Pensieve header: The rank 2 mod ϵ^2 invariant using integration techniques - further experiments in 2026.

Initialization

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
Once[<< "IType(SimplifiedFormatting).m"];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

pdf

In[∘]:=
```
T3 = T1 T2;  i_+ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /. {π_is__ :→ B^-1 π_is, x_is__ :→ B^-1 x_is, p_is__ :→ B p_is} /. ϵ B^b_ /; b < 0 → 0 /.
     B → 1) &;
```

pdf

In[∘]:=
```
ps_i_ := Sequence[p_{1,i}, p_{2,i}, p_{3,i}];
xs_i_ := Sequence[x_{1,i}, x_{2,i}, x_{3,i}];
vs_i_ := Sequence[ps_i, xs_i];
ℱ[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K]〚2〛];
vs[K_] := Union @@ Table[{vs_i}, {i, Features[K]〚1〛}]
```

In[∘]:= `vs_7`

Out[∘]=

$\text{Sequence}[p_{1,7}, p_{2,7}, p_{3,7}, x_{1,7}, x_{2,7}, x_{3,7}]$

The Lagrangian

tex

\needspace{30mm}
{\bf\red The Lagrangian.}

exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

In[∘]:=

$$\mathcal{L}[X_{i\_,j\_}[s\_]] := T_3^s\,\mathbb{E}\Big[\text{CF@Plus}\Big[\sum_{\nu=1}^{3}\left(x_{\nu i}\,(p_{\nu i^+}-p_{\nu i})+x_{\nu j}\,(p_{\nu j^+}-p_{\nu j})+\left(T_\nu^s-1\right)x_{\nu i}\,(p_{\nu i^+}-p_{\nu j^+})\right),$$
$$\left(T_1^s-1\right)p_{3j}\,x_{1i}\,\left(T_2^s\,x_{2i}-x_{2j}\right),$$
$$\epsilon\,s\,\left(T_3^s-1\right)p_{1j}\,(p_{2i}-p_{2j})\,x_{3i}\big/\left(T_2^s-1\right),$$
$$\epsilon\,s\,\Big(1/2+T_2^s\,p_{1i}\,p_{2j}\,x_{1i}\,x_{2i}-p_{1i}\,p_{2j}\,x_{1i}\,x_{2j}-p_{3i}\,x_{3i}-\left(T_2^s-1\right)p_{2j}\,p_{3i}\,x_{2i}\,x_{3i}+$$
$$\left(T_3^s-1\right)p_{2j}\,p_{3j}\,x_{2i}\,x_{3i}+2\,p_{2j}\,p_{3i}\,x_{2j}\,x_{3i}+p_{1i}\,p_{3j}\,x_{1i}\,x_{3j}-p_{2i}\,p_{3j}\,x_{2i}\,x_{3j}-T_2^s\,p_{2j}\,p_{3j}\,x_{2i}\,x_{3j}+$$
$$\Big(\left(T_1^s-1\right)p_{1j}\,x_{1i}\,\left(T_2^{2s}\,p_{2j}\,x_{2i}-T_2^s\,p_{2j}\,x_{2j}-\left(T_2^s+1\right)\left(T_3^s-1\right)p_{3j}\,x_{3i}+T_2^s\,p_{3j}\,x_{3j}\right)+$$
$$\left(T_3^s-1\right)p_{3j}\,x_{3i}\,\left(1-T_2^s\,p_{1i}\,x_{1i}+p_{2i}\,x_{2j}+\left(T_2^s-2\right)p_{2j}\,x_{2j}\right)\Big)\big/\left(T_2^s-1\right)\Big)\Big]\Big]$$

pdf

In[∘]:=

$$\mathcal{L}[C_{i\_}[\varphi\_]] := T_3^{\varphi}\,\mathbb{E}\left[\sum_{\nu=1}^{3}x_{\nu i}\,(p_{\nu i^+}-p_{\nu i})+\epsilon\,\varphi\,(p_{3i}\,x_{3i}-1/2)\right]$$

exec

```
nb2tex$PDFWidth /= 1.25;
```

Reidemeister 3 - Fourier

tex

{\bf\red Reidemeister 3.}

pdf

In[∘]:=

$$\text{Short}\left[\text{lhs} = \int \mathcal{F}[i,j,k]\;\mathcal{L}\,/@\,(X_{i,j}[1]\,X_{i^+,k}[1]\,X_{j^+,k^+}[1])\,d\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\}\right]$$

Out[∘]//Short=

pdf

$$T_1^3\,T_2^3\,\mathbb{E}\left[\frac{3\,\epsilon}{2}+T_1^2\,p_{1,2+i}\,\pi_{1,i}-(-1+T_1)\,T_1\,p_{1,2+j}\,\pi_{1,i}+\ll 150\gg\right]$$

pdf

In[∘]:=

$$\text{rhs} = \int \mathcal{F}[i,j,k]\;\mathcal{L}\,/@\,(X_{j,k}[1]\,X_{i,k^+}[1]\,X_{i^+,j^+}[1])\,d\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\};\ \text{lhs} == \text{rhs}$$

Out[∘]=

pdf

True

Reidemeister 3 - staggered integration

{\bf\red Reidemeister 3.}

In[∘]:=

$$\text{Short}\left[\text{lhs} = \int \mathcal{L}\,/@\,(X_{i,j}[1]\,X_{i^+,k}[1]\,X_{j^+,k^+}[1])\,d\{xs_i, xs_j, xs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}, ps_{i^{++}}, ps_{j^{++}}, ps_{k^{++}}\}\right]$$

Out[∘]//Short=

$$\frac{\mathbb{E}\left[\frac{3\,\epsilon}{2}\right]}{T_1^3\,T_2^3}$$

In[∘]:=
```
rhs = ∫ℒ /@ (X_{j,k}[1] X_{i,k⁺}[1] X_{i⁺,j⁺}[1]) 𝕕{xs_i, xs_j, xs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}, ps_{i⁺⁺}, ps_{j⁺⁺}, ps_{k⁺⁺}};
lhs == rhs
```
Out[∘]=
```
True
```

The Trefoil

tex

\needspace{25mm}
\parpic[r]{\includegraphics[width=0.6in]{../Beijing-2407/Trefoil.jpg}}
{\bf\red The Trefoil.}

pdf

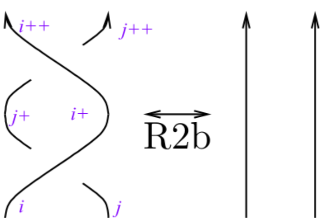
In[∘]:=
```
K = Knot[3, 1]; ∫ℒ[K] 𝕕 vs[K]
```
pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

$$-\frac{\mathbb{i}\, T_1^2\, T_2^2\, \mathbb{E}\left[-\frac{\epsilon\left(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}$$

Invariance Under Reidemeister 2b



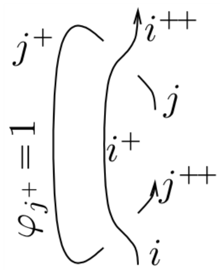
Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=
```
lhs = ∫ℱ[i, j] ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}}
rhs = ∫ℱ[i, j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}};
lhs == rhs
```
Out[∘]=

$$\mathbb{E}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}]$$

Out[∘]=
```
True
```

Invariance Under R2c



In[∘]:=

$$\text{lhs} = \int \mathcal{F}[i, j]\ \mathcal{L} \text{ /@ } (X_{i+1,j}[1]\ X_{i,j+2}[-1]\ C_{j+1}[1])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\}$$

$$\text{rhs} = \int \mathcal{F}[i, j]\ \mathcal{L} \text{ /@ } (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[1]\ C_{j+2}[0])\ \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\};$$

$$\text{lhs} == \text{rhs}$$

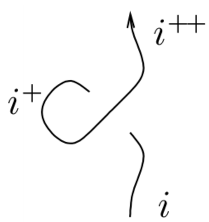
Out[∘]=

$$-\mathbb{i}\ T_1\ T_2\ \mathbb{E}\left[\frac{\epsilon}{2} + p_{1,2+i}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j} + \epsilon\, p_{3,3+j}\,\pi_{3,j}\right]$$

Out[∘]=

True

Invariance Under R1l



In[∘]:=

$$\text{lhs} = \int \mathcal{F}[i]\ \mathcal{L} \text{ /@ } (X_{i+2,i}[1]\ C_{i+1}[1])\ \mathbb{d}\{vs_i, vs_{i^+}, vs_{i+2}\}$$

$$\text{rhs} = \int \mathcal{F}[i]\ \mathcal{L} \text{ /@ } (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ \mathbb{d}\{vs_i, vs_{i^+}, vs_{i+2}\};$$

$$\text{lhs} == \text{rhs}$$

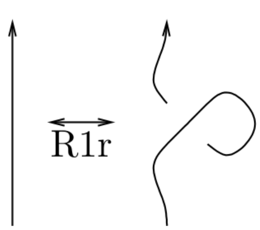
Out[∘]=

$$-\mathbb{i}\ \mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$$

Out[∘]=

True

Invariance Under R1r



In[∘]:=
```
lhs = ∫𝓕[i] ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) ⅆ{vs_i, vs_{i⁺}, vs_{i+2}}
rhs = ∫𝓕[i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{vs_i, vs_{i⁺}, vs_{i+2}};
lhs == rhs
```

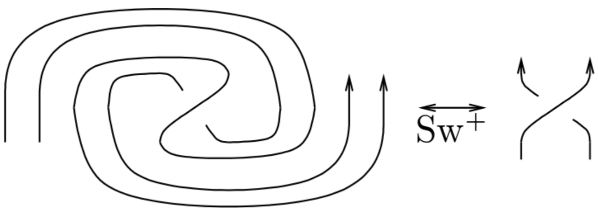
Out[∘]=

$-\mathbb{i}\,\mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$

Out[∘]=

True

Invariance Under Sw



In[∘]:=
```
lhs =
  ∫𝓕[i, j] ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1]) ⅆ{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{i+2}, vs_{j+2}}
rhs = ∫𝓕[i, j] ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0]) ⅆ{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{i+2}, vs_{j+2}};
lhs == rhs
```

Out[∘]=

$$T_1 T_2 \mathbb{E}\Big[\frac{\epsilon}{2} + T_1 p_{1,3+i} \pi_{1,i} + (1 - T_1) p_{1,3+j} \pi_{1,i} + p_{1,3+j} \pi_{1,j} + T_2 p_{2,3+i} \pi_{2,i} - \epsilon T_2 p_{2,3+i} \pi_{2,i} + (1 - T_2) p_{2,3+j} \pi_{2,i} + \epsilon T_1 T_2 p_{1,3+i} p_{2,3+j} \pi_{1,i} \pi_{2,i} + \frac{\epsilon (-1 + T_1) T_2 p_{1,3+j} p_{2,3+j} \pi_{1,i} \pi_{2,i}}{-1 + T_2} + (-1 + T_1) T_2 p_{3,3+j} \pi_{1,i} \pi_{2,i} + p_{2,3+j} \pi_{2,j} + \epsilon p_{2,3+j} \pi_{2,j} - \epsilon T_1 p_{1,3+i} p_{2,3+j} \pi_{1,i} \pi_{2,j} - \frac{\epsilon (-1 + T_1) p_{1,3+j} p_{2,3+j} \pi_{1,i} \pi_{2,j}}{-1 + T_2} + (1 - T_1) p_{3,3+j} \pi_{1,i} \pi_{2,j} + \frac{\epsilon T_2 (-1 + T_1 T_2) p_{1,3+j} p_{2,3+i} \pi_{3,i}}{-1 + T_2} - \frac{\epsilon T_2 (-1 + T_1 T_2) p_{1,3+j} p_{2,3+j} \pi_{3,i}}{-1 + T_2} + T_1 T_2 p_{3,3+i} \pi_{3,i} + \epsilon T_1 T_2 p_{3,3+i} \pi_{3,i} + (1 - T_1 T_2) p_{3,3+j} \pi_{3,i} - \frac{\epsilon T_2 (-1 + T_1 T_2) p_{3,3+j} \pi_{3,i}}{-1 + T_2} - \frac{\epsilon T_1 T_2 (-1 + T_1 T_2) p_{1,3+i} p_{3,3+j} \pi_{1,i} \pi_{3,i}}{-1 + T_2} + \frac{\epsilon (-1 + T_1) T_2 (-1 + T_1 T_2) p_{1,3+j} p_{3,3+j} \pi_{1,i} \pi_{3,i}}{-1 + T_2} - \epsilon T_1 (-1 + T_2) T_2 p_{2,3+j} p_{3,3+i} \pi_{2,i} \pi_{3,i} + \epsilon T_2 (-1 + T_1 T_2) p_{2,3+j} p_{3,3+j} \pi_{2,i} \pi_{3,i} + 2 \epsilon T_1 T_2 p_{2,3+j} p_{3,3+i} \pi_{2,j} \pi_{3,i} + \frac{\epsilon T_2 (-1 + T_1 T_2) p_{2,3+i} p_{3,3+j} \pi_{2,j} \pi_{3,i}}{-1 + T_2} - \frac{\epsilon (-1 + 2 T_2) (-1 + T_1 T_2) p_{2,3+j} p_{3,3+j} \pi_{2,j} \pi_{3,i}}{-1 + T_2} + p_{3,3+j} \pi_{3,j} + \epsilon T_1 p_{1,3+i} p_{3,3+j} \pi_{1,i} \pi_{3,j} + \frac{\epsilon (-1 + T_1) p_{1,3+j} p_{3,3+j} \pi_{1,i} \pi_{3,j}}{-1 + T_2} - \epsilon T_2 p_{2,3+i} p_{3,3+j} \pi_{2,i} \pi_{3,j} - \epsilon p_{2,3+j} p_{3,3+j} \pi_{2,i} \pi_{3,j} \Big]$$

Out[∘]=

True