Proof of the Conversion Theorem

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$E_l(\lambda;\,\omega) := \exp(l\lambda)\exp(\iota\omega)$ (1)

$$(\lambda\colon H \to FL(T);\ \omega \in FL(T)) \mapsto \exp_{\#}(e_s(\lambda;\omega)), \tag{12}$$

where $e_s(\lambda;\omega)$ is the sum over $x \in H$ of planting λ_x with its root on strand x and its leafs on the strands in T so that the labels match but at an arbitrary order on any T strand, plus the result of planting ω on just the T strands so that the labels match but at an arbitrary order on any T strand.

Given $(\lambda;\,\omega)$ as above and a scalar t, let $\Gamma(\lambda,t) = \{s \to \gamma_s(t)\} \in FL(S)^S$ be the unique solution of the system of ordinary differential equations

$$\forall s \in S \qquad \frac{d\gamma_s(t)}{dt} = \gamma_s(t) /\!\!/ e^{-\operatorname{der}(t\lambda)} /\!\!/ \frac{\operatorname{ad}\gamma_s(t)}{e^{\operatorname{ad}\gamma_s(t)}-1}; \qquad \gamma_s(0)=0, \qquad (2)$$

where $\operatorname{der}(t\lambda)$ denotes the tangential derivation in $\mathfrak{tder}_S$ corresponding to $t\lambda$ under the identification $FL(S)^S \simeq \mathfrak{a}_S \oplus \mathfrak{tder}_S$. Let $\Gamma(\lambda) := \Gamma(\lambda,1)$.

Theorem 2.15. *$\omega' = \Gamma(\lambda)$ and $\omega' = \omega$. Namely,*

$$E_l(\lambda;\,\omega) = E_s(\Gamma(\lambda);\,\omega) \qquad (3)$$

Proof: E_l & E_s both plant wheels at the top, and as tails commute, they do so in the same way. So $\omega' = \omega$ and we only need to show (3) at tree level (meaning, modulo wheels). We will show that for every scalar t,

$$\exp(l(t\lambda)) = \exp_{\#}(e_s(\Gamma(\lambda,t))) /\!\!/ f^{-1}; \qquad \text{[eq:treelevel]} \qquad (4)$$

the desired result is the specialization of [eq:treelevel] to $t=1$. It is clear that [ ] holds for some unique $\Gamma_0 = \{s \to \gamma_{0s}(t)\}$, that $\gamma_s(0)=0$, and that each coefficient of each $\gamma_{0s}(t)$ depends polynomially on t, and hence it is enough to show that Γ_0 satisfies the differential equation in (2).

Differentiating (4) w.r.t. t:

LHS: $l(\lambda)e^{l(t\lambda)} = e^{l(t\lambda)}l(\lambda)$

RHS: $\left[e_s(\Gamma_0'(\lambda,t)) /\!\!/ \frac{e^{\operatorname{ad}\Gamma_0}-1}{\ldots}\right)_{\#} e^{e_s(\Gamma_0(\lambda,t))}\Big] /\!\!/ f^{-1}$

RHS: $\left[e_S(\Gamma_0^{-1}(\lambda,t) // \frac{e^{\mathrm{ad}\Gamma_0}-1}{\mathrm{ad}\Gamma_0}) \# e_\#^{e_S(\Gamma_0^{-1}(\lambda,t))}\right] // \delta^{-1}$

<u>Claim</u> $(P \# e_\#^Q) // \delta^{-1} \overset{?}{=} \delta^{-1}(e^{\partial_Q} P) \cdot \delta^{-1}(e_\#^Q)$ in $\mathcal{A}^w(\uparrow_S)$

Primitive Exponential (for $P, Q \in FL(S)^S$)

<u>Lemma</u> $[P,Q]_\# // \delta^{-1} \overset{?}{=} [\delta^{-1}P, \delta^{-1}Q]_{\mathcal{A}^w} + \partial_P Q - \partial_Q P$

<u>Better Lemma</u> $(P \# Q) // \delta^{-1} \overset{?}{=} (\delta^{-1}P)(\delta^{-1}Q) + \partial_Q P$

$P \# e_\#^Q = P' e_\#^Q$

$$\begin{array}{c} \mathcal{A}^w(S;S) \quad \# \\ \downarrow \delta^{-1} \\ \mathcal{A}^w(\uparrow_S) \end{array}$$

$P \# Q = PQ + \partial_P Q$

$P \# Q \# Q = (PQ + \partial_P Q) \# Q$

$=$

Recycling:

$= e_S(\Gamma_0^{-1}(\lambda,t) // \frac{e^{\mathrm{ad}\Gamma_0}-1}{\mathrm{ad}\Gamma_0}) \# e^{l(t\lambda)}$