Quantum $\mathfrak{gl}_N$: Nilpotent Parts and PBW Coordinates

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This is a convention sheet for the positive and negative nilpotent parts of the Drinfeld–Jimbo quantum group. The point is that, after choosing an order of roots, these quantum nilpotent parts are *visibly polynomial as vector spaces.*

1. Strictly upper-triangular generators

Let $U_q(\mathfrak{n}_+)$ denote the positive nilpotent part. We write its root vectors as

$$x_{ab} \qquad (1 \le a < b \le N),$$

where the simple generators are

$$x_{a,a+1} \qquad (1 \le a < N).$$

For $a < b$, define recursively

$$x_{ab} = x_{ac}x_{cb} - q^{-1}x_{cb}x_{ac} = [x_{ac}, x_{cb}]_{q^{-1}}, \qquad a < c < b. \tag{1}$$

With the usual Drinfeld–Jimbo conventions, the right-hand side is independent of the choice of intermediate index c. Equivalently, one may take the adjacent recursion

$$x_{ab} = x_{a,a+1}x_{a+1,b} - q^{-1}x_{a+1,b}x_{a,a+1}.$$

For example,

$$x_{13} = x_{12}x_{23} - q^{-1}x_{23}x_{12},$$

and

$$x_{14} = x_{12}x_{24} - q^{-1}x_{24}x_{12} = x_{13}x_{34} - q^{-1}x_{34}x_{13}.$$

2. Strictly lower-triangular generators

Similarly, let $U_q(\mathfrak{n}_-)$ denote the negative nilpotent part. We label its root vectors by the same positive-root indices as the upper ones:

$$y_{ab} \qquad (1 \le a < b \le N),$$

so that y_{ab} is the adjoint/lower-triangular partner of x_{ab}. Thus the simple generators are $y_{a,a+1}$, meaning the usual lowering generators paired with $x_{a,a+1}$. The analogous recursion is

$$y_{ab} = y_{cb}y_{ac} - q^{-1}y_{ac}y_{cb}, \qquad a < c < b. \tag{2}$$

This is what one gets by applying an anti-involution/adjoint with $x_{ab}^* = y_{ab}$ to (**??**). For example,

$$y_{13} = y_{23}y_{12} - q^{-1}y_{12}y_{23}.$$

Some authors use the opposite normalization or replace q^{-1} by q; this changes the named root vectors but not the PBW/vector-space statement below.

3. Convex orderings of roots

A total ordering $<$ on the set Φ_+ of positive roots is called *convex* if whenever

$$\alpha, \beta, \alpha+\beta \in \Phi_+ \quad \text{and} \quad \alpha < \beta,$$

then

$$\alpha < \alpha+\beta < \beta. \tag{3}$$

For type A_{N-1}, identify the positive root $\varepsilon_a - \varepsilon_b$ with the pair (a,b), where $a<b$. A standard convex ordering is

$$(1,2) < (1,3) < \cdots < (1,N) < (2,3) < \cdots < (2,N) < \cdots < (N-1,N).$$

Another common convex ordering is the column order

$$(1,2) < (2,3) < (1,3) < (3,4) < (2,4) < (1,4) < \cdots .$$

Different convex orders give different PBW bases, but all give the same vector-space conclusion.

4. PBW bases

Fix a convex ordering of the positive roots. Then the ordered monomials

$$\overrightarrow{\prod_{a<b}} x_{ab}^{m_{ab}}, \qquad m_{ab} \in \mathbb{Z}_{\geq 0}, \tag{4}$$

form a basis of $U_q(\mathfrak{n}_+)$. Here the arrow means: multiply the factors in the chosen convex order.
Likewise, the ordered monomials

$$\overrightarrow{\prod_{a<b}} y_{ab}^{n_{ab}}, \qquad n_{ab} \in \mathbb{Z}_{\geq 0}, \tag{5}$$

form a basis of $U_q(\mathfrak{n}_-)$, with the chosen convention for ordering.

5. Polynomial vector-space identification

Therefore there are explicit vector-space isomorphisms

$$\mathbb{Q}(q)[X_{ab} : a<b] \xrightarrow{\sim} U_q(\mathfrak{n}_+), \qquad \overrightarrow{\prod_{a<b}} X_{ab}^{m_{ab}} \longmapsto \overrightarrow{\prod_{a<b}} x_{ab}^{m_{ab}},$$

and

$$\mathbb{Q}(q)[Y_{ab} : a<b] \xrightarrow{\sim} U_q(\mathfrak{n}_-), \qquad \overrightarrow{\prod_{a<b}} Y_{ba}^{n_{ab}} \longmapsto \overrightarrow{\prod_{a<b}} y_{ab}^{n_{ab}}.$$

These are **not** algebra isomorphisms to commutative polynomial rings. They are PBW, hence vector-space, identifications. The multiplication on the right is the noncommutative q-deformed multiplication, governed by straightening relations.

6. Universal R-matrix in the same PBW coordinates

With the same convex ordering and the same paired root vectors $x_{ab} \leftrightarrow y_{ab}$, the universal R-matrix has the ordered factorization

$$\mathcal{R} = \mathcal{R}_0 \, \mathcal{R}_{\mathrm{nil}}, \qquad \mathcal{R}_{\mathrm{nil}} = \overrightarrow{\prod_{a<b}} \exp_{q^{-2}}\big((q - q^{-1})\, x_{ab} \otimes y_{ab}\big) \,. \tag{6}$$

Here

$$\exp_t(z) := \sum_{m \geq 0} \frac{z^m}{[m]_t!}, \qquad [m]_t! := \prod_{j=1}^{m} \frac{1 - t^j}{1 - t}.$$

Equivalently, the nilpotent factor expands in the PBW basis as

$$\mathcal{R}_{\mathrm{nil}} = \sum_{(m_{ab})} \left(\overrightarrow{\prod_{a<b}} \frac{(q - q^{-1})^{m_{ab}}}{[m_{ab}]_{q^{-2}}!} \right) \left(\overrightarrow{\prod_{a<b}} x_{ab}^{m_{ab}} \right) \otimes \left(\overrightarrow{\prod_{a<b}} y_{ab}^{m_{ab}} \right), \tag{7}$$

where the sum is over all functions $(a, b) \mapsto m_{ab} \in \mathbb{Z}_{\geq 0}$. The order of the factors in both tensor components is the chosen convex order; using the opposite convention for the lower PBW order moves the reversal into the displayed formula.

For $\mathfrak{gl}_N$, the Cartan factor may be written, in the diagonal basis $h_1, \ldots, h_N$, as

$$\mathcal{R}_0 = q^{\sum_{i=1}^{N} h_i \otimes h_i}, \tag{8}$$

up to the usual completion and normalization conventions. In the $\mathfrak{sl}_N$ normalization one replaces this by the factor determined by the inverse Cartan matrix.

7. Tiny case: $N = 3$

For $N = 3$, the upper nilpotent part has root vectors

$$x_{12}, \qquad x_{13} = x_{12}x_{23} - q^{-1}x_{23}x_{12}, \qquad x_{23}.$$

For the convex order $(1, 2) < (1, 3) < (2, 3)$, the PBW basis is

$$x_{12}^a x_{13}^b x_{23}^c, \qquad a, b, c \geq 0.$$

Hence, as a vector space,

$$U_q(\mathfrak{n}_+) \cong \mathbb{Q}(q)[X_{12}, X_{13}, X_{23}].$$

Standard references: Lusztig, *Introduction to Quantum Groups*; Jantzen, *Lectures on Quantum Groups*; Chari–Pressley, *A Guide to Quantum Groups*.