Pensieve header: Mathematica notebook for Talks: Bonn-2505.

Ancestors in Talks/Geneva-2408, Talks/Beijing-2407 and in Projects/HigherRank.

exec

```
nb2tex$TeXFileName = "IType.tex";
```

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Bonn-2505"];
```

Preliminaries

tex

{\bf\red Implementation} (see IType.nb of \web{ap}).

pdf

In[∘]:=
```
Once[<< KnotTheory`; << Rot.m];
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.
```

pdf

In[∘]:=
```
CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
   vs = Cases[ℰ, (x | p | ξ | π | g)__, ∞] ⋃ {ϵ};
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> Factor[c] (Times @@ vs^ps)] ];
```

tex

\vskip 1mm\rule{\linewidth}{1pt}\vspace{-2mm}

Integration

tex

{\bf\red Integration} using Picard iteration. The \myyellow{core is in yellow} and \mpink{hacks are in pink}.

pdf

In[∘]:=
```
𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B];
```

pdf

In[∘]:=
```
$π = Identity; (* The Wisdom Projection *)
```

pdf

```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, DDZ, FZ, a, b},
   n = Length@vs; L0 = L /. ϵ → 0;
   Q = Table[-D[L0, vs⟦a⟧, vs⟦b⟧] /. Thread[vs → 0] /. (p | x)__ → 0, {a, n}, {b, n}];
   If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
   Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
   FixedPoint[(DZ = Table[∂_v Z, {v, vs}];
      DDZ = Table[∂_u DZ, {u, vs}];
      FZ = Sum[G⟦a, b⟧ (DDZ⟦a, b⟧ + DZ⟦a⟧ DZ⟦b⟧), {a, n}, {b, n}] / 2;
      Z = CF[Z0 + ∫_0^λ $π[FZ] ⅆλ]) &, Z];
   PowerExpand@Factor[ω Δ^(-1/2)] 𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

tex

\parpic[r]{\parbox{0.75in}{
 \includegraphics[width=0.75in]{../../Projects/Gallery/Fourier.jpg}
 \footnotesize Joseph Fourier
}}
\picskip{2}

pdf

In[∘]:=

```
∫𝔼[-μ x^2/2 + ⅈ ξ x] ⅆ{x}
```

Out[∘]=
pdf

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

tex

\needspace{12mm}

pdf

In[∘]:=

```
FofG = ∫𝔼[-μ (x - a)^2/2 + ⅈ ξ x] ⅆ{x}
```

Out[∘]=
pdf

$$\frac{\mathbb{E}\left[\frac{i\,(2\,a\,\mu+i\,\xi)\,\xi}{2\,\mu}\right]}{\sqrt{\mu}}$$

tex

\needspace{12mm}

pdf

In[∘]:= $\int \text{FofG}\ \mathbb{E}[-i\,\xi\,x]\ d\{\xi\}$

Out[∘]=
pdf

$$\mathbb{E}\left[-\frac{1}{2}\,(a-x)^2\,\mu\right]$$

tex

So we’ve tested and nearly proven the Fourier inversion formula!

pdf

In[∘]:= $L = -\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c\end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$

tex

\parpic[r]{\parbox{0.65in}{
\includegraphics[width=0.65in]{../../Projects/Gallery/Fubini.jpg}
\scriptsize Guido Fubini
}}
\picskip{2}

pdf

In[∘]:= $Z12 = \int \mathbb{E}[L]\ d\{x_1, x_2\}$

Out[∘]=
pdf

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\,(-b^2+a\,c)} + \frac{b\,\xi_1\,\xi_2}{b^2-a\,c} + \frac{a\,\xi_2^2}{2\,(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$$

pdf

In[∘]:= $\left\{Z1 = \int \mathbb{E}[L]\ d\{x_1\},\ Z12 == \int Z1\ d\{x_2\}\right\}$

Out[∘]=
pdf

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\,x_2^2}{2\,a} - \frac{b\,x_2\,\xi_1}{a} + \frac{\xi_1^2}{2\,a} + x_2\,\xi_2\right]}{\sqrt{a}},\ \text{True}\right\}$$

pdf

In[∘]:= $\$\pi = \text{Normal}\left[\# + O[\epsilon]^{13}\right]\ \&;\ \int \mathbb{E}\left[-\phi^2/2 + \epsilon\,\phi^3/6\right]\ d\{\phi\}$

Out[∘]=
pdf

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24} + \frac{5\,\epsilon^4}{16} + \frac{1105\,\epsilon^6}{1152} + \frac{565\,\epsilon^8}{128} + \frac{82\,825\,\epsilon^{10}}{3072} + \frac{19\,675\,\epsilon^{12}}{96}\right]$$

tex

\vfill
From \surl{oeis.org/A226260}:
\vskip 1mm
\includegraphics[width=\linewidth]{../Groningen-240530/OEIS.png}

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.

1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 38362676768845045751875, 281365778405032973125, 2824650747089425586152484375, 776632157034116712734375 (list; graph; refs; listen; history; text; internal format)

The Right-Handed Trefoil

tex

\vfill
\rule{\linewidth}{1pt}
{\bf\red The Right-Handed Trefoil.}

pdf

In[∘]:=
```
K = Mirror@Knot[3, 1]; Features[K]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
pdf

Features[7, $C_4[-1]\, X_{1,5}[1]\, X_{3,7}[1]\, X_{6,2}[1]$]

pdf

In[∘]:=

$$\mathcal{L}[X_{i\_,j\_}[s\_]] := T^{s/2}\,\mathbb{E}\Big[x_i\,(p_{i+1}-p_i) + x_j\,(p_{j+1}-p_j) + \left(T^s-1\right) x_i\,(p_{i+1}-p_{j+1}) + (\epsilon\, s/2)\times\left(x_i\,(p_i-p_j)\,\left(\left(T^s-1\right) x_i\, p_j + 2\,(1-x_j\, p_j)\right)-1\right)\Big]$$

$$\mathcal{L}[C_{i\_}[\varphi\_]] := T^{\varphi/2}\,\mathbb{E}\left[x_i\,(p_{i+1}-p_i) + \epsilon\,\varphi\left(\frac{1}{2}-x_i\,p_i\right)\right]$$

```
ℒ[K_] := CF[ℒ /@ Features[K][[2]]]
vs[K_] := Join @@ Table[{p_i, x_i}, {i, Features[K][[1]]}]
```

exec

In[∘]:=
```
nb2tex$PDFWidth *= 1.25;
```

tex

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pdf

In[∘]:= `{vs[K], ℒ[K]}`

Out[∘]=
pdf

$$\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-2\in - p_1 x_1 + \in p_1 x_1 + T p_2 x_1 - \in p_5 x_1 + (1-T)\, p_6 x_1 + \frac{1}{2}\,(-1+T)\in p_1 p_5 x_1^2 + \frac{1}{2}\,(1-T)\in p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 - p_3 x_3 + \in p_3 x_3 + T p_4 x_3 - \in p_7 x_3 + (1-T)\, p_8 x_3 + \frac{1}{2}\,(-1+T)\in p_3 p_7 x_3^2 + \frac{1}{2}\,(1-T)\in p_7^2 x_3^2 - p_4 x_4 + \in p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \in p_1 p_5 x_1 x_5 + \in p_5^2 x_1 x_5 - \in p_2 x_6 + (1-T)\, p_3 x_6 - p_6 x_6 + \in p_6 x_6 + T p_7 x_6 + \in p_2^2 x_2 x_6 - \in p_2 p_6 x_2 x_6 + \frac{1}{2}\,(1-T)\in p_2^2 x_6^2 + \frac{1}{2}\,(-1+T)\in p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \in p_3 p_7 x_3 x_7 + \in p_7^2 x_3 x_7\Big]\}$$

exec

In[∘]:= `nb2tex$PDFWidth /= 1.25;`

tex

\needspace{10mm}

pdf

In[∘]:= $\$\pi = \mathrm{Normal}[\# + O[\in]^2]\ \&;\ \int \mathcal{L}[K]\, d\!l\ \mathrm{vs}[K]$

Out[∘]=
pdf

$$-\frac{i\, T\, \mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^2)\in}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

In[∘]:= $\int (\mathcal{L}[K]\ /.\ x_{i\_} \Rightarrow i\, x_i)\, d\!l\, (\mathrm{vs@K})$

Out[∘]=

$$\frac{T\, \mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^2)\in}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

tex

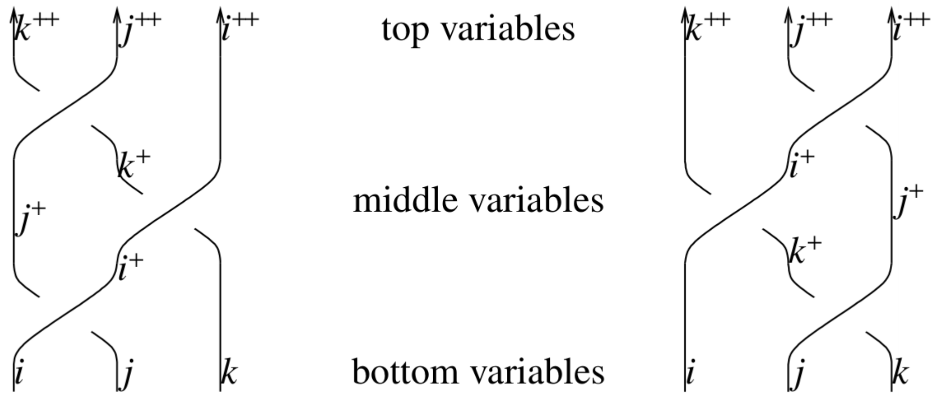
\vskip 1mm
A faster program to compute ρ_1, and more stories about it, are at~\cite{APAI}.

\rule{\linewidth}{1pt}\vspace{2mm}\vskip -2mm
%\newcolumn

Invariance Under Reidemeister 3

tex

{\bf\red Invariance Under Reidemeister 3.}
\vskip 2mm
\def\ip{{i^+}} \def\jp{{j^+}} \def\kp{{k^+}}
\def\ipp{{i^{+\!+}}} \def\jpp{{j^{+\!+}}} \def\kpp{{k^{+\!+}}}
\import{../Beijing-2407/figs}{R3.pdf_t}



pdf

In[∘]:= $\mathbf{lhs} = \int (\mathcal{L} /@ (X_{i,j}[1]\; X_{i+1,k}[1]\; X_{j+1,k+1}[1]))\; d\{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$

$\mathbf{rhs} = \int (\mathcal{L} /@ (X_{j,k}[1]\; X_{i,k+1}[1]\; X_{i+1,j+1}[1]))\; d\{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$

lhs === rhs

Out[∘]=
pdf

False

Invariance Under Reidemeister 3, Take 2

tex

\vskip 1mm\rule{\linewidth}{1pt}
\par{\bf\red Invariance Under Reidemeister 3, Take 2.}

pdf

In[∘]:= $\mathbf{lhs} = \int (\mathcal{L} /@ (X_{i,j}[1]\; X_{i+1,k}[1]\; X_{j+1,k+1}[1]))\; d\{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$

$\mathbf{rhs} = \int (\mathcal{L} /@ (X_{j,k}[1]\; X_{i,k+1}[1]\; X_{i+1,j+1}[1]))\; d\{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$

lhs === rhs

Out[∘]=
pdf

True

pdf

In[∘]:= lhs

Out[∘]=
pdf

Degenerate Q!

tex

%\newcolumn

Invariance Under Reidemeister 3, Take 3

tex

\vskip 1mm\rule{\linewidth}{1pt}
\par{\bf\red Invariance Under Reidemeister 3, Take 3.}

exec

In[◦]:=
```
nb2tex$PDFWidth *= 1.25;
```

pdf

In[◦]:=
```
lhs = ∫(𝔼[𝕚 π_i p_i + 𝕚 π_j p_j + 𝕚 π_k p_k] ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))
      𝕕{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
rhs = ∫(𝔼[𝕚 π_i p_i + 𝕚 π_j p_j + 𝕚 π_k p_k] ℒ /@ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1]))
      𝕕{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}};
lhs == rhs
```

Out[◦]=
pdf

True

tex

\needspace{20mm}

pdf

In[◦]:=
```
lhs
```

Out[◦]=
pdf

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2} + i\,T^2 p_{2+i}\pi_i - i(-1+T)T p_{2+j}\pi_i + i\,T^2\epsilon p_{2+j}\pi_i - i(-1+T)p_{2+k}\pi_i + i\,T\epsilon p_{2+k}\pi_i - \frac{1}{2}(-1+T)T^3\epsilon p_{2+i}p_{2+j}\pi_i^2 + \frac{1}{2}(-1+T)T^3\epsilon p_{2+j}^2\pi_i^2 - \frac{1}{2}(-1+T)T^2\epsilon p_{2+i}p_{2+k}\pi_i^2 + \frac{1}{2}(-1+T)^2 T\epsilon p_{2+j}p_{2+k}\pi_i^2 + \frac{1}{2}(-1+T)T\epsilon p_{2+k}^2\pi_i^2 + i\,T p_{2+j}\pi_j - i\,T\epsilon p_{2+j}\pi_j - i(-1+T)p_{2+k}\pi_j + i(-1+2T)\epsilon p_{2+k}\pi_j + T^3\epsilon p_{2+i}p_{2+j}\pi_i\pi_j - T^3\epsilon p_{2+j}^2\pi_i\pi_j - (-1+T)T^2\epsilon p_{2+i}p_{2+k}\pi_i\pi_j + (-1+T)^2 T\epsilon p_{2+j}p_{2+k}\pi_i\pi_j + (-1+T)T\epsilon p_{2+k}^2\pi_i\pi_j - \frac{1}{2}(-1+T)T\epsilon p_{2+j}p_{2+k}\pi_j^2 + \frac{1}{2}(-1+T)T\epsilon p_{2+k}^2\pi_j^2 + i\,p_{2+k}\pi_k - 2i\,\epsilon p_{2+k}\pi_k + T^2\epsilon p_{2+i}p_{2+k}\pi_i\pi_k - (-1+T)T\epsilon p_{2+j}p_{2+k}\pi_i\pi_k - T\epsilon p_{2+k}^2\pi_i\pi_k + T\epsilon p_{2+j}p_{2+k}\pi_j\pi_k - T\epsilon p_{2+k}^2\pi_j\pi_k\Big]$$

exec

In[◦]:=
```
nb2tex$PDFWidth /= 1.25;
```

tex

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at \web{ap}.

Invariance Under Reidemeister 3, Take 4 (just for fun)

In[∘]:=
```
lhs = ∫(𝔼[ⅈ πi pi + ⅈ πj pj + ⅈ πk pk + ⅈ πi+2 pi+2 + ⅈ πj+2 pj+2 + ⅈ πk+2 pk+2 +
      ⅈ ξi+2 xi+2 + ⅈ ξj+2 xj+2 + ⅈ ξk+2 xk+2] ℒ /@ (Xi,j[1] Xi+1,k[1] Xj+1,k+1[1]))
    ⅆ{pi, pj, pk, xi, xj, xk, pi+1, pj+1, pk+1, xi+1, xj+1, xk+1, pi+2, pj+2, pk+2, xi+2, xj+2, xk+2};
rhs = ∫(𝔼[ⅈ πi pi + ⅈ πj pj + ⅈ πk pk + ⅈ πi+2 pi+2 + ⅈ πj+2 pj+2 + ⅈ πk+2 pk+2 +
      ⅈ ξi+2 xi+2 + ⅈ ξj+2 xj+2 + ⅈ ξk+2 xk+2] ℒ /@ (Xj,k[1] Xi,k+1[1] Xi+1,j+1[1]))
    ⅆ{pi, pj, pk, xi, xj, xk, pi+1, pj+1, pk+1, xi+1, xj+1, xk+1, pi+2, pj+2, pk+2, xi+2, xj+2, xk+2};
lhs == rhs
```

Out[∘]=

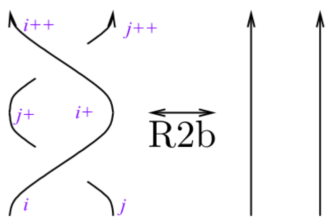
True

In[∘]:=
```
lhs
```

Out[∘]=

Degenerate Q!

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=
```
lhs = ∫𝔼[ⅈ πi pi + ⅈ πj pj] ℒ /@ (Xi,j[1] Xi+1,j+1[-1]) ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1}
rhs =
  ∫𝔼[ⅈ πi pi + ⅈ πj pj] ℒ /@ (Ci[0] Ci+1[0] Cj[0] Cj+1[0]) ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1};
lhs == rhs
```

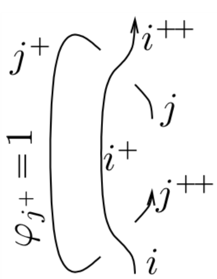
Out[∘]=

$\mathbb{E}[\mathbb{i}\, p_{2+i}\, \pi_i + \mathbb{i}\, p_{2+j}\, \pi_j]$

Out[∘]=

True

Invariance Under R2c



In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[ⅈ π_i p_i + ⅈ π_j p_j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
lhs == rhs
```

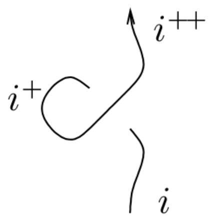
Out[∘]=

$$-\mathbb{i}\sqrt{T}\,\mathbb{E}\left[-\frac{\epsilon}{2}+\mathbb{i}\,p_{2+i}\,\pi_i+\mathbb{i}\,p_{3+j}\,\pi_j-\mathbb{i}\,\epsilon\,p_{3+j}\,\pi_j\right]$$

Out[∘]=

True

Invariance Under R1l



In[∘]:=
```
lhs = ∫𝔼[ⅈ π_i p_i] ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[ⅈ π_i p_i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs
```

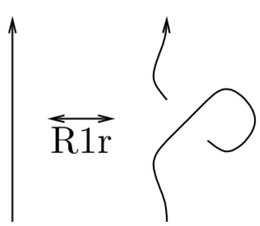
Out[∘]=

$$-\mathbb{i}\,\mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$$

Out[∘]=

True

Invariance Under R1r



In[◦]:=
```
lhs = ∫𝔼[𝕚 πi pi] ℒ /@ (Xi,i+2[1] Ci+1[-1]) ⅆ{xi, pi, xi+1, pi+1, xi+2, pi+2}
rhs = ∫𝔼[𝕚 πi pi] ℒ /@ (Ci[0] Ci+1[0] Ci+2[0]) ⅆ{xi, pi, xi+1, pi+1, xi+2, pi+2};
lhs == rhs
```

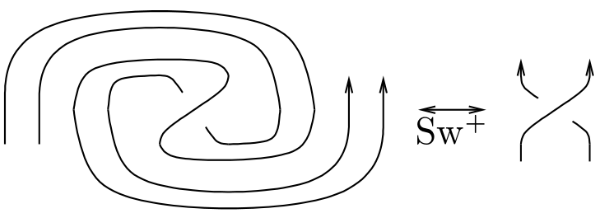
Out[◦]=

$-\mathbb{i}\,\mathbb{E}[\mathbb{i}\,p_{3+i}\,\pi_i]$

Out[◦]=

True

Invariance Under Sw



In[◦]:=
```
lhs = ∫𝔼[𝕚 πi pi + 𝕚 πj pj] ℒ /@ (Xi+1,j+1[1] Ci[-1] Cj[-1] Ci+2[1] Cj+2[1])
    ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1, xi+2, pi+2, xj+2, pj+2}
rhs = ∫𝔼[𝕚 πi pi + 𝕚 πj pj] ℒ /@ (Xi+1,j+1[1] Ci[0] Cj[0] Ci+2[0] Cj+2[0])
    ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1, xi+2, pi+2, xj+2, pj+2};
lhs == rhs
```

Out[◦]=

$$\sqrt{T}\,\mathbb{E}\Big[-\frac{\epsilon}{2}+\mathbb{i}\,T\,p_{3+i}\,\pi_i-\mathbb{i}\,(-1+T)\,p_{3+j}\,\pi_i+\mathbb{i}\,T\,\epsilon\,p_{3+j}\,\pi_i-\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i^2+\frac{1}{2}\,(-1+T)\,T\,\epsilon\,p_{3+j}^2\,\pi_i^2+\mathbb{i}\,p_{3+j}\,\pi_j-\mathbb{i}\,\epsilon\,p_{3+j}\,\pi_j+T\,\epsilon\,p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j-T\,\epsilon\,p_{3+j}^2\,\pi_i\,\pi_j\Big]$$

Out[◦]=

True