Pensieve header: Experiments in 2026: Alternative integrands for ρ_1.

Initialization

```
In[∘]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
        Once[<< "IType(SimplifiedFormatting).m"];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

Minor utilities

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
        CCF[ℰ_] := Factor[ℰ];
        CF[ℰ_List] := CF /@ ℰ;
        CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
        CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, ϵ | (p | x | π | ξ | g)__, F] /. F → CCF];
        CF[ℰ_𝔼] := CF /@ ℰ;
        CF[𝔼sp___[ℰs___]] := CF /@ 𝔼sp[ℰs];
```

```
In[∘]:= ℒi_[K_] := CF[ℒi /@ Features[K][[2]]]
        vs[K_] := Join @@ Table[{pi, xi}, {i, Features[K][[1]]}]
```

```
In[∘]:= {p*, x*, π*, ξ*} = {π, ξ, p, x};
        (vs_List)* := (v ↦ v*) /@ vs;
        (u_vi__)* := (u*)vi;
```

```
In[∘]:= HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

g2px and px2g

Modified from pensieve://Talks/Geneva-2408/DataConversions.nb

```
In[∘]:= g2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {gi_,j_ :→ λ pi xj}] /. {λ^k_. :→ 1/k!}]
        gv2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. {gν_,i_,j_ :→ λν pν,i xν,j}] /. {λ^k_. :→ 1/k!}]
```

```
In[∘]:= Zip{}[ℰ_] := ℰ;
        Zip{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip{ξs}, ξ] /. f_. ξ^d_. :→ (D[f, {ξ*, d}])) /. ξ* → 0
```

```
In[•]:= px2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
          ps = Union[Cases[ℰ, p_, ∞]]; xs = Union[Cases[ℰ, x_, ∞]];
          Q = Sum[p0* x0* g_{p0[[2]],x0[[2]]}, {p0, ps}, {x0, xs}];
          Expand[Zip_{ps⋃xs}[ℰ e^Q]]
        ]
        pxν2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
          ps = Union[Cases[ℰ, p__, ∞]]; xs = Union[Cases[ℰ, x__, ∞]];
          Q = Sum[p0* x0* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
          Expand[Zip_{ps⋃xs}[ℰ e^Q] /. g_{α_,β_,i_,j_} :> If[α === β, g_{α,i,j}, 0]]
        ]
```

gRules

From pensieve://Talks/MonteVerita-2604/Theta.nb

```
In[•]:= gR_{s_,i_,j_} := {
          g_{jβ_} :> g_{j⁺β} + δ_{jβ}, g_{iβ_} :> T^s g_{i⁺β} + (1 - T^s) g_{j⁺β} + δ_{iβ},
          g_{α_i⁺} :> T^s g_{αi} + δ_{αi⁺}, g_{α_j⁺} :> g_{αj} + (1 - T^s) g_{αi} + δ_{αj⁺},
          g_{ν_jβ_} :> g_{νj⁺β} + δ_{jβ}, g_{ν_iβ_} :> T_ν^s g_{νi⁺β} + (1 - T_ν^s) g_{νj⁺β} + δ_{iβ},
          g_{ν_α_i⁺} :> T_ν^s g_{ναi} + δ_{αi⁺}, g_{ν_α_j⁺} :> g_{ναj} + (1 - T_ν^s) g_{ναi} + δ_{αj⁺}
        }
        gR_{i_} := {g_{i,β_} :> g_{i⁺,β} + δ_{i,β}, g_{α_,i⁺} :> g_{α,i} + δ_{α,i⁺}}
```

Inverse gRules written locally:

```
In[•]:= igR_{s_,i_,j_} := {
          g_{j⁺,β_} :> g_{j,β} - δ_{j,β}, g_{i⁺,β_} :> T^-s g_{iβ} - (T^-s - 1) g_{j⁺β} - T^-s δ_{iβ},
          g_{α_,i} :> T^-s (g_{α,i⁺} - δ_{α,i⁺}), g_{α_,j} :> g_{α,j⁺} - (1 - T^s) g_{α,i} - δ_{α,j⁺},
          g_{ν_,j⁺,β_} :> g_{ν,j,β} - δ_{j,β}, g_{ν_,i⁺,β_} :> T_ν^-s g_{νiβ} - (T_ν^-s - 1) g_{νj⁺β} - T_ν^-s δ_{iβ},
          g_{ν_,α_,i} :> T_ν^-s (g_{ν,α,i⁺} - δ_{α,i⁺}), g_{ν_,α_,j} :> g_{ν,α,j⁺} - (1 - T_ν^s) g_{ν,α,i} - δ_{α,j⁺}
        }
```

The Base Lagrangian

From pensieve://Talks/Bonn-2505.

```
In[•]:= ℒ1[X_{i_,j_}[s_]] := T^{s/2} 𝔼[
          x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
            (ϵ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
        ℒ1[C_{i_}[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
        $ℒ = ℒ1;
```

The Trefoil

In[∘]:=
```
K = Mirror@Knot[3, 1]; Features[K]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$\text{Features}[7, C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]]$

In[∘]:=
```
{vs[K], $ℒ[K]}
```

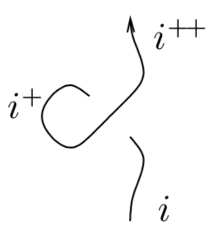
Out[∘]=

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-2\in + (-1+\in)\, p_1 x_1 + T\, p_2 x_1 - \in p_5 x_1 + (1-T)\, p_6 x_1 + \frac{1}{2}(-1+T)\in p_1 p_5 x_1^2 - \frac{1}{2}(-1+T)\in p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 + (-1+\in)\, p_3 x_3 + T\, p_4 x_3 - \in p_7 x_3 + (1-T)\, p_8 x_3 + \frac{1}{2}(-1+T)\in p_3 p_7 x_3^2 - \frac{1}{2}(-1+T)\in p_7^2 x_3^2 + (-1+\in)\, p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \in p_1 p_5 x_1 x_5 + \in p_5^2 x_1 x_5 - \in p_2 x_6 + (1-T)\, p_3 x_6 + (-1+\in)\, p_6 x_6 + T\, p_7 x_6 + \in p_2^2 x_2 x_6 - \in p_2 p_6 x_2 x_6 - \frac{1}{2}(-1+T)\in p_2^2 x_6^2 + \frac{1}{2}(-1+T)\in p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \in p_3 p_7 x_3 x_7 + \in p_7^2 x_3 x_7\Big]\Big\}$$

In[∘]:=
```
$π = Normal[# + O[∊]^2] &; ∫$ℒ[K] ⅆ vs[K]
```

Out[∘]=

$$-\frac{i\, T\, \mathbb{E}\left[-\frac{(-1+T)^2 (1+T^2)\in}{(1-T+T^2)^2}\right]}{1-T+T^2}$$



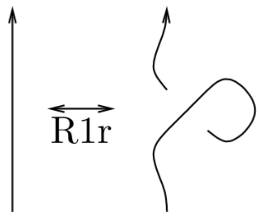
In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i] $ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[ⅈ π_i p_i] $ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs // HL
```

Out[∘]=

$-i\,\mathbb{E}[i\, p_{3+i}\, \pi_i]$

Out[∘]=

True



In[∘]:=

$$\texttt{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \$\mathcal{L}\ /@\ (X_{i,i+2}[1]\ C_{i+1}[-1])\ d\{x_i,\ p_i,\ x_{i+1},\ p_{i+1},\ x_{i+2},\ p_{i+2}\}$$

$$\texttt{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{x_i,\ p_i,\ x_{i+1},\ p_{i+1},\ x_{i+2},\ p_{i+2}\};$$

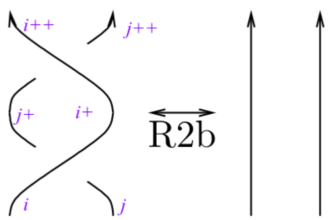
$$\texttt{lhs} == \texttt{rhs}\ //\ \texttt{HL}$$

Out[∘]=

$$-\mathbb{i}\,\mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Out[∘]=

True



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=

$$\texttt{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ d\{x_i,\ x_j,\ p_i,\ p_j,\ x_{i+1},\ x_{j+1},\ p_{i+1},\ p_{j+1}\}$$

$$\texttt{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ d\{x_i,\ x_j,\ p_i,\ p_j,\ x_{i+1},\ x_{j+1},\ p_{i+1},\ p_{j+1}\};$$

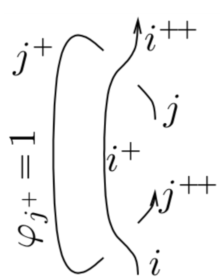
$$\texttt{lhs} == \texttt{rhs}\ //\ \texttt{HL}$$

Out[∘]=

$$\mathbb{E}[\mathbb{i}\,p_{2+i}\,\pi_i + \mathbb{i}\,p_{2+j}\,\pi_j]$$

Out[∘]=

True



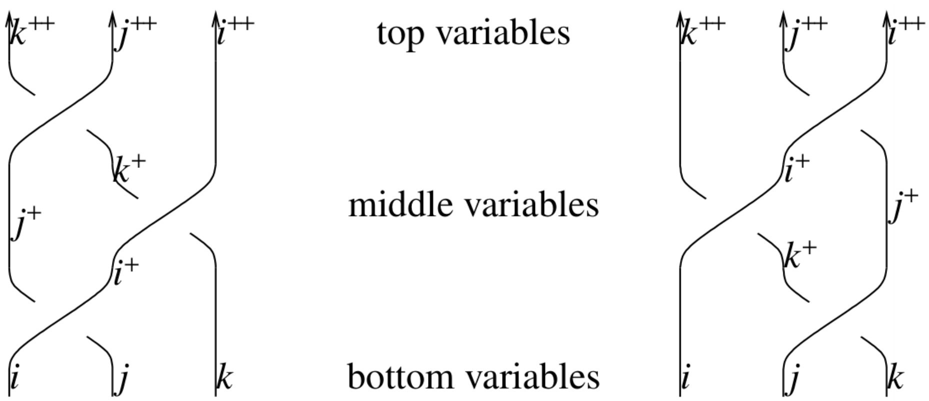
In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] $ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
lhs == rhs // HL
```

Out[∘]=

$$-\mathbb{i}\,\sqrt{T}\;\mathbb{E}\left[-\frac{\epsilon}{2}+\mathbb{i}\,p_{2+i}\,\pi_i-\mathbb{i}\,(-1+\epsilon)\,p_{3+j}\,\pi_j\right]$$

Out[∘]=

True



In[∘]:=
```
CF[$ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])]
```

Out[∘]=

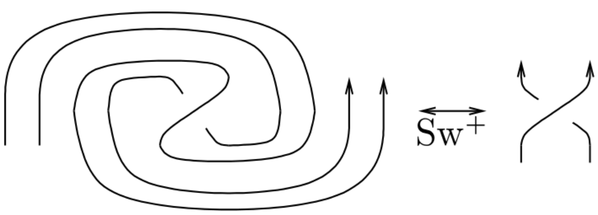
$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2}-p_i\,x_i+\epsilon\,p_i\,x_i+T\,p_{1+i}\,x_i-\epsilon\,p_j\,x_i+(1-T)\,p_{1+j}\,x_i+\frac{1}{2}\,(-1+T)\,\epsilon\,p_i\,p_j\,x_i^2+\frac{1}{2}\,(1-T)\,\epsilon\,p_j^2\,x_i^2-p_{1+i}\,x_{1+i}+\epsilon\,p_{1+i}\,x_{1+i}+T\,p_{2+i}\,x_{1+i}-\epsilon\,p_k\,x_{1+i}+(1-T)\,p_{1+k}\,x_{1+i}+\frac{1}{2}\,(-1+T)\,\epsilon\,p_{1+i}\,p_k\,x_{1+i}^2+\frac{1}{2}\,(1-T)\,\epsilon\,p_k^2\,x_{1+i}^2-p_j\,x_j+p_{1+j}\,x_j-\epsilon\,p_i\,p_j\,x_i\,x_j+\epsilon\,p_j^2\,x_i\,x_j-p_{1+j}\,x_{1+j}+\epsilon\,p_{1+j}\,x_{1+j}+T\,p_{2+j}\,x_{1+j}-\epsilon\,p_{1+k}\,x_{1+j}+(1-T)\,p_{2+k}\,x_{1+j}+\frac{1}{2}\,(-1+T)\,\epsilon\,p_{1+j}\,p_{1+k}\,x_{1+j}^2+\frac{1}{2}\,(1-T)\,\epsilon\,p_{1+k}^2\,x_{1+j}^2-p_k\,x_k+p_{1+k}\,x_k-\epsilon\,p_{1+i}\,p_k\,x_{1+i}\,x_k+\epsilon\,p_k^2\,x_{1+i}\,x_k-p_{1+k}\,x_{1+k}+p_{2+k}\,x_{1+k}-\epsilon\,p_{1+j}\,p_{1+k}\,x_{1+j}\,x_{1+k}+\epsilon\,p_{1+k}^2\,x_{1+j}\,x_{1+k}\Big]$$

In[∘]:= $\text{lhs} = \int (\mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j + \mathbb{i}\,\pi_k\,p_k]\ \$\mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1]))$
$\quad d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\}$
$\text{rhs} = \int (\mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j + \mathbb{i}\,\pi_k\,p_k]\ \$\mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1]))$
$\quad d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$
$\text{lhs} == \text{rhs}\ //\ \text{HL}$

Out[∘]=

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\in}{2} + \mathbb{i}\,T^2\,p_{2+i}\,\pi_i - \mathbb{i}\,(-1+T)\,T\,p_{2+j}\,\pi_i + \mathbb{i}\,T^2\in p_{2+j}\,\pi_i - \mathbb{i}\,(-1+T)\,p_{2+k}\,\pi_i + \mathbb{i}\,T\in p_{2+k}\,\pi_i - \frac{1}{2}\,(-1+T)\,T^3\in p_{2+i}\,p_{2+j}\,\pi_i^2 + \frac{1}{2}\,(-1+T)\,T^3\in p_{2+j}^2\,\pi_i^2 - \frac{1}{2}\,(-1+T)\,T^2\in p_{2+i}\,p_{2+k}\,\pi_i^2 + \frac{1}{2}\,(-1+T)^2\,T\in p_{2+j}\,p_{2+k}\,\pi_i^2 + \frac{1}{2}\,(-1+T)\,T\in p_{2+k}^2\,\pi_i^2 + \mathbb{i}\,T\,p_{2+j}\,\pi_j - \mathbb{i}\,T\in p_{2+j}\,\pi_j - \mathbb{i}\,(-1+T)\,p_{2+k}\,\pi_j + \mathbb{i}\,(-1+2T)\in p_{2+k}\,\pi_j + T^3\in p_{2+i}\,p_{2+j}\,\pi_i\,\pi_j - T^3\in p_{2+j}^2\,\pi_i\,\pi_j - (-1+T)\,T^2\in p_{2+i}\,p_{2+k}\,\pi_i\,\pi_j + (-1+T)^2\,T\in p_{2+j}\,p_{2+k}\,\pi_i\,\pi_j + (-1+T)\,T\in p_{2+k}^2\,\pi_i\,\pi_j - \frac{1}{2}\,(-1+T)\,T\in p_{2+j}\,p_{2+k}\,\pi_j^2 + \frac{1}{2}\,(-1+T)\,T\in p_{2+k}^2\,\pi_j^2 + \mathbb{i}\,p_{2+k}\,\pi_k - 2\,\mathbb{i}\in p_{2+k}\,\pi_k + T^2\in p_{2+i}\,p_{2+k}\,\pi_i\,\pi_k - (-1+T)\,T\in p_{2+j}\,p_{2+k}\,\pi_i\,\pi_k - T\in p_{2+k}^2\,\pi_i\,\pi_k + T\in p_{2+j}\,p_{2+k}\,\pi_j\,\pi_k - T\in p_{2+k}^2\,\pi_j\,\pi_k\Big]$$

Out[∘]=
True



In[∘]:= $\text{lhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \$\mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])$
$\quad d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\}$
$\text{rhs} = \int \mathbb{E}[\mathbb{i}\,\pi_i\,p_i + \mathbb{i}\,\pi_j\,p_j]\ \$\mathcal{L}\ /@\ (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])$
$\quad d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\};$
$\text{lhs} == \text{rhs}\ //\ \text{HL}$

Out[∘]=

$$\sqrt{T}\,\mathbb{E}\Big[-\frac{\in}{2} + \mathbb{i}\,T\,p_{3+i}\,\pi_i + \mathbb{i}\,(1-T+T\in)\,p_{3+j}\,\pi_i - \frac{1}{2}\,(-1+T)\,T\in p_{3+i}\,p_{3+j}\,\pi_i^2 + \frac{1}{2}\,(-1+T)\,T\in p_{3+j}^2\,\pi_i^2 - \mathbb{i}\,(-1+\in)\,p_{3+j}\,\pi_j + T\in p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j - T\in p_{3+j}^2\,\pi_i\,\pi_j\Big]$$

Out[∘]=
True

The pPush Lagrangian

In[∘]:= $\text{pPush}_{loc\_\_}[\mathcal{E}\_] := \text{g2px}[\text{px2g}[\mathcal{E}]\ /.\ \text{gR}_{loc}\ /.\ \{\delta_{\alpha\_,\alpha\_} \to 1,\ \delta_{i,j} \to 0,\ \delta_{j,i} \to 0\}\ /.\ \alpha\_^{+} :\to \alpha + 1];$

In[ ]:=
```
{pPush_{s,i,j}[x_i p_i], pPush_i[x_i p_i]}
```
Out[ ]=
$$\{1 + T^s p_{1+i} x_i + (1 - T^s) p_{1+j} x_i,\ 1 + p_{1+i} x_i\}$$

In[ ]:=
```
𝓛_pPush[X_{i_,j_}[s_]] := T^{s/2} CF@𝔼[Plus[
    x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}),
    pPush_{s,i,j}[(ϵ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
  ]]
𝓛_pPush[C_{i_}[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + pPush_i[ϵ φ (1/2 - x_i p_i)]]
$𝓛 = 𝓛_pPush;
```

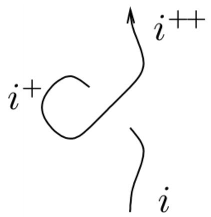
In[ ]:=
```
$𝓛[X_{i,j}[1]]
$𝓛[C_i[1]]
```
Out[ ]=
$$\sqrt{T}\ \mathbb{E}\Big[-\frac{\epsilon}{2} - p_i x_i + T p_{1+i} x_i + (1 - T) p_{1+j} x_i + T \epsilon p_{1+j} x_i + \frac{1}{2} (-1 + T) T \epsilon p_{1+i} p_{1+j} x_i^2 - \frac{1}{2} (-1 + T) T \epsilon p_{1+j}^2 x_i^2 - p_j x_j + p_{1+j} x_j - \epsilon p_{1+j} x_j - T \epsilon p_{1+i} p_{1+j} x_i x_j + T \epsilon p_{1+j}^2 x_i x_j\Big]$$

Out[ ]=
$$\sqrt{T}\ \mathbb{E}\Big[-\frac{\epsilon}{2} - \epsilon p_{1+i} x_i + (-p_i + p_{1+i}) x_i\Big]$$

The Trefoil

In[ ]:=
```
K = Mirror@Knot[3, 1]; {vs[K], $𝓛[K]}
```
Out[ ]=
$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\ \mathbb{E}\Big[-\epsilon - p_1 x_1 + T p_2 x_1 + (1 - T) p_6 x_1 + T \epsilon p_6 x_1 + \frac{1}{2} (-1 + T) T \epsilon p_2 p_6 x_1^2 - \frac{1}{2} (-1 + T) T \epsilon p_6^2 x_1^2 - p_2 x_2 + p_3 x_2 - \epsilon p_3 x_2 - p_3 x_3 + T p_4 x_3 + (1 - T) p_8 x_3 + T \epsilon p_8 x_3 + \frac{1}{2} (-1 + T) T \epsilon p_4 p_8 x_3^2 - \frac{1}{2} (-1 + T) T \epsilon p_8^2 x_3^2 - p_4 x_4 + p_5 x_4 + \epsilon p_5 x_4 - p_5 x_5 + p_6 x_5 - \epsilon p_6 x_5 - T \epsilon p_2 p_6 x_1 x_5 + T \epsilon p_6^2 x_1 x_5 + (1 - T) p_3 x_6 + T \epsilon p_3 x_6 - p_6 x_6 + T p_7 x_6 + T \epsilon p_3^2 x_2 x_6 - T \epsilon p_3 p_7 x_2 x_6 - \frac{1}{2} (-1 + T) T \epsilon p_3^2 x_6^2 + \frac{1}{2} (-1 + T) T \epsilon p_3 p_7 x_6^2 - p_7 x_7 + p_8 x_7 - \epsilon p_8 x_7 - T \epsilon p_4 p_8 x_3 x_7 + T \epsilon p_8^2 x_3 x_7\Big]\Big\}$$

In[ ]:=
```
$π = Normal[# + O[ϵ]^2] &; ∫$𝓛[K] ⅆ vs[K]
```
Out[ ]=
$$-\frac{\mathbb{i}\ T\ \mathbb{E}\Big[-\frac{(-1+T)^2 (1+T^2) \epsilon}{(1-T+T^2)^2}\Big]}{1 - T + T^2}$$



In[∘]:= $\text{lhs} = \int \mathbb{E}[\mathbb{i}\, \pi_i\, p_i]\ \$\mathcal{L}\ /@\ (X_{i+2,i}[1]\ C_{i+1}[1])\ d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$

$\text{rhs} = \int \mathbb{E}[\mathbb{i}\, \pi_i\, p_i]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\};$

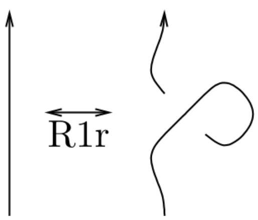
lhs == rhs // HL

Out[∘]=

$-\mathbb{i}\, \mathbb{E}[\mathbb{i}\, p_{3+i}\, \pi_i]$

Out[∘]=

True



In[∘]:= $\text{lhs} = \int \mathbb{E}[\mathbb{i}\, \pi_i\, p_i]\ \$\mathcal{L}\ /@\ (X_{i,i+2}[1]\ C_{i+1}[-1])\ d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$

$\text{rhs} = \int \mathbb{E}[\mathbb{i}\, \pi_i\, p_i]\ \$\mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\};$

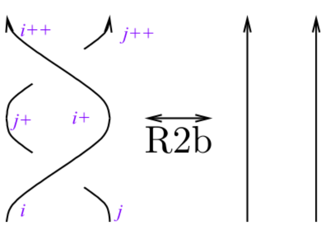
lhs == rhs // HL

Out[∘]=

$-\mathbb{i}\, \mathbb{E}[\mathbb{i}\, p_{3+i}\, \pi_i]$

Out[∘]=

True



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

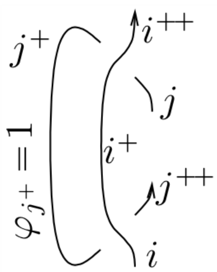
```
In[•]:= lhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
        rhs =
          ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
        lhs == rhs // HL
```

Out[•]=

$\mathbb{E}[\mathbb{i}\, p_{2+i}\, \pi_i + \mathbb{i}\, p_{2+j}\, \pi_j]$

Out[•]=

True



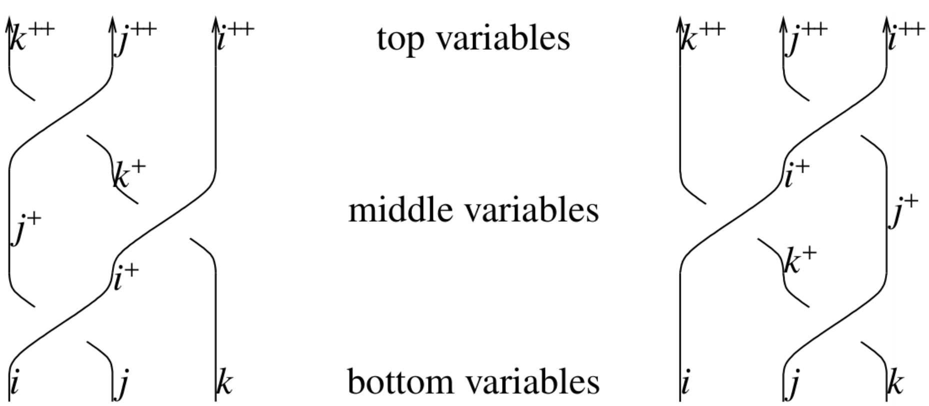

```
In[•]:= lhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
            ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
        rhs = ∫𝔼[𝕚 π_i p_i + 𝕚 π_j p_j] $ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
            ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
        lhs == rhs // HL
```

Out[•]=

$-\mathbb{i}\,\sqrt{T}\;\mathbb{E}\left[-\frac{\epsilon}{2} + \mathbb{i}\, p_{2+i}\, \pi_i + \mathbb{i}\, p_{3+j}\, \pi_j - \mathbb{i}\,\epsilon\, p_{3+j}\, \pi_j\right]$

Out[•]=

True



In[∘]:=
```
$ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])
```

Out[∘]=

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2}-p_i x_i+T p_{1+i} x_i+(1-T)\,p_{1+j} x_i+T\epsilon\,p_{1+j} x_i+\frac{1}{2}(-1+T)\,T\epsilon\,p_{1+i} p_{1+j} x_i^2-\frac{1}{2}(-1+T)\,T\epsilon\,p_{1+j}^2 x_i^2-p_{1+i} x_{1+i}+T p_{2+i} x_{1+i}+(1-T)\,p_{1+k} x_{1+i}+T\epsilon\,p_{1+k} x_{1+i}+\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+i} p_{1+k} x_{1+i}^2-\frac{1}{2}(-1+T)\,T\epsilon\,p_{1+k}^2 x_{1+i}^2-p_j x_j+p_{1+j} x_j-\epsilon\,p_{1+j} x_j-T\epsilon\,p_{1+i} p_{1+j} x_i x_j+T\epsilon\,p_{1+j}^2 x_i x_j-p_{1+j} x_{1+j}+T p_{2+j} x_{1+j}+(1-T)\,p_{2+k} x_{1+j}+T\epsilon\,p_{2+k} x_{1+j}+\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+j} p_{2+k} x_{1+j}^2-\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+k}^2 x_{1+j}^2-p_k x_k+p_{1+k} x_k-\epsilon\,p_{1+k} x_k-T\epsilon\,p_{2+i} p_{1+k} x_{1+i} x_k+T\epsilon\,p_{1+k}^2 x_{1+i} x_k-p_{1+k} x_{1+k}+p_{2+k} x_{1+k}-\epsilon\,p_{2+k} x_{1+k}-T\epsilon\,p_{2+j} p_{2+k} x_{1+j} x_{1+k}+T\epsilon\,p_{2+k}^2 x_{1+j} x_{1+k}\Big]$$

pdf

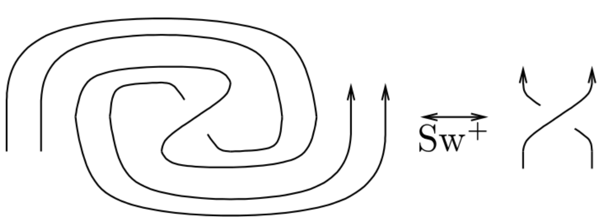
In[∘]:=
```
lhs = ∫(𝔼[𝕚 π_i p_i + 𝕚 π_j p_j + 𝕚 π_k p_k] $ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
    ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}}
rhs = ∫(𝔼[𝕚 π_i p_i + 𝕚 π_j p_j + 𝕚 π_k p_k] $ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
    ⅆ{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
lhs == rhs // HL
```

Out[∘]=
pdf

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2}+i\,T^2 p_{2+i}\pi_i-i(-1+T)\,T p_{2+j}\pi_i+i\,T^2\epsilon\,p_{2+j}\pi_i-i(-1+T)\,p_{2+k}\pi_i+i\,T\epsilon\,p_{2+k}\pi_i-\frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+i}p_{2+j}\pi_i^2+\frac{1}{2}(-1+T)\,T^3\epsilon\,p_{2+j}^2\pi_i^2-\frac{1}{2}(-1+T)\,T^2\epsilon\,p_{2+i}p_{2+k}\pi_i^2+\frac{1}{2}(-1+T)^2\,T\epsilon\,p_{2+j}p_{2+k}\pi_i^2+\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+k}^2\pi_i^2+i\,T p_{2+j}\pi_j-i\,T\epsilon\,p_{2+j}\pi_j-i(-1+T)\,p_{2+k}\pi_j+i(-1+2T)\,\epsilon\,p_{2+k}\pi_j+T^3\epsilon\,p_{2+i}p_{2+j}\pi_i\pi_j-T^3\epsilon\,p_{2+j}^2\pi_i\pi_j-(-1+T)\,T^2\epsilon\,p_{2+i}p_{2+k}\pi_i\pi_j+(-1+T)^2\,T\epsilon\,p_{2+j}p_{2+k}\pi_i\pi_j+(-1+T)\,T\epsilon\,p_{2+k}^2\pi_i\pi_j-\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+j}p_{2+k}\pi_j^2+\frac{1}{2}(-1+T)\,T\epsilon\,p_{2+k}^2\pi_j^2+i\,p_{2+k}\pi_k-2i\,\epsilon\,p_{2+k}\pi_k+T^2\epsilon\,p_{2+i}p_{2+k}\pi_i\pi_k-(-1+T)\,T\epsilon\,p_{2+j}p_{2+k}\pi_i\pi_k-T\epsilon\,p_{2+k}^2\pi_i\pi_k+T\epsilon\,p_{2+j}p_{2+k}\pi_j\pi_k-T\epsilon\,p_{2+k}^2\pi_j\pi_k\Big]$$

Out[∘]=
pdf

True



In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] $ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] $ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}};
lhs == rhs // HL
```

Out[∘]=

$$\sqrt{T}\ \mathbb{E}\Big[-\frac{\in}{2} + i\,T\,p_{3+i}\,\pi_i - i\,(-1+T)\,p_{3+j}\,\pi_i + i\,T\in p_{3+j}\,\pi_i - \frac{1}{2}\,(-1+T)\,T\in p_{3+i}\,p_{3+j}\,\pi_i^2 + \frac{1}{2}\,(-1+T)\,T\in p_{3+j}^2\,\pi_i^2 + i\,p_{3+j}\,\pi_j - i\in p_{3+j}\,\pi_j + T\in p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j - T\in p_{3+j}^2\,\pi_i\,\pi_j\Big]$$

Out[∘]=

True