

Quantum \mathfrak{gl}_N Cheat Sheet

For the Drinfeld–Jimbo quantum group $U_q(\mathfrak{gl}_N)$, let $U_q(\mathfrak{n}_+)$ and $U_q(\mathfrak{n}_-)$ denote the positive and negative nilpotent parts.

1. Simple generators.

$$x_{a,a+1} := E_a, \quad y_{a,a+1} := E_a \quad (1 \leq a \leq N-1).$$

2. Root vectors.

For $a < b$, define recursively

$$x_{ab} = x_{ac}x_{cb} - q^{-1}x_{cb}x_{ac},$$

where $a < c < b$. The resulting element is independent of the choice of c . Similarly,

$$y_{ab} = y_{ac}y_{cb} - q^{-1}y_{cb}y_{ac}.$$

3. Examples.

$$\begin{aligned} x_{13} &= x_{12}x_{23} - q^{-1}x_{23}x_{12}, \\ x_{14} &= x_{12}x_{24} - q^{-1}x_{24}x_{12} = x_{13}x_{34} - q^{-1}x_{34}x_{13}. \end{aligned}$$

4. Convex ordering of positive roots.

A total ordering $<$ on the positive roots Φ_+ is called *convex* if whenever

$$\alpha, \beta, \alpha + \beta \in \Phi_+ \quad \text{and} \quad \alpha < \beta,$$

then

$$\alpha < \alpha + \beta < \beta.$$

For \mathfrak{gl}_N , identifying positive roots with pairs (a, b) , $a < b$, a standard convex ordering is

$$(1, 2) < (1, 3) < \cdots < (1, N) < (2, 3) < \cdots < (N-1, N).$$

6. PBW basis.

Fix a convex ordering of the positive roots. Then the ordered monomials

$$\prod_{a < b}^{\rightarrow} x_{ab}^{m_{ab}} \quad (m_{ab} \geq 0)$$

form a basis of $U_q(\mathfrak{n}_+)$. Likewise,

$$\prod_{a < b}^{\rightarrow} y_{ab}^{n_{ab}}$$

form a basis of $U_q(\mathfrak{n}_-)$.

7. Universal R -matrix.

With the same convex ordering,

$$\mathcal{R} = q^{\sum h_i \otimes h_i} \prod_{a < b}^{\rightarrow} \exp_q((q - q^{-1})x_{ab} \otimes y_{ab}).$$

The product is taken in the chosen convex order.