p_3\,p_7\,x_2 x_6-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_3^2\,x_6^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_3\,p_7\,x_6^2-p_7 x_7+p_8 x_7-\epsilon\,p_8 x_7-T\,\epsilon\,p_4\,p_8\,x_3 x_7+T\,\epsilon\,p_8^2\,x_3 x_7\Big]\Big\}$$

In[∘]:=
```
$π = Normal[# + O[ϵ]^2] &; ∫$ℒ[K] ⅆ vs[K]
```

Out[∘]=

$$-\frac{\mathbb{i}\ T\ \mathbb{E}\left[-\frac{(-1+T)^2\ (1+T^2)\ \epsilon}{(1-T+T^2)^2}\right]}{1-T+T^2}$$



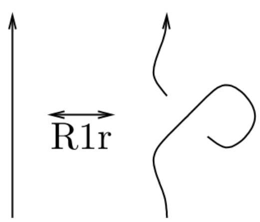
In[∘]:=
```
lhs = ∫𝔼[𝕚 π_i p_i] $ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[𝕚 π_i p_i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs // HL
```

Out[∘]=

$-\mathbb{i}\ \mathbb{E}[\mathbb{i}\ p_{3+i}\ \pi_i]$

Out[∘]=

True



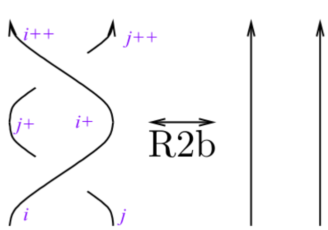
In[∘]:=
```
lhs = ∫𝔼[𝕚 π_i p_i] $ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[𝕚 π_i p_i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs // HL
```

Out[∘]=

$-\mathbb{i}\ \mathbb{E}[\mathbb{i}\ p_{3+i}\ \pi_i]$

Out[∘]=

True



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

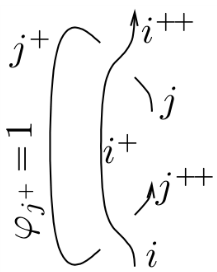
In[∘]:=
```
lhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) 𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
rhs =
  ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) 𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
lhs == rhs // HL
```

Out[∘]=

$\mathbb{E}[\mathbb{i}\, p_{2+i}\, \pi_i + \mathbb{i}\, p_{2+j}\, \pi_j]$

Out[∘]=

True



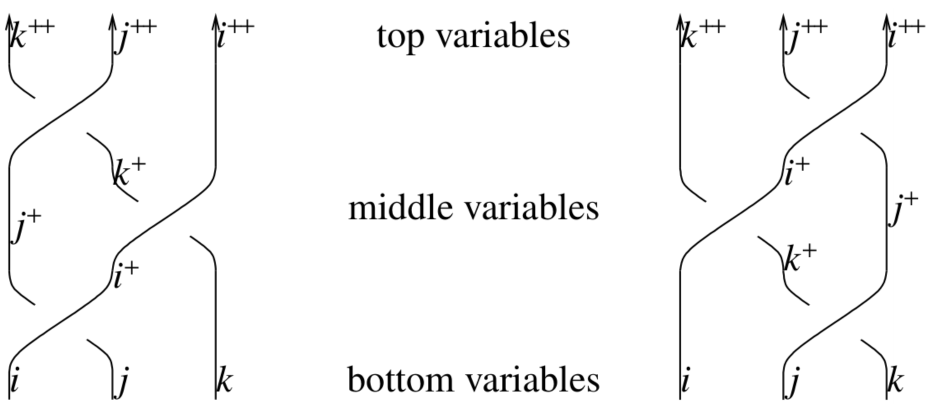
In[∘]:=
```
lhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
    𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
     𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
lhs == rhs // HL
```

Out[∘]=

$-\mathbb{i}\, \sqrt{T}\, \mathbb{E}\left[-\frac{\epsilon}{2} + \mathbb{i}\, p_{2+i}\, \pi_i + \mathbb{i}\, p_{3+j}\, \pi_j - \mathbb{i}\, \epsilon\, p_{3+j}\, \pi_j\right]$

Out[∘]=

True



In[]:=
```
$ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])
```

Out[]=

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2}-p_i x_i+T p_{1+i} x_i+(1-T)\,p_{1+j} x_i+T\epsilon\, p_{1+j} x_i+\frac{1}{2}(-1+T)\,T\epsilon\, p_{1+i} p_{1+j} x_i^2-\frac{1}{2}(-1+T)\,T\epsilon\, p_{1+j}^2 x_i^2-p_{1+i} x_{1+i}+T p_{2+i} x_{1+i}+(1-T)\,p_{1+k} x_{1+i}+T\epsilon\, p_{1+k} x_{1+i}+\frac{1}{2}(-1+T)\,T\epsilon\, p_{2+i} p_{1+k} x_{1+i}^2-\frac{1}{2}(-1+T)\,T\epsilon\, p_{1+k}^2 x_{1+i}^2-p_j x_j+p_{1+j} x_j-\epsilon\, p_{1+j} x_j-T\epsilon\, p_{1+i} p_{1+j} x_i x_j+T\epsilon\, p_{1+j}^2 x_i x_j-p_{1+j} x_{1+j}+T p_{2+j} x_{1+j}+(1-T)\,p_{2+k} x_{1+j}+T\epsilon\, p_{2+k} x_{1+j}+\frac{1}{2}(-1+T)\,T\epsilon\, p_{2+j} p_{2+k} x_{1+j}^2-\frac{1}{2}(-1+T)\,T\epsilon\, p_{2+k}^2 x_{1+j}^2-p_k x_k+p_{1+k} x_k-\epsilon\, p_{1+k} x_k-T\epsilon\, p_{2+i} p_{1+k} x_{1+i} x_k+T\epsilon\, p_{1+k}^2 x_{1+i} x_k-p_{1+k} x_{1+k}+p_{2+k} x_{1+k}-\epsilon\, p_{2+k} x_{1+k}-T\epsilon\, p_{2+j} p_{2+k} x_{1+j} x_{1+k}+T\epsilon\, p_{2+k}^2 x_{1+j} x_{1+k}\Big]$$

pdf

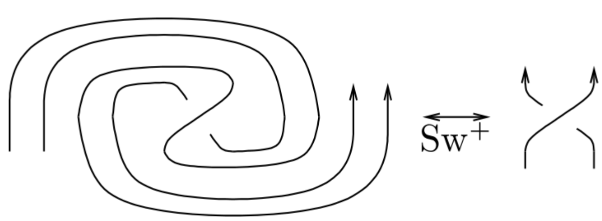
In[]:=
```
lhs = ∫(𝔼[ⅈ π_i p_i + ⅈ π_j p_j + ⅈ π_k p_k] $ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
    ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}}
rhs = ∫(𝔼[ⅈ π_i p_i + ⅈ π_j p_j + ⅈ π_k p_k] $ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
    ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
lhs == rhs // HL
```

Out[]=
pdf

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2}+i\,T^2 p_{2+i}\,\pi_i-i\,(-1+T)\,T\, p_{2+j}\,\pi_i+i\,T^2\epsilon\, p_{2+j}\,\pi_i-i\,(-1+T)\,p_{2+k}\,\pi_i+i\,T\epsilon\, p_{2+k}\,\pi_i-\frac{1}{2}(-1+T)\,T^3\epsilon\, p_{2+i} p_{2+j}\,\pi_i^2+\frac{1}{2}(-1+T)\,T^3\epsilon\, p_{2+j}^2\,\pi_i^2-\frac{1}{2}(-1+T)\,T^2\epsilon\, p_{2+i} p_{2+k}\,\pi_i^2+\frac{1}{2}(-1+T)^2\,T\epsilon\, p_{2+j} p_{2+k}\,\pi_i^2+\frac{1}{2}(-1+T)\,T\epsilon\, p_{2+k}^2\,\pi_i^2+i\,T p_{2+j}\,\pi_j-i\,T\epsilon\, p_{2+j}\,\pi_j-i\,(-1+T)\,p_{2+k}\,\pi_j+i\,(-1+2T)\,\epsilon\, p_{2+k}\,\pi_j+T^3\epsilon\, p_{2+i} p_{2+j}\,\pi_i\pi_j-T^3\epsilon\, p_{2+j}^2\,\pi_i\pi_j-(-1+T)\,T^2\epsilon\, p_{2+i} p_{2+k}\,\pi_i\pi_j+(-1+T)^2\,T\epsilon\, p_{2+j} p_{2+k}\,\pi_i\pi_j+(-1+T)\,T\epsilon\, p_{2+k}^2\,\pi_i\pi_j-\frac{1}{2}(-1+T)\,T\epsilon\, p_{2+j} p_{2+k}\,\pi_j^2+\frac{1}{2}(-1+T)\,T\epsilon\, p_{2+k}^2\,\pi_j^2+i\, p_{2+k}\,\pi_k-2\,i\,\epsilon\, p_{2+k}\,\pi_k+T^2\epsilon\, p_{2+i} p_{2+k}\,\pi_i\pi_k-(-1+T)\,T\epsilon\, p_{2+j} p_{2+k}\,\pi_i\pi_k-T\epsilon\, p_{2+k}^2\,\pi_i\pi_k+T\epsilon\, p_{2+j} p_{2+k}\,\pi_j\pi_k-T\epsilon\, p_{2+k}^2\,\pi_j\pi_k\Big]$$

Out[]=
pdf

True



In[∘]:=
```
lhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}};
lhs == rhs // HL
```

Out[∘]=

$$\sqrt{T}\,\mathbb{E}\left[-\frac{\epsilon}{2}+i\,T\,p_{3+i}\,\pi_i-i\,(-1+T)\,p_{3+j}\,\pi_i+i\,T\,\epsilon\,p_{3+j}\,\pi_i-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+j}^2\,\pi_i^2+i\,p_{3+j}\,\pi_j-i\,\epsilon\,p_{3+j}\,\pi_j+T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j-T\,\epsilon\,p_{3+j}^2\,\pi_i\,\pi_j\right]$$

Out[∘]=

True