Pensieve header: Verifying the PBW coherence, following https://chatgpt.com/c/6a0dc8f8-dd54-83ea-9134-36a5dbdbd6d6.

In[∘]:=
```
SR = { (* Straightening Rules *)
   w[l___, x_{c_,d_}, x_{a_,b_}, r___] /; a < b < c < d ⧴ w[l, x_{a,b}, x_{c,d}, r],
   w[l___, x_{a_,c_}, x_{a_,b_}, r___] /; a < b < c ⧴ q^{-1} w[l, x_{a,b}, x_{a,c}, r],
   w[l___, x_{b_,c_}, x_{a_,c_}, r___] /; a < b < c ⧴ q^{-1} w[l, x_{a,c}, x_{b,c}, r],
   w[l___, x_{b_,c_}, x_{a_,b_}, r___] /; a < b < c ⧴ q w[l, x_{a,b}, x_{b,c}, r] - q w[l, x_{a,c}, r],
   w[l___, x_{a_,d_}, x_{b_,c_}, r___] /; a < b < c < d ⧴ w[l, x_{b,c}, x_{a,d}, r],
   w[l___, x_{b_,d_}, x_{a_,c_}, r___] /; a < b < c < d ⧴
    w[l, x_{a,c}, x_{b,d}, r] - (q - q^{-1}) w[l, x_{b,c}, x_{a,d}, r]
  };
```

In[∘]:=
```
Straighten[ℰ_] := FixedPoint[Expand[# /. SR] &, ℰ, 10]
```

In[∘]:=
```
Straighten[w[x_{1,2}, x_{2,3}]]
```

Out[∘]=

$$w[x_{1,2},\ x_{2,3}]$$

In[∘]:=
```
Straighten[w[x_{1,3}, x_{1,2}]]
```

Out[∘]=

$$\frac{w[x_{1,2},\ x_{1,3}]}{q}$$

In[∘]:=
```
Straighten[w[x_{1,4}, x_{1,3}, x_{1,2}]]
```

Out[∘]=

$$\frac{w[x_{1,2},\ x_{1,3},\ x_{1,4}]}{q^3}$$

In[∘]:=
```
Straighten[w[x_{2,4}, x_{1,3}, x_{1,2}]]
```

Out[∘]=

$$-\frac{w[x_{1,3},\ x_{1,4}]}{q}+w[x_{1,2},\ x_{1,3},\ x_{2,4}]+\frac{w[x_{1,2},\ x_{2,3},\ x_{1,4}]}{q}-q\,w[x_{1,2},\ x_{2,3},\ x_{1,4}]$$

In[∘]:=
```
AllSingleSteps[ℰ_] := Union@Expand@Flatten[
      Map[
       p ↦ MapAt[
         ω ↦ Replace[{} → {ω}]@ReplaceList[SR]@ω,
         {ℰ}, p
        ],
       Position[{ℰ}, _w, ∞]
      ]
     ];
AllSingleSteps[ℒ_List] := Union@Flatten[AllSingleSteps /@ ℒ]
```

In[∘]:=

$$\text{AllSingleSteps}[w[x_{1,4},\ x_{1,3},\ x_{1,2}]]$$

Out[∘]=

$$\left\{\frac{w[x_{1,3},\ x_{1,4},\ x_{1,2}]}{q},\ \frac{w[x_{1,4},\ x_{1,2},\ x_{1,3}]}{q}\right\}$$

In[∘]:=

$$\text{AllSingleSteps}@\left\{\frac{w[x_{1,3},\ x_{1,4},\ x_{1,2}]}{q},\ \frac{w[x_{1,4},\ x_{1,2},\ x_{1,3}]}{q}\right\}$$

Out[∘]=

$$\left\{\frac{w[x_{1,2},\ x_{1,4},\ x_{1,3}]}{q^2},\ \frac{w[x_{1,3},\ x_{1,2},\ x_{1,4}]}{q^2}\right\}$$

In[∘]:=

$$\text{AllSingleSteps}@\left\{\frac{w[x_{1,2},\ x_{1,4},\ x_{1,3}]}{q^2},\ \frac{w[x_{1,3},\ x_{1,2},\ x_{1,4}]}{q^2}\right\}$$

Out[∘]=

$$\left\{\frac{w[x_{1,2},\ x_{1,3},\ x_{1,4}]}{q^3}\right\}$$

In[∘]:=

```
Straightenings[ℰ_] := FixedPoint[(Echo[Test /@ #]; AllSingleSteps[#]) &, ℰ, 50]
```

In[∘]:=

$$\text{Straightenings}[w[x_{1,3},\ x_{1,2}]]$$

Out[∘]=

$$\left\{\frac{w[x_{1,2},\ x_{1,3}]}{q}\right\}$$

In[∘]:=

$$\text{Straightenings}[w[x_{1,4},\ x_{1,3},\ x_{1,2}]]$$

Out[∘]=

$$\left\{\frac{w[x_{1,2},\ x_{1,3},\ x_{1,4}]}{q^3}\right\}$$

In[∘]:=

$$\text{Straighten}@w[x_{2,4},\ x_{1,3},\ x_{1,2}]$$

Out[∘]=

$$-\frac{w[x_{1,3},\ x_{1,4}]}{q}+w[x_{1,2},\ x_{1,3},\ x_{2,4}]+\frac{w[x_{1,2},\ x_{2,3},\ x_{1,4}]}{q}-q\,w[x_{1,2},\ x_{2,3},\ x_{1,4}]$$

In[∘]:=

$$\text{Straightenings}[w[x_{2,4},\ x_{1,3},\ x_{1,2}]]$$

```
» w[0, 0, 0]
» {1, 1}
» {1, 1, 1, 1}
» {1, 1, 1, 1, 1, 1, 1, 1, 1}
» {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
» {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
» {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Out[∘]=

$$\left\{-\frac{w[x_{1,3},\ x_{1,4}]}{q}+w[x_{1,2},\ x_{1,3},\ x_{2,4}]+\frac{w[x_{1,2},\ x_{2,3},\ x_{1,4}]}{q}-q\,w[x_{1,2},\ x_{2,3},\ x_{1,4}],\right.$$

$$-w[x_{1,4},\ x_{1,3}]+w[x_{1,2},\ x_{1,3},\ x_{2,4}]+\frac{w[x_{1,2},\ x_{2,3},\ x_{1,4}]}{q}-q\,w[x_{1,2},\ x_{2,3},\ x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{2,4}, x_{1,3}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + q\,w[x_{1,3}, x_{1,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + w[x_{1,3}, x_{2,4}, x_{1,2}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} - q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - w[x_{2,3}, x_{1,2}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} - q\,w[x_{1,3}, x_{1,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} + w[x_{1,3}, x_{2,4}, x_{1,2}] - w[x_{2,3}, x_{1,2}, x_{1,4}],$$

$$w[x_{1,2}, x_{1,3}, x_{2,4}] - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$-q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$q\,w[x_{1,3}, x_{1,4}] - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + w[x_{1,3}, x_{2,4}, x_{1,2}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$-q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$-q\,w[x_{1,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$w[x_{1,2}, x_{1,3}, x_{2,4}] - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$-q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$q\,w[x_{1,3}, x_{1,4}] - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + w[x_{1,3}, x_{2,4}, x_{1,2}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$-q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$-q\,w[x_{1,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} - q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} - q\,w[x_{1,3}, x_{1,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} + w[x_{1,3}, x_{2,4}, x_{1,2}] - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2} - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-q\,w[x_{1,3},\,x_{1,4}]+q\,w[x_{1,3},\,x_{1,2},\,x_{2,4}]+\frac{w[x_{2,3},\,x_{1,2},\,x_{1,4}]}{q^2}-q\,w[x_{2,3},\,x_{1,4},\,x_{1,2}],$$

$$-q\,w[x_{1,3},\,x_{1,4}]+w[x_{1,2},\,x_{1,3},\,x_{2,4}]+\frac{w[x_{2,3},\,x_{1,4},\,x_{1,2}]}{q}-q\,w[x_{2,3},\,x_{1,4},\,x_{1,2}],$$

$$-q\,w[x_{1,3},\,x_{1,4}]+q\,w[x_{1,3},\,x_{1,2},\,x_{2,4}]+\frac{w[x_{2,3},\,x_{1,4},\,x_{1,2}]}{q}-q\,w[x_{2,3},\,x_{1,4},\,x_{1,2}]\Big\}$$

In[∘]:= **AllSingleSteps[w[$x_{2,4}$, $x_{1,3}$]]**

Out[∘]=

$$\Big\{w[x_{1,3},\,x_{2,4}]+\frac{w[x_{2,3},\,x_{1,4}]}{q}-q\,w[x_{2,3},\,x_{1,4}]\Big\}$$

In[∘]:= **AllSingleSteps**$\Big[\Big\{w[x_{1,3},\,x_{2,4}]+\frac{w[x_{2,3},\,x_{1,4}]}{q}-q\,w[x_{2,3},\,x_{1,4}]\Big\}\Big]$

Out[∘]=

$$\Big\{w[x_{1,3},\,x_{2,4}]+\frac{w[x_{2,3},\,x_{1,4}]}{q}-q\,w[x_{2,3},\,x_{1,4}]\Big\}$$

In[∘]:= **AllSingleSteps[w[$x_{2,4}$, $x_{1,3}$, $x_{1,2}$]]**

Out[∘]=

$$\Big\{w[x_{1,3},\,x_{2,4},\,x_{1,2}]+\frac{w[x_{2,3},\,x_{1,4},\,x_{1,2}]}{q}-q\,w[x_{2,3},\,x_{1,4},\,x_{1,2}],\ \frac{w[x_{2,4},\,x_{1,2},\,x_{1,3}]}{q}\Big\}$$

In[∘]:= **AllSingleSteps /@**

$$\Big\{w[x_{1,3},\,x_{2,4},\,x_{1,2}]+\frac{w[x_{2,3},\,x_{1,4},\,x_{1,2}]}{q}-q\,w[x_{2,3},\,x_{1,4},\,x_{1,2}],\ \frac{w[x_{2,4},\,x_{1,2},\,x_{1,3}]}{q}\Big\}$$

Out[∘]=

$$\Big\{\Big\{w[x_{1,3},\,x_{2,4},\,x_{1,2}]-w[x_{2,3},\,x_{1,2},\,x_{1,4}]+\frac{w[x_{2,3},\,x_{1,4},\,x_{1,2}]}{q},$$

$$w[x_{1,3},\,x_{2,4},\,x_{1,2}]+\frac{w[x_{2,3},\,x_{1,2},\,x_{1,4}]}{q^2}-q\,w[x_{2,3},\,x_{1,4},\,x_{1,2}],$$

$$-q\,w[x_{1,3},\,x_{1,4}]+q\,w[x_{1,3},\,x_{1,2},\,x_{2,4}]+\frac{w[x_{2,3},\,x_{1,4},\,x_{1,2}]}{q}-q\,w[x_{2,3},\,x_{1,4},\,x_{1,2}]\Big\},$$

$$\{-w[x_{1,4},\,x_{1,3}]+w[x_{1,2},\,x_{2,4},\,x_{1,3}]\}\Big\}$$

In[∘]:= **Straightenings[w[$x_{2,4}$, $x_{1,3}$, $x_{1,2}$]]**

Out[∘]=

$$\Big\{-\frac{w[x_{1,3},\,x_{1,4}]}{q}+w[x_{1,2},\,x_{1,3},\,x_{2,4}]+\frac{w[x_{1,2},\,x_{2,3},\,x_{1,4}]}{q}-q\,w[x_{1,2},\,x_{2,3},\,x_{1,4}],$$

$$-w[x_{1,4},\,x_{1,3}]+w[x_{1,2},\,x_{1,3},\,x_{2,4}]+\frac{w[x_{1,2},\,x_{2,3},\,x_{1,4}]}{q}-q\,w[x_{1,2},\,x_{2,3},\,x_{1,4}],$$

$$-\frac{w[x_{1,3},\,x_{1,4}]}{q}+w[x_{1,2},\,x_{2,4},\,x_{1,3}],$$

$$-\frac{w[x_{1,3},\,x_{1,4}]}{q}+\frac{w[x_{1,2},\,x_{2,3},\,x_{1,4}]}{q}-q\,w[x_{1,2},\,x_{2,3},\,x_{1,4}]+q\,w[x_{1,3},\,x_{1,2},\,x_{2,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + q\,w[x_{1,3}, x_{1,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + w[x_{1,3}, x_{2,4}, x_{1,2}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} - q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - w[x_{2,3}, x_{1,2}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} - q\,w[x_{1,3}, x_{1,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} + w[x_{1,3}, x_{2,4}, x_{1,2}] - w[x_{2,3}, x_{1,2}, x_{1,4}],$$

$$w[x_{1,2}, x_{1,3}, x_{2,4}] - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$-q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$q\,w[x_{1,3}, x_{1,4}] - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + w[x_{1,3}, x_{2,4}, x_{1,2}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$-q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$-q\,w[x_{1,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2},$$

$$w[x_{1,2}, x_{1,3}, x_{2,4}] - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$-q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$q\,w[x_{1,3}, x_{1,4}] - q\,w[x_{1,2}, x_{2,3}, x_{1,4}] + w[x_{1,3}, x_{2,4}, x_{1,2}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$-q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$-q\,w[x_{1,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] - w[x_{2,3}, x_{1,2}, x_{1,4}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q},$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} - q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} - q\,w[x_{1,3}, x_{1,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} + w[x_{1,3}, x_{2,4}, x_{1,2}] - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2} - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-q\,w[x_{1,3}, x_{1,4}] + q\,w[x_{1,3}, x_{1,2}, x_{2,4}] + \frac{w[x_{2,3}, x_{1,2}, x_{1,4}]}{q^2} - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-q\,w[x_{1,3}, x_{1,4}] + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{2,3}, x_{1,4}, x_{1,2}]}{q} - q\,w[x_{2,3}, x_{1,4}, x_{1,2}],$$

$$-q\,w[x_{1,3},\ x_{1,4}]+q\,w[x_{1,3},\ x_{1,2},\ x_{2,4}]+\frac{w[x_{2,3},\ x_{1,4},\ x_{1,2}]}{q}-q\,w[x_{2,3},\ x_{1,4},\ x_{1,2}]\Big\}$$

In[∘]:=
```
Test[ℰ_] := Length@Union[Straighten /@ AllSingleSteps[ℰ]]
```

In[∘]:=
$$\texttt{Test}[w[x_{2,4},\ x_{1,3},\ x_{1,2}]]$$

Out[∘]=
```
1
```

In[∘]:=
$$\texttt{Straighten /@ AllSingleSteps}[w[x_{2,4},\ x_{1,3},\ x_{1,2}]]$$

Out[∘]=
$$\Big\{-\frac{w[x_{1,3},\ x_{1,4}]}{q}+w[x_{1,2},\ x_{1,3},\ x_{2,4}]+\frac{w[x_{1,2},\ x_{2,3},\ x_{1,4}]}{q}-q\,w[x_{1,2},\ x_{2,3},\ x_{1,4}],$$
$$-\frac{w[x_{1,3},\ x_{1,4}]}{q}+w[x_{1,2},\ x_{1,3},\ x_{2,4}]+\frac{w[x_{1,2},\ x_{2,3},\ x_{1,4}]}{q}-q\,w[x_{1,2},\ x_{2,3},\ x_{1,4}]\Big\}$$

In[∘]:=
$$\texttt{AllSingleSteps}[w[x_{2,4},\ x_{1,3},\ x_{1,2}]]$$

Out[∘]=
$$\Big\{w[x_{1,3},\ x_{2,4},\ x_{1,2}]+\frac{w[x_{2,3},\ x_{1,4},\ x_{1,2}]}{q}-q\,w[x_{2,3},\ x_{1,4},\ x_{1,2}],\ \frac{w[x_{2,4},\ x_{1,2},\ x_{1,3}]}{q}\Big\}$$

In[∘]:=
$$\texttt{Test}\Big[w[x_{1,3},\ x_{2,4},\ x_{1,2}]+\frac{w[x_{2,3},\ x_{1,4},\ x_{1,2}]}{q}-q\,w[x_{2,3},\ x_{1,4},\ x_{1,2}]\Big]$$

Out[∘]=
```
1
```

In[∘]:=
$$\texttt{Test}\Big[\frac{w[x_{2,4},\ x_{1,2},\ x_{1,3}]}{q}\Big]$$

Out[∘]=
```
1
```

In[∘]:=
$$\texttt{Test /@ AllSingleSteps}\Big[w[x_{1,3},\ x_{2,4},\ x_{1,2}]+\frac{w[x_{2,3},\ x_{1,4},\ x_{1,2}]}{q}-q\,w[x_{2,3},\ x_{1,4},\ x_{1,2}]\Big]$$

Out[∘]=
```
{1, 1, 1}
```

In[∘]:=
$$\texttt{Test /@ AllSingleSteps}\Big[\frac{w[x_{2,4},\ x_{1,2},\ x_{1,3}]}{q}\Big]$$

Out[∘]=
```
{1}
```

In[∘]:=
$$\texttt{Straighten /@ Straightenings}[w[x_{2,4},\ x_{1,3},\ x_{1,2}]]$$

```
» w[0, 0, 0]
» {1, 1}
» {1, 1, 1, 1}
» {1, 1, 1, 1, 1, 1, 1, 1, 1}
» {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
» {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
» {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Out[∘]=

$$\begin{gathered}
\Big\{-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}], \\
-\frac{w[x_{1,3}, x_{1,4}]}{q}+w[x_{1,2}, x_{1,3}, x_{2,4}]+\frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q}-q\,w[x_{1,2}, x_{2,3}, x_{1,4}],
\end{gathered}$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\, w[x_{1,2}, x_{2,3}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\, w[x_{1,2}, x_{2,3}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\, w[x_{1,2}, x_{2,3}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\, w[x_{1,2}, x_{2,3}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\, w[x_{1,2}, x_{2,3}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\, w[x_{1,2}, x_{2,3}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\, w[x_{1,2}, x_{2,3}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\, w[x_{1,2}, x_{2,3}, x_{1,4}],$$

$$-\frac{w[x_{1,3}, x_{1,4}]}{q} + w[x_{1,2}, x_{1,3}, x_{2,4}] + \frac{w[x_{1,2}, x_{2,3}, x_{1,4}]}{q} - q\, w[x_{1,2}, x_{2,3}, x_{1,4}]\}$$