Pensieve header: Proof of invariance of ρ_2 using integration techniques; continues pensieve://Talks/Groningen-240530.

Initialization

In[◦]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Beijing-2407"];
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/AP/Talks/Beijing-2407 to compute rotation numbers.
```

Initialization

In[◦]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CCF[ℰ_] := Factor[ℰ];
CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs = Cases[ℰ, (x | p | π)__, ∞] ⋃ {x, p, ϵ}, ps, c},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :→ CCF[c] (Times @@ vs^ps)]];
```

The Basic Feynman Ring

In[◦]:=
```
𝒮 = {x, x_, y, z};
q_{x_,y_}[f_] := (∂_{x,y} f) /. Thread[𝒮 → 0];
θ_{x_,y_} := x y;
f_ ≡ 0 := f === 0;
Ev_{vs_List→0}[f_] := CF[f /. Thread[vs → 0]]
```

The ϵSeries Feynman Ring

```
In[•]:= 𝒮 = {x, y, z, ϕ, x_, p_, x̲_, p̲_};
q_{x_,y_}[ser_ϵSeries] := (∂_{x,y} ser⟦1⟧) /. Thread[𝒮 → 0];
θ_{x_,y_} := x y;
ϵSeries /: D[ser_ϵSeries, vs___] := D[#, vs] & /@ ser;
ϵSeries /: Plus[ss___ϵSeries] /; Length[{ss}] > 1 := Module[{l = Min[Length /@ {ss}]},
   ϵSeries @@ Total[Take[List @@ #, l] & /@ {ss}]]
ϵSeries /: t_ + ser_ϵSeries := MapAt[(# + t) &, ser, 1];
ϵSeries /: s1_ϵSeries * s2_ϵSeries := ϵSeries @@ Table[
    Sum[s1⟦ii + 1⟧ s2⟦kk - ii + 1⟧, {ii, 0, kk}], {kk, 0, Min[Length@s1, Length@s2] - 1}];
ϵSeries /: c_ * ser_ϵSeries := (c #) & /@ ser;
ser_ϵSeries ≡ 0 := And @@ ((# === 0) & /@ ser);
ϵSeries /: Integrate[ser_ϵSeries, pars__] := ϵSeries @@ (Integrate[#, pars] & /@ ser);
ϵSeries /: Ev_{vs_List→0}[ser_ϵSeries] := ser /. Thread[vs → 0];
CF[ser_ϵSeries] := CF /@ ser;
```

Integration

Using Picard Iteration!

```
In[•]:= 𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B]
```

```
In[•]:= 𝔼[sd_SeriesData] /; (List @@ sd)⟦{1, 2, 4, 6}⟧ === {ϵ, 0, 0, 1} :=
  𝔼[ϵSeries @@ PadRight[sd⟦3⟧, sd⟦5⟧, 0]]
```

Following a program in Projects/FullDoPeGDO/Engine.nb, we write $Z_\lambda = \sum Z[m]\,\lambda^m$.

In[∘]:=
```
Unprotect[Integrate];
Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, Q, Δ, G, a, b, m, m1, $m}, Clear[Z];
  n = Length@vs;
  Q = Table[q_{vs⟦a⟧,vs⟦b⟧}[L], {a, n}, {b, n}];
  If[(Δ = CF@Det[-Q]) == 0, Message[Integrate::sing]; Return[]];
  G = CF[-Inverse[Q] / 2];
  Z[] = Z[0] = CF[L - Sum[Q⟦a, b⟧ θ_{vs⟦a⟧,vs⟦b⟧}, {a, n}, {b, n}] / 2];
  Z[m_, a_] := Z[m, a] = CF@D[Z[m], vs⟦a⟧];
  Z[m_, a_, b_] /; a ≤ b := Z[m, a, b] = CF@D[Z[m, a], vs⟦b⟧];
  Z[m_, a_, b_] /; a > b := Z[m, b, a];
  For[$m = m = 0, m ≤ 2 $m, ++m,
   Z[m + 1] = CF@Sum[Sum[If[G⟦a, b⟧ === 0, 0,
        G⟦a, b⟧/(m + 1) (Z[m, a, b] + Sum[Z[m1, a] Z[m - m1, b], {m1, 0, m}])],
       {a, n}], {b, n}];
   If[! (Z[m + 1] ≡ 0), $m = m + 1; Z[] += Z[m + 1]];
  ];
  PowerExpand@Factor[ω Δ^(-1/2)] 𝔼[CF[Ev_{vs→0}[Z[]]]]
 ];
Protect[Integrate];
```

In[∘]:= $\int \mathbb{E}\left[-\mu x^2/2 + i\,\xi\,x\right] \mathrm{d}\{x\}$

Out[∘]=

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

In[∘]:= $L = -\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$

$Z12 = \int \mathbb{E}[L]\,\mathrm{d}\{x_1, x_2\}$

Out[∘]=

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2 - 2\,b\,\xi_1\,\xi_2 + a\,\xi_2^2}{2\left(-b^2 + a\,c\right)}\right]}{\sqrt{-b^2 + a\,c}}$$

In[∘]:= $\left\{Z1 = \int \mathbb{E}[L]\,\mathrm{d}\{x_1\},\ Z12 == \int Z1\,\mathrm{d}\{x_2\}\right\}$

Out[∘]=

$$\left\{\frac{\mathbb{E}\left[-\frac{\left(-b^2 + a\,c\right) x_2^2}{2\,a} + \frac{\xi_1^2}{2\,a} + \frac{x_2\left(-b\,\xi_1 + a\,\xi_2\right)}{a}\right]}{\sqrt{a}},\ \text{True}\right\}$$

Integration of ϵSeries

In[∘]:=
```
∫𝔼[-x^2/2 + ϵ x^3/6 + O[ϵ]^13] ⅆ{x}
```

Out[∘]=

$$\mathbb{E}\left[\epsilon\text{Series}\left[0,\ 0,\ \frac{5}{24},\ 0,\ \frac{5}{16},\ 0,\ \frac{1105}{1152},\ 0,\ \frac{565}{128},\ 0,\ \frac{82\,825}{3072},\ 0,\ \frac{19\,675}{96}\right]\right]$$

The ρ_2 Integrand

Adopted from pensieve://Talks//Oaxaca-2210/Rho.nb.

In[∘]:=
```
𝒮 = {x_, p_};
q[s_, i_, j_] := x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1});
r1[s_, i_, j_] := s/2 (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - p_j x_j)) - 1);
r2[1, i_, j_] :=
  (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
     2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j -
     6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12;
r2[-1, i_, j_] :=
  (-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 - 4 (-1 + T) T p_i^2 p_j x_i^3 +
     2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j -
     18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j -
     6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2);
γ1[φ_, k_] := φ (1 / 2 - x_k p_k);
γ2[φ_, k_] := -φ^2 p_k x_k / 2;
ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[q[s, i, j] + ϵ r1[s, i, j] + ϵ^2 r2[s, i, j] + O[ϵ]^3];
ℒ[C_k_[φ_]] := T^{φ/2} 𝔼[-x_k (p_k - p_{k+1}) + ϵ γ1[φ, k] + ϵ^2 γ2[φ, k] + O[ϵ]^3];
ℒ[K_] := (2 π)^(-Features[K][[1]]) CF[ℒ /@ Features[K][[2]]];
vs[K_] := Union @@ Table[{p_i, x_i}, {i, Features[K][[1]]}]
```

pdf

In[∘]:=
```
ϵ^2 r2[1, i, j]
```

Out[∘]=
pdf

$$\frac{1}{12}\epsilon^2\left(-6\,p_i\,x_i + 6\,p_j\,x_i - 3\,(-1+3\,T)\,p_i\,p_j\,x_i^2 + 3\,(-1+3\,T)\,p_j^2\,x_i^2 + 4\,(-1+T)\,p_i^2\,p_j\,x_i^3 - 2\,(-1+T)\,(5+T)\,p_i\,p_j^2\,x_i^3 + 2\,(-1+T)\,(3+T)\,p_j^3\,x_i^3 + 18\,p_i\,p_j\,x_i\,x_j - 18\,p_j^2\,x_i\,x_j - 6\,p_i^2\,p_j\,x_i^2\,x_j + 6\,(2+T)\,p_i\,p_j^2\,x_i^2\,x_j - 6\,(1+T)\,p_j^3\,x_i^2\,x_j - 6\,p_i\,p_j^2\,x_i\,x_j^2 + 6\,p_j^3\,x_i\,x_j^2\right)$$

pdf

In[∘]:= $\epsilon^2\, r_2[-1, i, j]$

Out[∘]=
pdf

$$\frac{1}{12\,T^2}\,\epsilon^2\left(-6\,T^2\,p_i\,x_i+6\,T^2\,p_j\,x_i+3\,(-3+T)\,T\,p_i\,p_j\,x_i^2-3\,(-3+T)\,T\,p_j^2\,x_i^2-4\,(-1+T)\,T\,p_i^2\,p_j\,x_i^3+2\,(-1+T)\,(1+5\,T)\,p_i\,p_j^2\,x_i^3-2\,(-1+T)\,(1+3\,T)\,p_j^3\,x_i^3+18\,T^2\,p_i\,p_j\,x_i\,x_j-18\,T^2\,p_j^2\,x_i\,x_j-6\,T^2\,p_i^2\,p_j\,x_i^2\,x_j+6\,T\,(1+2\,T)\,p_i\,p_j^2\,x_i^2\,x_j-6\,T\,(1+T)\,p_j^3\,x_i^2\,x_j-6\,T^2\,p_i\,p_j^2\,x_i\,x_j^2+6\,T^2\,p_j^3\,x_i\,x_j^2\right)$$

pdf

In[∘]:= $\epsilon^2\,\gamma_2[\varphi, i]$

Out[∘]=
pdf

$$-\frac{1}{2}\,\epsilon^2\,\varphi^2\,p_i\,x_i$$

In[∘]:= Features[Knot[3, 1]]

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

Features[7, $C_4[-1]\,X_{2,6}[-1]\,X_{5,1}[-1]\,X_{7,3}[-1]$]

In[∘]:= **ℒ[Knot[3, 1]]**

Out[∘]=

$$\frac{1}{128\,\pi^7\,T^2}$$

$$\mathbb{E}\Big[\epsilon\text{Series}\Big[-p_1 x_1 + p_2 x_1 - p_2 x_2 + \frac{p_3 x_2}{T} + \frac{(-1+T)\,p_7 x_2}{T} - p_3 x_3 + p_4 x_3 - p_4 x_4 + p_5 x_4 + \frac{(-1+T)\,p_2 x_5}{T} -$$

$$p_5 x_5 + \frac{p_6 x_5}{T} - p_6 x_6 + p_7 x_6 + \frac{(-1+T)\,p_4 x_7}{T} - p_7 x_7 + \frac{p_8 x_7}{T},\; 1 - p_2 x_2 + p_6 x_2 + \frac{(-1+T)\,p_2 p_6 x_2^2}{2\,T} -$$

$$\frac{(-1+T)\,p_6^2 x_2^2}{2\,T} + p_4 x_4 + p_1 x_5 - p_5 x_5 - p_1^2 x_1 x_5 + p_1 p_5 x_1 x_5 - \frac{(-1+T)\,p_1^2 x_5^2}{2\,T} + \frac{(-1+T)\,p_1 p_5 x_5^2}{2\,T} +$$

$$p_2 p_6 x_2 x_6 - p_6^2 x_2 x_6 + p_3 x_7 - p_7 x_7 - p_3^2 x_3 x_7 + p_3 p_7 x_3 x_7 - \frac{(-1+T)\,p_3^2 x_7^2}{2\,T} + \frac{(-1+T)\,p_3 p_7 x_7^2}{2\,T},$$

$$-\frac{1}{2} p_2 x_2 + \frac{p_6 x_2}{2} + \frac{(-3+T)\,p_2 p_6 x_2^2}{4\,T} - \frac{(-3+T)\,p_6^2 x_2^2}{4\,T} - \frac{(-1+T)\,p_2^2 p_6 x_2^3}{3\,T} +$$

$$\frac{(-1+T)\,(1+5\,T)\,p_2 p_6^2 x_2^3}{6\,T^2} - \frac{(-1+T)\,(1+3\,T)\,p_6^3 x_2^3}{6\,T^2} - \frac{p_4 x_4}{2} + \frac{p_1 x_5}{2} - \frac{p_5 x_5}{2} - \frac{3}{2} p_1^2 x_1 x_5 +$$

$$\frac{3}{2} p_1 p_5 x_1 x_5 + \frac{1}{2} p_1^3 x_1^2 x_5 - \frac{1}{2} p_1^2 p_5 x_1^2 x_5 - \frac{(-3+T)\,p_1^2 x_5^2}{4\,T} + \frac{(-3+T)\,p_1 p_5 x_5^2}{4\,T} - \frac{(1+T)\,p_1^3 x_1 x_5^2}{2\,T} +$$

$$\frac{(1+2\,T)\,p_1^2 p_5 x_1 x_5^2}{2\,T} - \frac{1}{2} p_1 p_5^2 x_1 x_5^2 - \frac{(-1+T)\,(1+3\,T)\,p_1^3 x_5^3}{6\,T^2} + \frac{(-1+T)\,(1+5\,T)\,p_1^2 p_5 x_5^3}{6\,T^2} -$$

$$\frac{(-1+T)\,p_1 p_5^2 x_5^3}{3\,T} + \frac{3}{2} p_2 p_6 x_2 x_6 - \frac{3}{2} p_6^2 x_2 x_6 - \frac{1}{2} p_2^2 p_6 x_2^2 x_6 + \frac{(1+2\,T)\,p_2 p_6^2 x_2^2 x_6}{2\,T} -$$

$$\frac{(1+T)\,p_6^3 x_2^2 x_6}{2\,T} - \frac{1}{2} p_2 p_6^2 x_2 x_6^2 + \frac{1}{2} p_6^3 x_2 x_6^2 + \frac{p_3 x_7}{2} - \frac{p_7 x_7}{2} - \frac{3}{2} p_3^2 x_3 x_7 + \frac{3}{2} p_3 p_7 x_3 x_7 + \frac{1}{2} p_3^3 x_3^2 x_7 -$$

$$\frac{1}{2} p_3^2 p_7 x_3^2 x_7 - \frac{(-3+T)\,p_3^2 x_7^2}{4\,T} + \frac{(-3+T)\,p_3 p_7 x_7^2}{4\,T} - \frac{(1+T)\,p_3^3 x_3 x_7^2}{2\,T} + \frac{(1+2\,T)\,p_3^2 p_7 x_3 x_7^2}{2\,T} -$$

$$\frac{1}{2} p_3 p_7^2 x_3 x_7^2 - \frac{(-1+T)\,(1+3\,T)\,p_3^3 x_7^3}{6\,T^2} + \frac{(-1+T)\,(1+5\,T)\,p_3^2 p_7 x_7^3}{6\,T^2} - \frac{(-1+T)\,p_3 p_7^2 x_7^3}{3\,T}\Big]\Big]$$

In[∘]:= **vs[Knot[3, 1]]**

Out[∘]=

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

In[∘]:= **K = Knot[3, 1]; ∫ℒ[K] ⅆ(vs@K)**

Out[∘]=

$$-\frac{i\,T\,\mathbb{E}\Big[\epsilon\text{Series}\Big[0,\; \frac{(-1+T)^2\,(1+T^2)}{(1-T+T^2)^2},\; -\frac{T^2\,(1-4\,T^2+T^4)}{2\,(1-T+T^2)^4}\Big]\Big]}{128\,\pi^7\,(1-T+T^2)}$$

Invariance Under Reidemeister 3

In[•]:=

$$\text{lhs} = \int \left(\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j + \pi_k\, p_k]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1])\right) \mathbb{d}\{p_i, p_j, p_k, p_{i+1}, p_{j+1}, p_{k+1}, x_i, x_j, x_k, x_{i+1}, x_{j+1}, x_{k+1}\}$$

$$\text{rhs} = \int \left(\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j + \pi_k\, p_k]\ \mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1])\right) \mathbb{d}\{p_i, p_j, p_k, p_{i+1}, p_{j+1}, p_{k+1}, x_i, x_j, x_k, x_{i+1}, x_{j+1}, x_{k+1}\};$$

$$\text{lhs} == \text{rhs}$$

Out[•]=

$$T^{3/2}\, \mathbb{E}\Big[\in\text{Series}\Big[T^2 p_{2+i} \pi_i - (-1+T)\, T\, p_{2+j} \pi_i + (1-T)\, p_{2+k} \pi_i + T\, p_{2+j} \pi_j + (1-T)\, p_{2+k} \pi_j + p_{2+k} \pi_k,$$

$$-\frac{3}{2} + T^2 p_{2+j} \pi_i + T\, p_{2+k} \pi_i + \frac{1}{2} (-1+T)\, T^3 p_{2+i} p_{2+j} \pi_i^2 - \frac{1}{2} (-1+T)\, T^3 p_{2+j}^2 \pi_i^2 +$$

$$\frac{1}{2} (-1+T)\, T^2 p_{2+i} p_{2+k} \pi_i^2 - \frac{1}{2} (-1+T)^2\, T\, p_{2+j} p_{2+k} \pi_i^2 - \frac{1}{2} (-1+T)\, T\, p_{2+k}^2 \pi_i^2 - T\, p_{2+j} \pi_j +$$

$$(-1+2\,T)\, p_{2+k} \pi_j - T^3 p_{2+i} p_{2+j} \pi_i \pi_j + T^3 p_{2+j}^2 \pi_i \pi_j + (-1+T)\, T^2 p_{2+i} p_{2+k} \pi_i \pi_j -$$

$$(-1+T)^2\, T\, p_{2+j} p_{2+k} \pi_i \pi_j - (-1+T)\, T\, p_{2+k}^2 \pi_i \pi_j + \frac{1}{2} (-1+T)\, T\, p_{2+j} p_{2+k} \pi_j^2 - \frac{1}{2} (-1+T)\, T\, p_{2+k}^2 \pi_j^2 -$$

$$2\, p_{2+k} \pi_k - T^2 p_{2+i} p_{2+k} \pi_i \pi_k + (-1+T)\, T\, p_{2+j} p_{2+k} \pi_i \pi_k + T\, p_{2+k}^2 \pi_i \pi_k - T\, p_{2+j} p_{2+k} \pi_j \pi_k + T\, p_{2+k}^2 \pi_j \pi_k,$$

$$-\frac{1}{2} T^2 p_{2+j} \pi_i - \frac{1}{2} T\, p_{2+k} \pi_i - \frac{1}{4} T^3 (-1+3\,T)\, p_{2+i} p_{2+j} \pi_i^2 + \frac{1}{4} T^3 (-3+5\,T)\, p_{2+j}^2 \pi_i^2 -$$

$$\frac{1}{4} T^2 (-1+3\,T)\, p_{2+i} p_{2+k} \pi_i^2 + \frac{1}{4} (-1+T)\, T\, (-1+5\,T)\, p_{2+j} p_{2+k} \pi_i^2 +$$

$$\frac{1}{4} T\, (-3+5\,T)\, p_{2+k}^2 \pi_i^2 - \frac{1}{6} (-1+T)\, T^5 p_{2+i}^2 p_{2+j} \pi_i^3 + \frac{1}{6} (-1+T)\, T^4 (-1+4\,T)\, p_{2+i} p_{2+j}^2 \pi_i^3 -$$

$$\frac{1}{6} (-1+T)\, T^4 (-1+3\,T)\, p_{2+j}^3 \pi_i^3 - \frac{1}{6} (-1+T)\, T^4 p_{2+i}^2 p_{2+k} \pi_i^3 + \frac{5}{6} (-1+T)^2\, T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i^3 -$$

$$\frac{1}{6} (-1+T)^2\, T^2 (-1+4\,T)\, p_{2+j}^2 p_{2+k} \pi_i^3 + \frac{1}{6} (-1+T)\, T^2 (-1+4\,T)\, p_{2+i} p_{2+k}^2 \pi_i^3 -$$

$$\frac{1}{6} (-1+T)^2\, T\, (-1+4\,T)\, p_{2+j} p_{2+k}^2 \pi_i^3 - \frac{1}{6} (-1+T)\, T\, (-1+3\,T)\, p_{2+k}^3 \pi_i^3 + \frac{1}{2} T\, p_{2+j} \pi_j +$$

$$\frac{1}{2} (1-4\,T)\, p_{2+k} \pi_j + \frac{3}{2} T^3 p_{2+i} p_{2+j} \pi_i \pi_j - \frac{5}{2} T^3 p_{2+j}^2 \pi_i \pi_j - \frac{1}{2} T^2 (-3+5\,T)\, p_{2+i} p_{2+k} \pi_i \pi_j +$$

$$\frac{1}{2} (-1+T)\, T\, (-3+7\,T)\, p_{2+j} p_{2+k} \pi_i \pi_j + \frac{1}{2} T\, (-5+7\,T)\, p_{2+k}^2 \pi_i \pi_j + \frac{1}{2} T^5 p_{2+i}^2 p_{2+j} \pi_i^2 \pi_j -$$

$$\frac{1}{2} T^4 (-1+4\,T)\, p_{2+i} p_{2+j}^2 \pi_i^2 \pi_j + \frac{1}{2} T^4 (-1+3\,T)\, p_{2+j}^3 \pi_i^2 \pi_j - \frac{1}{2} (-1+T)\, T^4 p_{2+i}^2 p_{2+k} \pi_i^2 \pi_j +$$

$$\frac{1}{2} (-1+T)\, T^3 (-5+4\,T)\, p_{2+i} p_{2+j} p_{2+k} \pi_i^2 \pi_j - \frac{1}{2} (-1+T)\, T^2 \left(1-5\,T+3\,T^2\right) p_{2+j}^2 p_{2+k} \pi_i^2 \pi_j +$$

$$\frac{1}{2} (-1+T)\, T^2 (-1+4\,T)\, p_{2+i} p_{2+k}^2 \pi_i^2 \pi_j - \frac{1}{2} (-1+T)^2\, T\, (-1+4\,T)\, p_{2+j} p_{2+k}^2 \pi_i^2 \pi_j -$$

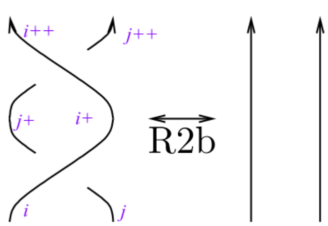
$$\frac{1}{2} (-1+T)\, T\, (-1+3\,T)\, p_{2+k}^3 \pi_i^2 \pi_j - \frac{1}{4} T\, (-5+7\,T)\, p_{2+j} p_{2+k} \pi_j^2 + \frac{1}{4} T\, (-7+9\,T)\, p_{2+k}^2 \pi_j^2 +$$

$$\frac{1}{2} T^4 p_{2+i} p_{2+j}^2 \pi_i \pi_j^2 - \frac{1}{2} T^4 p_{2+j}^3 \pi_i \pi_j^2 - 2 (-1+T) T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i \pi_j^2 +$$
$$\frac{1}{2} (-1+T) T^2 (-1+4T) p_{2+j}^2 p_{2+k} \pi_i \pi_j^2 + \frac{1}{2} (-1+T) T^2 (-1+3T) p_{2+i} p_{2+k}^2 \pi_i \pi_j^2 -$$
$$\frac{1}{2} (-1+T) T \left(1-5T+3T^2\right) p_{2+j} p_{2+k}^2 \pi_i \pi_j^2 - \frac{1}{2} (-1+T) T (-1+3T) p_{2+k}^3 \pi_i \pi_j^2 -$$
$$\frac{1}{6} (-1+T) T^2 p_{2+j}^2 p_{2+k} \pi_j^3 + \frac{1}{6} (-1+T) T (-1+4T) p_{2+j} p_{2+k}^2 \pi_j^3 - \frac{1}{6} (-1+T) T (-1+3T) p_{2+k}^3 \pi_j^3 +$$
$$2 p_{2+k} \pi_k + \frac{5}{2} T^2 p_{2+i} p_{2+k} \pi_i \pi_k - \frac{1}{2} T (-5+7T) p_{2+j} p_{2+k} \pi_i \pi_k - \frac{7}{2} T p_{2+k}^2 \pi_i \pi_k +$$
$$\frac{1}{2} T^4 p_{2+i}^2 p_{2+k} \pi_i^2 \pi_k - 2 (-1+T) T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i^2 \pi_k + \frac{1}{2} (-1+T) T^2 (-1+3T) p_{2+j}^2 p_{2+k} \pi_i^2 \pi_k -$$
$$\frac{1}{2} T^2 (-1+4T) p_{2+i} p_{2+k}^2 \pi_i^2 \pi_k + \frac{1}{2} (-1+T) T (-1+4T) p_{2+j} p_{2+k}^2 \pi_i^2 \pi_k + \frac{1}{2} T (-1+3T) p_{2+k}^3 \pi_i^2 \pi_k +$$
$$\frac{7}{2} T p_{2+j} p_{2+k} \pi_j \pi_k - \frac{9}{2} T p_{2+k}^2 \pi_j \pi_k + 3 T^3 p_{2+i} p_{2+j} p_{2+k} \pi_i \pi_j \pi_k - T^2 (-1+3T) p_{2+j}^2 p_{2+k} \pi_i \pi_j \pi_k -$$
$$T^2 (-1+3T) p_{2+i} p_{2+k}^2 \pi_i \pi_j \pi_k + T \left(1-5T+3T^2\right) p_{2+j} p_{2+k}^2 \pi_i \pi_j \pi_k + T (-1+3T) p_{2+k}^3 \pi_i \pi_j \pi_k +$$
$$\frac{1}{2} T^2 p_{2+j}^2 p_{2+k} \pi_j^2 \pi_k - \frac{1}{2} T (-1+4T) p_{2+j} p_{2+k}^2 \pi_j^2 \pi_k + \frac{1}{2} T (-1+3T) p_{2+k}^3 \pi_j^2 \pi_k + \frac{1}{2} T^2 p_{2+i} p_{2+k}^2 \pi_i \pi_k^2 -$$
$$\frac{1}{2} (-1+T) T p_{2+j} p_{2+k}^2 \pi_i \pi_k^2 - \frac{1}{2} T p_{2+k}^3 \pi_i \pi_k^2 + \frac{1}{2} T p_{2+j} p_{2+k}^2 \pi_j \pi_k^2 - \frac{1}{2} T p_{2+k}^3 \pi_j \pi_k^2 \Big] \Big]$$

Out[∘]=

```
True
```

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=

```
lhs = ∫𝔼[πi pi + πj pj] ℒ /@ (Xi,j[1] Xi+1,j+1[-1]) ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1}
rhs = ∫𝔼[πi pi + πj pj] ℒ /@ (Ci[0] Ci+1[0] Cj[0] Cj+1[0]) ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1};
lhs == rhs
```

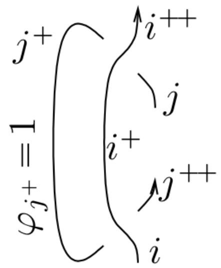
Out[∘]=

```
𝔼[∈Series[p2+i πi + p2+j πj, 0, 0]]
```

Out[∘]=

```
True
```

Invariance Under R2c



In[∘]:=
```
lhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
lhs == rhs
```

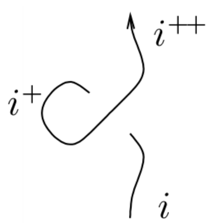
Out[∘]=

$-\mathbb{i}\,\sqrt{T}\,\mathbb{E}\left[\epsilon\text{Series}\left[p_{2+i}\,\pi_i + p_{3+j}\,\pi_j,\ -\frac{1}{2} - p_{3+j}\,\pi_j,\ \frac{1}{2}\,p_{3+j}\,\pi_j\right]\right]$

Out[∘]=

True

Invariance Under R1l



In[∘]:=
```
lhs = ∫𝔼[π_i p_i] ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[π_i p_i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs
```

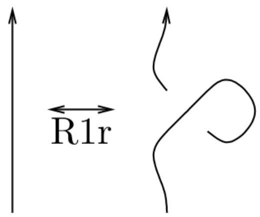
Out[∘]=

$-\mathbb{i}\,\mathbb{E}[\epsilon\text{Series}[p_{3+i}\,\pi_i, 0, 0]]$

Out[∘]=

True

Invariance Under R1r



In[◦]:=
```
lhs = ∫𝔼[π_i p_i] ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) 𝕕{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[π_i p_i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) 𝕕{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs
```

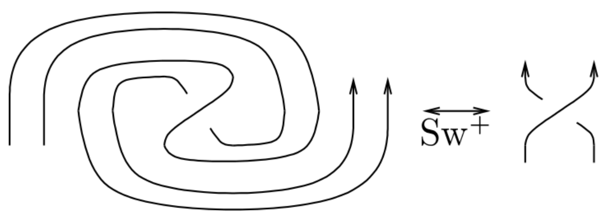
Out[◦]=

$-\mathbb{i}\,\mathbb{E}[\in\text{Series}[p_{3+i}\,\pi_i, 0, 0]]$

Out[◦]=

True

Invariance Under Sw



In[◦]:=
```
lhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}};
lhs == rhs
```

Out[◦]=

$$\sqrt{T}\ \mathbb{E}\Big[\epsilon\text{Series}\Big[T\,p_{3+i}\,\pi_i + (1-T)\,p_{3+j}\,\pi_i + p_{3+j}\,\pi_j,\ -\frac{1}{2} + T\,p_{3+j}\,\pi_i + \frac{1}{2}(-1+T)\,T\,p_{3+i}\,p_{3+j}\,\pi_i^2 - \frac{1}{2}(-1+T)\,T\,p_{3+j}^2\,\pi_i^2 - p_{3+j}\,\pi_j - T\,p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j + T\,p_{3+j}^2\,\pi_i\,\pi_j,\ -\frac{1}{2}T\,p_{3+j}\,\pi_i - \frac{1}{4}T(-1+3T)\,p_{3+i}\,p_{3+j}\,\pi_i^2 + \frac{1}{4}T(-3+5T)\,p_{3+j}^2\,\pi_i^2 - \frac{1}{6}(-1+T)\,T^2\,p_{3+i}^2\,p_{3+j}\,\pi_i^3 + \frac{1}{6}(-1+T)\,T(-1+4T)\,p_{3+i}\,p_{3+j}^2\,\pi_i^3 - \frac{1}{6}(-1+T)\,T(-1+3T)\,p_{3+j}^3\,\pi_i^3 + \frac{1}{2}p_{3+j}\,\pi_j + \frac{3}{2}T\,p_{3+i}\,p_{3+j}\,\pi_i\,\pi_j - \frac{5}{2}T\,p_{3+j}^2\,\pi_i\,\pi_j + \frac{1}{2}T^2\,p_{3+i}^2\,p_{3+j}\,\pi_i^2\,\pi_j - \frac{1}{2}T(-1+4T)\,p_{3+i}\,p_{3+j}^2\,\pi_i^2\,\pi_j + \frac{1}{2}T(-1+3T)\,p_{3+j}^3\,\pi_i^2\,\pi_j + \frac{1}{2}T\,p_{3+i}\,p_{3+j}^2\,\pi_i\,\pi_j^2 - \frac{1}{2}T\,p_{3+j}^3\,\pi_i\,\pi_j^2\Big]\Big]$$

Out[◦]=

True