In[∘]:= `n = 2`

Out[∘]=

2

In[∘]:= $\xi_{\alpha\_,\alpha\_}$ `:= Log[`Ξ_α`]`

In[∘]:= `A = B = IdentityMatrix[n]`

Out[∘]=

{{1, 0}, {0, 1}}

In[∘]:= `Do[A = A.MatrixExp[SparseArray[{α, β} → ξ_{α,β} /. ξ_{α_,α_} :→ Log[Ξ_α], {n, n}]], {β, 1, n}, {α, 1, β}]; (* This specifies the PBW ordering *)`

In[∘]:= `Do[B = MatrixExp[SparseArray[{β, α} → η_{α,β}, {n, n}]].B, {β, 2, n}, {α, 1, β - 1}];`

In[∘]:= `MatrixForm /@ {A, B}`

Out[∘]=

$$\left\{\begin{pmatrix}\Xi_1 & \Xi_1\Xi_2\xi_{1,2}\\ 0 & \Xi_2\end{pmatrix},\ \begin{pmatrix}1 & 0\\ \eta_{1,2} & 1\end{pmatrix}\right\}$$

In[∘]:= `M = (A /. {ξ_{αβ__} :→ ξ_{αβ}[i], Ξ_{α_} :→ Ξ_α[i]}).(B /. η_{αβ__} :→ η_{αβ}[j])`

Out[∘]=

$$\{\{\Xi_1[i]+\Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i],\ \Xi_1[i]\,\Xi_2[i]\,\xi_{1,2}[i]\},\ \{\Xi_2[i]\,\eta_{1,2}[j],\ \Xi_2[i]\}\}$$

In[∘]:= `LU = First@LUDecomposition[M, Pivoting → False]`

Out[∘]=

$$\left\{\{\Xi_1[i]+\Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i],\ \Xi_1[i]\,\Xi_2[i]\,\xi_{1,2}[i]\},\ \left\{\frac{\Xi_2[i]\,\eta_{1,2}[j]}{\Xi_1[i]+\Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]},\ \Xi_2[i]-\frac{\Xi_1[i]\,\Xi_2[i]^2\,\eta_{1,2}[j]\,\xi_{1,2}[i]}{\Xi_1[i]+\Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]}\right\}\right\}$$

In[∘]:= `MatrixForm[L = LowerTriangularize[LU, -1] + IdentityMatrix[n]]`

Out[∘]//MatrixForm=

$$\begin{pmatrix}1 & 0\\ \frac{\Xi_2[i]\,\eta_{1,2}[j]}{\Xi_1[i]+\Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]} & 1\end{pmatrix}$$

In[∘]:= `MatrixForm[U = UpperTriangularize[LU]]`

Out[∘]//MatrixForm=

$$\begin{pmatrix}\Xi_1[i]+\Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i] & \Xi_1[i]\,\Xi_2[i]\,\xi_{1,2}[i]\\ 0 & \Xi_2[i]-\frac{\Xi_1[i]\,\Xi_2[i]^2\,\eta_{1,2}[j]\,\xi_{1,2}[i]}{\Xi_1[i]+\Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]}\end{pmatrix}$$

In[∘]:= `Simplify[L.U - M]`

Out[∘]=

{{0, 0}, {0, 0}}

In[∘]:= `A /. {ξ → ξ1, Ξ → Ξ1} // MatrixForm`

Out[∘]//MatrixForm=

$$\begin{pmatrix}\Xi1_1 & \Xi1_1\,\Xi1_2\,\xi1_{1,2} & \Xi1_3\,(\Xi1_1\,\xi1_{1,3}+\Xi1_1\,\Xi1_2\,\xi1_{1,2}\,\xi1_{2,3})\\ 0 & \Xi1_2 & \Xi1_2\,\Xi1_3\,\xi1_{2,3}\\ 0 & 0 & \Xi1_3\end{pmatrix}$$

```
In[•]:= B /. η → η1 // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ \eta 1_{1,2} & 1 & 0 \\ \eta 1_{1,3} + \eta 1_{1,2}\, \eta 1_{2,3} & \eta 1_{2,3} & 1 \end{pmatrix}$$

```
In[•]:= eqnsU = Table[U⟦α, α + δα⟧ == (A⟦α, α + δα⟧ /. {ξ → ξ1, Ξ → Ξ1}), {δα, 0, n - 1}, {α, 1, n - δα}]
```

Out[•]=

$$\left\{\left\{\Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i] == \Xi 1_1,\ \Xi_2[i] - \frac{\Xi_1[i]\, \Xi_2[i]^2\, \eta_{1,2}[j]\, \xi_{1,2}[i]}{\Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]} == \Xi 1_2\right\},\ \{\Xi_1[i]\, \Xi_2[i]\, \xi_{1,2}[i] == \Xi 1_1\, \Xi 1_2\, \xi 1_{1,2}\}\right\}$$

```
In[•]:= varsU = Flatten@{Table[Ξ1_α, {α, 1, n}], Table[ξ1_{α,α+δα}, {δα, 1, n - 1}, {α, 1, n - δα}]}
```

Out[•]=

$$\{\Xi 1_1, \Xi 1_2, \xi 1_{1,2}\}$$

```
In[•]:= solU = TriangularSolve[eqnsU, varsU]
```

Out[•]=

$$\left\{\Xi 1_1 \to \Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i],\ \Xi 1_2 \to \frac{\Xi_2[i]}{1 + \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]},\ \xi 1_{1,2} \to \xi_{1,2}[i]\right\}$$

```
In[•]:= eqnsL = Table[(L⟦α + δα, α⟧ /. solU) == (B⟦α + δα, α⟧ /. {η → η1}), {δα, 1, n - 1}, {α, 1, n - δα}]
```

Out[•]=

$$\left\{\left\{\frac{\Xi_2[i]\, \eta_{1,2}[j]}{\Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]} == \eta 1_{1,2}\right\}\right\}$$

```
In[•]:= varsL = Flatten@Table[η1_{α,α+δα}, {δα, 1, n - 1}, {α, 1, n - δα}]
```

Out[•]=

$$\{\eta 1_{1,2}\}$$

```
In[•]:= solL = TriangularSolve[eqnsL, varsL]
```

Out[•]=

$$\left\{\eta 1_{1,2} \to \frac{\Xi_2[i]\, \eta_{1,2}[j]}{\Xi_1[i]\, (1 + \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i])}\right\}$$

```
In[•]:= vars = varsU ⋃ varsL /. Ξ1_α_ ⧴ ξ1_{α,α}
```

Out[•]=

$$\{\xi 1_{1,1}, \xi 1_{2,2}, \eta 1_{1,2}, \xi 1_{1,2}\}$$

```
In[•]:= sol = solU ⋃ solL /. (Ξ1_α_ → ℰ_) ⧴ (ξ1_{α,α} → Log[ℰ])
```

Out[•]=

$$\left\{\xi 1_{1,1} \to \text{Log}[\Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]],\ \xi 1_{2,2} \to \text{Log}\left[\frac{\Xi_2[i]}{1 + \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]}\right],\ \eta 1_{1,2} \to \frac{\Xi_2[i]\, \eta_{1,2}[j]}{\Xi_1[i]\, (1 + \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i])},\ \xi 1_{1,2} \to \xi_{1,2}[i]\right\}$$

In[∘]:= $\text{CF}\left[\left(\text{vars} \mathbin{/.} \{\eta 1_{\alpha\beta\_\_} \to y_{\alpha\beta}[i],\ \xi 1_{\alpha\beta\_\_} \to x_{\alpha\beta}[j]\}\right).(\text{vars} \mathbin{/.} \text{sol})\right]$

Out[∘]=

$$\text{Log}[\Xi_1[i] + \Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]]\,x_{1,1}[j] + \text{Log}\left[\frac{\Xi_2[i]}{1 + \Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]}\right] x_{2,2}[j] + x_{1,2}[j]\,\xi_{1,2}[i] + \frac{\Xi_2[i]\,y_{1,2}[i]\,\eta_{1,2}[j]}{\Xi_1[i]\,(1 + \Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i])}$$

In[∘]:= $\{\eta_{1,2}[4]\}^*$

Out[∘]=

$\text{List}^*[\eta_{1,2}[4]]$

In[∘]:=
```
With[{n = 2, $k = 0},
  sw_{ut,ln,n,$k}[2, 4] // Echo // sw_{ut,ln,n,$k}[1, 4] // Echo // utm_n[1, 2 → 3]
]
```

» $\mathbb{E}_{\{2,4\}\to\{2,4\}}\left[e^{-\xi_{1,1}[2]+\xi_{2,2}[2]}\,y_{1,2}[4]\,\eta_{1,2}[4] + x_{1,1}[2]\,\xi_{1,1}[2] + x_{1,2}[2]\,\xi_{1,2}[2] + x_{2,2}[2]\,\xi_{2,2}[2]\right]$

» {4}

» $\{\eta_{1,2}[4]\}$

Intersection: Heads List* and List at positions 2 and 1 are expected to be the same.

» $\{y_{1,2}[4]\} \cap \text{List}^*[\eta_{1,2}[4]]$

Intersection: Heads List* and List at positions 2 and 1 are expected to be the same.

Union: Heads SuperStar and Intersection at positions 2 and 1 are expected to be the same.

Complement: Heads List and Union at positions 2 and 1 are expected to be the same.

Table: Iterator {u$181277, {$[$y_{1,2}$[4]]} ∩ List*[$[$\eta_{1,2}$[4]]] ∪ ({$[$y_{1,2}$[4]]} ∩ List*[$[$\eta_{1,2}$[4]]])*} does not have appropriate bounds.

Table: Iterator {u$181277, {$[$y_{1,2}$[4]]} ∩ List*[$[$\eta_{1,2}$[4]]] ∪ ({$[$y_{1,2}$[4]]} ∩ List*[$[$\eta_{1,2}$[4]]])*} does not have appropriate bounds.

Table: Iterator {u$181277, {$[$y_{1,2}$[4]]} ∩ List*[$[$\eta_{1,2}$[4]]] ∪ ({$[$y_{1,2}$[4]]} ∩ List*[$[$\eta_{1,2}$[4]]])*} does not have appropriate bounds.

General: Further output of Table::iterb will be suppressed during this calculation.

Out[∘]=

$Aborted

In[∘]:= `{n = 2, $k = 0}`

Out[∘]=

{2, 0}

In[∘]:= $\text{sw}_{ut,ln,n,\$k}[1, 4]$

Out[∘]=

$$\mathbb{E}_{\{1,4\}\to\{1,4\}}\left[e^{-\xi_{1,1}[1]+\xi_{2,2}[1]}\,y_{1,2}[4]\,\eta_{1,2}[4] + x_{1,1}[1]\,\xi_{1,1}[1] + x_{1,2}[1]\,\xi_{1,2}[1] + x_{2,2}[1]\,\xi_{2,2}[1]\right]$$

In[∘]:= $\text{utm}_n[1, 2 \to 3]$

Out[∘]=

$$\mathbb{E}_{\{1,2\}\to\{3\}}\left[x_{1,1}[3]\,\xi_{1,1}[1] + x_{1,1}[3]\,\xi_{1,1}[2] + e^{-\xi_{1,1}[2]}\,x_{1,2}[3]\,\xi_{1,2}[1] + e^{-\xi_{2,2}[1]}\,x_{1,2}[3]\,\xi_{1,2}[2] + x_{2,2}[3]\,\xi_{2,2}[1] + x_{2,2}[3]\,\xi_{2,2}[2]\right]$$

In[∘]:= $\text{sw}_{ut,ln,n,\$k}[1, 4] \mathbin{//} \text{utm}_n[1, 2 \to 3]$

» $\{x_{1,1}[1],\ x_{1,2}[1],\ x_{2,2}[1]\}$

» $\{x_{1,1}[1],\ x_{1,2}[1],\ x_{2,2}[1],\ x_{2,2}[1]\}$

» $\{\{1\},\ \{x_{1,1}[1],\ x_{1,2}[1],\ x_{2,2}[1]\},\ \{x_{1,1}[\$[1]],\ x_{1,2}[\$[1]],\ x_{2,2}[\$[1]]\}\}$

Out[∘]=

$$\mathbb{E}_{\{1,2,4\}\to\{3,4\}}\left[e^{-\xi_{1,1}[1]+\xi_{2,2}[1]}\ y_{1,2}[4]\ \eta_{1,2}[4]+x_{1,1}[3]\ \xi_{1,1}[1]+x_{1,1}[3]\ \xi_{1,1}[2]+e^{-\xi_{1,1}[2]}\ x_{1,2}[3]\ \xi_{1,2}[1]+e^{-\xi_{2,2}[1]}\ x_{1,2}[3]\ \xi_{1,2}[2]+x_{2,2}[3]\ \xi_{2,2}[1]+x_{2,2}[3]\ \xi_{2,2}[2]\right]$$