Quantum $\mathfrak{gl}_N$: Nilpotent Parts and PBW Coordinates

Dror Bar-Natan: AcademicPensieve: Projects: glneps: ChatGPT: qgln-260525-03.pdf

This is a convention sheet for the positive and negative nilpotent parts of the Drinfeld–Jimbo quantum group, with optional diagonal generators added for the upper Borel. The point is that, after choosing an order of roots, the nilpotent parts are *visibly polynomial as vector spaces*, while the Borel is obtained by adjoining a commuting Laurent torus.

1. Strictly upper-triangular generators

Let $U_q(\mathfrak{n}_+)$ denote the positive nilpotent part. We write its root vectors as

$$x_{ab} \qquad (1 \le a < b \le N),$$

where the simple generators are

$$x_{a,a+1} \qquad (1 \le a < N).$$

For $a < b$, define recursively

$$x_{ab} = x_{ac}x_{cb} - q^{-1}x_{cb}x_{ac} = [x_{ac}, x_{cb}]_{q^{-1}}, \qquad a < c < b. \tag{1}$$

With the usual Drinfeld–Jimbo conventions, the right-hand side is independent of the choice of intermediate index c. Equivalently, one may take the adjacent recursion

$$x_{ab} = x_{a,a+1}x_{a+1,b} - q^{-1}x_{a+1,b}x_{a,a+1}.$$

For example,

$$x_{13} = x_{12}x_{23} - q^{-1}x_{23}x_{12},$$

and

$$x_{14} = x_{12}x_{24} - q^{-1}x_{24}x_{12} = x_{13}x_{34} - q^{-1}x_{34}x_{13}.$$

Optional diagonal generators. To pass from the nilpotent part $U_q(\mathfrak{n}_+)$ to an upper triangular version with primitive diagonal variables, adjoin commuting generators

$$x_{aa} \qquad (1 \le a \le N).$$

They commute among themselves,

$$[x_{aa}, x_{bb}] = 0,$$

and act on the positive root vector x_{bc} by its weight:

$$[x_{aa}, x_{bc}] = (\delta_{ab} - \delta_{ac})x_{bc} \qquad (b < c). \tag{2}$$

Equivalently,

$$e^{\hbar x_{aa}}x_{bc}e^{-\hbar x_{aa}} = q^{\delta_{ab}-\delta_{ac}}x_{bc}, \qquad q = e^{\hbar}.$$

Thus x_{aa} contributes $+1$ when it meets a root starting at a, contributes -1 when it meets a root ending at a, and otherwise commutes. For example, for $a < b < c$,

$$[x_{aa}, x_{ab}] = x_{ab}, \qquad [x_{bb}, x_{ab}] = -x_{ab}, \qquad [x_{aa}, x_{bc}] = 0.$$

With these generators included, ordered monomials in the x_{aa}'s and the PBW monomials in the x_{ab}'s give the usual vector-space basis of the upper triangular algebra.

Straightening relations.

The following table is oriented for the row convex order $(1,2) < (1,3) < \cdots < (1,N) < (2,3) < \cdots$. Thus the left side is an *inversion*; the right side is closer to PBW order. Here all displayed indices are strictly increasing.

configuration	rewrite rule
$a < b < c < d$	$x_{cd}x_{ab} = x_{ab}x_{cd}$
$a < b < c$	$x_{ac}x_{ab} = q^{-1}x_{ab}x_{ac}$
$a < b < c$	$x_{bc}x_{ac} = q^{-1}x_{ac}x_{bc}$
$a < b < c$	$x_{bc}x_{ab} = q\,x_{ab}x_{bc} - q\,x_{ac}$
$a < b < c < d$	$x_{bc}x_{ad} = x_{ad}x_{bc}$
$a < b < c < d$	$x_{bd}x_{ac} = x_{ac}x_{bd} - (q - q^{-1})x_{ad}x_{bc}$

Equivalently, the defining concatenation relation is

$$x_{ab}x_{bc} - q^{-1}x_{bc}x_{ab} = x_{ac},$$

and the last row is the interlaced case. These are the local rewrite rules used for straightening words in the generators x_{ab}; confluence is the PBW theorem.

2. Strictly lower-triangular generators

Similarly, let $U_q(\mathfrak{n}_-)$ denote the negative nilpotent part. We label its root vectors by the same positive-root indices as the upper ones:

$$y_{ab} \qquad (1 \le a < b \le N),$$

so that y_{ab} is the adjoint/lower-triangular partner of x_{ab}. Thus the simple generators are $y_{a,a+1}$, meaning the usual lowering generators paired with $x_{a,a+1}$. The analogous recursion is

$$y_{ab} = y_{cb}y_{ac} - q^{-1}y_{ac}y_{cb}, \qquad a < c < b. \tag{3}$$

This is what one gets by applying an anti-involution/adjoint with $x^*_{ab} = y_{ab}$ to (1). For example,

$$y_{13} = y_{23}y_{12} - q^{-1}y_{12}y_{23}.$$

Some authors use the opposite normalization or replace q^{-1} by q; this changes the named root vectors but not the PBW/vector-space statement below.

3. Convex orderings of roots

A total ordering $<$ on the set Φ_+ of positive roots is called *convex* if whenever

$$\alpha, \beta, \alpha + \beta \in \Phi_+ \quad \text{and} \quad \alpha < \beta,$$

then

$$\alpha < \alpha + \beta < \beta. \tag{4}$$

For type A_{N-1}, identify the positive root $\varepsilon_a - \varepsilon_b$ with the pair (a, b), where $a < b$. A standard convex ordering is

$$(1,2) < (1,3) < \cdots < (1,N) < (2,3) < \cdots < (2,N) < \cdots < (N-1,N).$$

Another common convex ordering is the column order

$$(1,2) < (2,3) < (1,3) < (3,4) < (2,4) < (1,4) < \cdots.$$

Different convex orders give different PBW bases, but all give the same vector-space conclusion.

4. PBW bases

Fix a convex ordering of the positive roots. Then the ordered monomials

$$\overrightarrow{\prod_{a<b}} x_{ab}^{m_{ab}}, \qquad m_{ab} \in \mathbb{Z}_{\geq 0}, \tag{5}$$

form a basis of $U_q(\mathfrak{n}_+)$. Here the arrow means: multiply the factors in the chosen convex order.

Likewise, the ordered monomials

$$\overrightarrow{\prod_{a<b}} y_{ab}^{n_{ab}}, \qquad n_{ab} \in \mathbb{Z}_{\geq 0}, \tag{6}$$

form a basis of $U_q(\mathfrak{n}_-)$, with the chosen convention for ordering.

With diagonal generators included, a corresponding basis is

$$x_{11}^{k_1} \cdots x_{NN}^{k_N} \overrightarrow{\prod_{a<b}} x_{ab}^{m_{ab}}, \qquad k_i, m_{ab} \in \mathbb{Z}_{\geq 0}.$$

6. Universal R-matrix in the same PBW coordinates

With the same convex ordering and the same paired root vectors $x_{ab} \leftrightarrow y_{ab}$, the universal R-matrix has the ordered factorization

$$\mathcal{R} = \mathcal{R}_0 \, \mathcal{R}_{\mathrm{nil}}, \qquad \mathcal{R}_{\mathrm{nil}} = \overrightarrow{\prod_{a<b}} \exp_{q^{-2}}\big((q - q^{-1})\, x_{ab} \otimes y_{ab}\big)\,. \tag{7}$$

Here

$$\exp_t(z) := \sum_{m \geq 0} \frac{z^m}{[m]_t!}, \qquad [m]_t! := \prod_{j=1}^{m} \frac{1 - t^j}{1 - t}.$$

Equivalently, the nilpotent factor expands in the PBW basis as

$$\mathcal{R}_{\mathrm{nil}} = \sum_{(m_{ab})} \left(\overrightarrow{\prod_{a<b}} \frac{(q - q^{-1})^{m_{ab}}}{[m_{ab}]_{q^{-2}}!} \right) \left(\overrightarrow{\prod_{a<b}} x_{ab}^{m_{ab}} \right) \otimes \left(\overrightarrow{\prod_{a<b}} y_{ab}^{m_{ab}} \right), \tag{8}$$

where the sum is over all functions $(a, b) \mapsto m_{ab} \in \mathbb{Z}_{\geq 0}$. The order of the factors in both tensor components is the chosen convex order; using the opposite convention for the lower PBW order moves the reversal into the displayed formula.

For $\mathfrak{gl}_N$, the Cartan factor may be written, in the diagonal basis $h_1, \ldots, h_N$, as

$$\mathcal{R}_0 = q^{\sum_{i=1}^{N} h_i \otimes h_i}, \tag{9}$$

up to the usual completion and normalization conventions. In the $\mathfrak{sl}_N$ normalization one replaces this by the factor determined by the inverse Cartan matrix.

7. Tiny case: $N = 3$

For $N = 3$, the upper nilpotent part has root vectors

$$x_{12}, \qquad x_{13} = x_{12}x_{23} - q^{-1}x_{23}x_{12}, \qquad x_{23}.$$

For the convex order $(1,2) < (1,3) < (2,3)$, the PBW basis is

$$x_{12}^a x_{13}^b x_{23}^c, \qquad a, b, c \geq 0.$$

Hence, as a vector space,

$$U_q(\mathfrak{n}_+) \cong \mathbb{Q}(q)[X_{12}, X_{13}, X_{23}].$$

Standard references: Lusztig, *Introduction to Quantum Groups*; Jantzen, *Lectures on Quantum Groups*; Chari–Pressley, *A Guide to Quantum Groups*.