

Proof of the Conversion Theorem

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$$E_l(\lambda; \omega) := \exp(l\lambda) \exp(\omega) \quad (1)$$

$$(\lambda: H \rightarrow FL(T); \omega \in FL(T)) \mapsto \exp_{\#}(e_s(\lambda; \omega)), \quad (12)$$

where $e_s(\lambda; \omega)$ is the sum over $x \in H$ of planting λ_x with its root on strand x and its leafs on the strands in T so that the labels match but at an arbitrary order on any T strand, plus the result of planting ω on just the T strands so that the labels match but at an arbitrary order on any T strand.

Given $(\lambda; \omega)$ as above and a scalar t , let $\Gamma(\lambda, t) = \{s \rightarrow \gamma_s(t)\} \in FL(S)^S$ be the unique solution of the system of ordinary differential equations

$$\forall s \in S \quad \frac{d\gamma_s(t)}{dt} = \gamma_s(t) // e^{-\text{der}(t\lambda)} // \frac{\text{ad } \gamma_s(t)}{e^{\text{ad } \gamma_s(t)} - 1}; \quad \gamma_s(0) = 0, \quad (2)$$

where $\text{der}(t\lambda)$ denotes the tangential derivation in \mathfrak{tder}_S corresponding to $t\lambda$ under the identification $FL(S)^S \simeq \mathfrak{a}_S \oplus \mathfrak{tder}_S$. Let $\Gamma(\lambda) := \Gamma(\lambda, 1)$.

Theorem 2.15. $\omega' = \Gamma(\lambda)$ and $\omega' = \omega$. Namely,

$$E_l(\lambda; \omega) = E_s(\Gamma(\lambda); \omega) \quad (3)$$

Proof E_l & E_s both plant wheels at the top, and as tails commute, they do so in the same way. So $\omega' = \omega$ and we only need to show (3) at tree level (meaning, modulo wheels). We will show that for every scalar t ,

$$\exp(l(t\lambda)) = \exp_{\#}(e_s(\Gamma(\lambda, t))) \quad [\text{eq: tree level}] \quad (4)$$

The desired result is the specialization of [eq: tree level] to $t=1$. It is clear that [] holds for some unique $\Gamma_0 = \{s \rightarrow \gamma_{0s}(t)\}$, that $\gamma_{0s}(0) = 0$, and that each coefficient of each $\gamma_{0s}(t)$ depends polynomially on t , and hence it is enough to show that Γ_0 satisfies the differential equation in (2).

Differentiating (4) w.r.t t :

$$\text{LHS: } l(\lambda) e^{l(t\lambda)} = e^{l(t\lambda)} l(t\lambda)$$

$$\text{RHS: } \left[e_s(\Gamma_0(\lambda, t)) // e^{\text{ad } \Gamma_0} - 1 \right] // \# \left[e_s(\Gamma_0(\lambda, t)) \right] // f^{-1}$$

RHS: $\left[\ell_s(\Gamma'_0(\lambda, t)) // \frac{e^{\text{ad}_{\Gamma'_0} - 1}}{\text{ad}_{\Gamma'_0}} \right] \# \ell_{\#}^{\ell_s(\Gamma'_0(\lambda, t))} // f^{-1}$

claim $(P \# \ell_{\#}^Q) // f^{-1} \cong f^{-1}(e^{\partial_Q} P) \cdot f^{-1}(\ell_{\#}^Q)$
 Primitive \uparrow Exponential \downarrow (for $P, Q \in FL(S)^S$)
 in $\mathcal{A}^w(\Gamma_S)$

Lemma $[P, Q]_{\#} // f^{-1} \cong [f^{-1}P, f^{-1}Q]_{\mathcal{A}^w} + \partial_P Q - \partial_Q P$

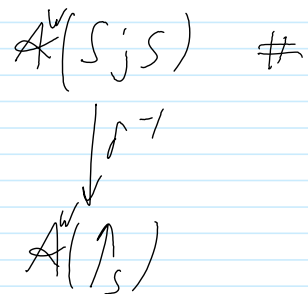
Better Lemma $(P \# Q) // f^{-1} \cong (f^{-1}P)(f^{-1}Q) + \partial_Q P$

$$P \# \ell_{\#}^Q = P' \ell_{\#}^Q$$

$$P \# Q = PQ + \partial_P Q$$

$$P \# Q \# Q = (PQ + \partial_P Q) \# Q$$

=



Recycling:

$$= \ell_s(\Gamma'_0(\lambda, t)) // \frac{e^{\text{ad}_{\Gamma'_0} - 1}}{\text{ad}_{\Gamma'_0}} \# \ell^{\ell(\lambda, t)}$$