Pensieve header: The rank 2 mod ϵ^2 invariant using integration techniques; continues UC4A2.nb and Theta.nb at pensieve://Projects/HigherRank/.

Initialization

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Beijing-2407"];
Once[<< IType.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Talks/Beijing-2407 to compute rotation numbers.
```

pdf

In[∘]:=
```
T3 = T1 T2;  i_+ := i + 1;
$π = (CF@Normal[# + O[ϵ]^2] /. {π_is__ ⧴ B^-1 π_is, x_is__ ⧴ B^-1 x_is, p_is__ ⧴ B p_is} /.
      ϵ B^b_ /; b < 0 → 0 /. B → 1) &;
```

The Lagrangian

tex

\needspace{30mm}
{\bf\red The Lagrangian.}

exec

```
nb2tex$PDFWidth *= 1.25;
```

pdf

```
ℒ[X_i_,j_[1]] := T3 𝔼[Plus[
    Sum_{ν=1}^{3} (x_νi (p_νi+ - p_νi) + x_νj (p_νj+ - p_νj) + (T_ν - 1) x_νi (p_νi+ - p_νj+)),
    p_3j x_2i (T1 x_1i - x_1j),
    ϵ (T3 - 1) p_1j x_3i (p_2j - p_2i),
    ϵ (1 / 2 - p_3i x_3i - T3 p_1j p_2j x_1i x_2i + p_2j p_3i x_2j x_3i - T2 p_2j p_3j x_2i x_3j +
       (T3 - 1) p_3j x_3i (T1 p_1j x_1i + T2 p_2j x_2i) + (p_1j x_1j (p_2i x_2i - p_3i x_3i) +
          T1 p_1i x_1i (p_3j x_3j - p_2j x_2j) + (T3 - 1) p_1j x_1j (p_2j x_2i - T1 p_3j x_3i)) / (T1 - 1)) ]]
```

pdf

```
ℒ[X_{i_,j_}[-1]] := T₃^{-1} 𝔼[Plus[
    Σ_{ν=1}^{3} (x_{νi} (p_{νi⁺} - p_{νi}) + x_{νj} (p_{νj⁺} - p_{νj}) + (T_ν^{-1} - 1) x_{νi} (p_{νi⁺} - p_{νj⁺})),
    T₂^{-1} (p_{3j} x_{1j} x_{2i} - T₁^{-1} p_{3j} x_{1i} x_{2i}),
    ∈ T₁^{-1} ((T₃ - 1) p_{1j} p_{2i} x_{3i} - (T₃ - 1) p_{1j} p_{2j} x_{3i}),
    ∈ (-1 / 2 + p_{3i} x_{3i} - T₁^{-1} p_{1j} p_{2i} x_{1i} x_{2i} - (1 - T₁^{-1} - T₂^{-1}) p_{1j} p_{2j} x_{1i} x_{2i} - p_{1j} p_{2j} x_{1j} x_{2i} -
       p_{1j} p_{2j} x_{1i} x_{2j} + T₁^{-1} p_{1j} p_{3i} x_{1i} x_{3i} - (1 - T₂^{-1}) p_{2j} p_{3i} x_{2i} x_{3i} - p_{2j} p_{3i} x_{2j} x_{3i} + p_{1j} p_{3j} x_{1i} x_{3j} +
       p_{2j} p_{3j} x_{2i} x_{3j} + (1 - T₃^{-1}) p_{3j} x_{3i} (p_{2j} x_{2j} + p_{1j} x_{1i} - p_{2i} x_{2i} + (2 - T₂^{-1}) p_{2j} x_{2i}) +
       (T₁ (1 - T₂^{-1}) p_{1i} p_{2j} x_{1i} x_{2i} - p_{1j} p_{2i} x_{1j} x_{2i} + T₁ p_{1i} p_{2j} x_{1i} x_{2j} -
          T₂^{-1} (T₃ - 1) p_{1i} p_{3j} x_{1i} x_{3i} + p_{1j} p_{3i} x_{1j} x_{3i} - T₁ p_{1i} p_{3j} x_{1i} x_{3j}) / (T₁ - 1))]]
```

pdf

In[∘]:=
```
ℒ[C_{i_}[φ_]] := T₃^φ 𝔼[Σ_{ν=1}^{3} x_{νi} (p_{νi⁺} - p_{νi}) + ∈ φ (p_{3,i} x_{3,i} - 1 / 2)]
```

pdf

In[∘]:=
```
vs_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
ℱ[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K][[2]]];
vs[K_] := Union @@ Table[{vs_i}, {i, Features[K][[1]]}]
```

exec

```
nb2tex$PDFWidth /= 1.25;
```

Reidemeister 3

tex

{\bf\red Reidemeister 3.}

pdf

In[∘]:=
```
Short[lhs = ∫ℱ[i, j, k] ℒ /@ (X_{i,j}[1] X_{i⁺,k}[1] X_{j⁺,k⁺}[1]) 𝕕{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}}]
```

Out[∘]//Short=

pdf

$$T_1^3\, T_2^3\, \mathbb{E}\left[\frac{3\in}{2} + \ll 138 \gg\right]$$

pdf

In[∘]:=
```
rhs = ∫ℱ[i, j, k] ℒ /@ (X_{j,k}[1] X_{i,k⁺}[1] X_{i⁺,j⁺}[1]) 𝕕{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}}; lhs == rhs
```

Out[∘]=

pdf

True

The Trefoil

tex

\needspace{25mm}

{\bf\red The Trefoil.}

pdf

In[ ]:=
```
K = Knot[3, 1]; ∫ℒ[K] ⅆ vs[K]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[ ]=
pdf

$$-\frac{\mathbb{i}\, T_1^2 T_2^2\, \mathbb{E}\left[-\frac{\in\left(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}$$

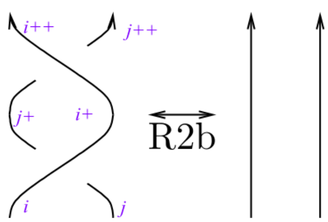
In[ ]:=

$$-\frac{\mathbb{i}\, T_1^2 T_2^2\, \mathbb{E}\left[-\frac{\in\left(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}$$

Out[ ]=

$$-\frac{\mathbb{i}\, T_1^2 T_2^2\, \mathbb{E}\left[-\frac{\in\left(1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4\right)}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}$$

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[ ]:=
```
lhs = ∫𝓕[i, j] ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}}
rhs = ∫𝓕[i, j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}};
lhs == rhs
```

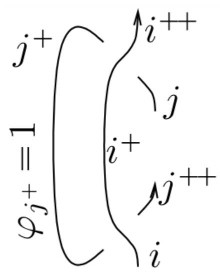
Out[ ]=

$$\mathbb{E}\left[p_{1,2+i}\,\pi_{1,i}+p_{1,2+j}\,\pi_{1,j}+p_{2,2+i}\,\pi_{2,i}+p_{2,2+j}\,\pi_{2,j}+p_{3,2+i}\,\pi_{3,i}+p_{3,2+j}\,\pi_{3,j}\right]$$

Out[ ]=

```
True
```

Invariance Under R2c



In[∘]:=
```
lhs = ∫ℱ[i, j] ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{j+2}}
rhs = ∫ℱ[i, j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0]) ⅆ{vs_i, vs_j, vs_{i+}, vs_{j+}, vs_{j+2}};
lhs == rhs
```

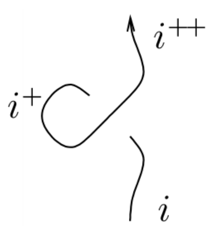
Out[∘]=

$$-\mathbb{i}\, T_1 T_2\, \mathbb{E}\left[\frac{\epsilon}{2} + p_{1,2+i}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j} + \epsilon\, p_{3,3+j}\,\pi_{3,j}\right]$$

Out[∘]=

True

Invariance Under R1l



In[∘]:=
```
lhs = ∫ℱ[i] ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{vs_i, vs_{i+}, vs_{i+2}}
rhs = ∫ℱ[i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{vs_i, vs_{i+}, vs_{i+2}};
lhs == rhs
```

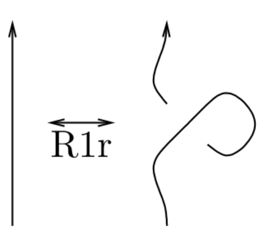
Out[∘]=

$$-\mathbb{i}\, \mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$$

Out[∘]=

True

Invariance Under R1r



In[∘]:=

$$\text{lhs} = \int \mathcal{F}[i]\ \mathcal{L} \,/@\, (X_{i,i+2}[1]\ C_{i+1}[-1])\ d\{vs_i, vs_{i^+}, vs_{i+2}\}$$

$$\text{rhs} = \int \mathcal{F}[i]\ \mathcal{L} \,/@\, (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{vs_i, vs_{i^+}, vs_{i+2}\};$$

$$\text{lhs} == \text{rhs}$$

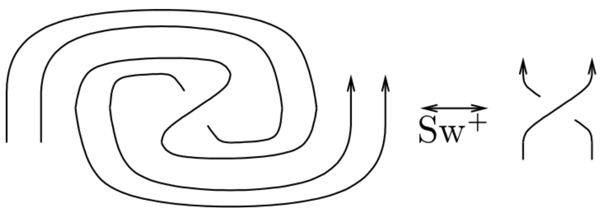
Out[∘]=

$$-\mathbb{i}\ \mathbb{E}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i}]$$

Out[∘]=

True

Invariance Under Sw



In[∘]:=

$$\text{lhs} = \int \mathcal{F}[i, j]\ \mathcal{L} \,/@\, (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}\}$$

$$\text{rhs} = \int \mathcal{F}[i, j]\ \mathcal{L} \,/@\, (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])\ d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{i+2}, vs_{j+2}\};$$

$$\text{lhs} == \text{rhs}$$

Out[∘]=

$$T_1 T_2\, \mathbb{E}\Big[\frac{\in}{2} + T_1 p_{1,3+i}\pi_{1,i} + (1-T_1)\, p_{1,3+j}\pi_{1,i} + p_{1,3+j}\pi_{1,j} + T_2 p_{2,3+i}\pi_{2,i} +$$
$$\frac{\in T_2 p_{2,3+i}\pi_{2,i}}{-1+T_1} + (1-T_2)\, p_{2,3+j}\pi_{2,i} - \in T_1 T_2 p_{1,3+j} p_{2,3+j}\pi_{1,i}\pi_{2,i} + T_1 p_{3,3+j}\pi_{1,i}\pi_{2,i} +$$
$$\frac{\in T_2 p_{1,3+j} p_{2,3+i}\pi_{1,j}\pi_{2,i}}{-1+T_1} + \in T_2 p_{1,3+j} p_{2,3+j}\pi_{1,j}\pi_{2,i} - p_{3,3+j}\pi_{1,j}\pi_{2,i} + p_{2,3+j}\pi_{2,j} - \frac{\in p_{2,3+j}\pi_{2,j}}{-1+T_1} -$$
$$\frac{\in T_1^2 p_{1,3+i} p_{2,3+j}\pi_{1,i}\pi_{2,j}}{-1+T_1} + \in T_1 p_{1,3+j} p_{2,3+j}\pi_{1,i}\pi_{2,j} - \in T_2 (-1+T_1 T_2)\, p_{1,3+j} p_{2,3+i}\pi_{3,i} +$$
$$\in T_2 (-1+T_1 T_2)\, p_{1,3+j} p_{2,3+j}\pi_{3,i} + T_1 T_2 p_{3,3+i}\pi_{3,i} - \frac{\in T_1 T_2 p_{3,3+i}\pi_{3,i}}{-1+T_1} +$$
$$(1-T_1 T_2)\, p_{3,3+j}\pi_{3,i} - \frac{\in T_1 T_2 p_{1,3+j} p_{3,3+i}\pi_{1,j}\pi_{3,i}}{-1+T_1} + \in T_2 (-1+T_1 T_2)\, p_{2,3+i} p_{3,3+j}\pi_{2,i}\pi_{3,i} +$$
$$\in T_1 T_2 p_{2,3+j} p_{3,3+i}\pi_{2,j}\pi_{3,i} + \in (1-T_1 T_2)\, p_{2,3+j} p_{3,3+j}\pi_{2,j}\pi_{3,i} + p_{3,3+j}\pi_{3,j} + \frac{\in T_1 p_{3,3+j}\pi_{3,j}}{-1+T_1} +$$
$$\frac{\in T_1^2 p_{1,3+i} p_{3,3+j}\pi_{1,i}\pi_{3,j}}{-1+T_1} - \in T_1 p_{1,3+j} p_{3,3+j}\pi_{1,i}\pi_{3,j} - \in T_2 p_{2,3+j} p_{3,3+j}\pi_{2,i}\pi_{3,j}\Big]$$

Out[∘]=

True