

Pensieve header: Fuller output of bch. Even fuller: <bch-to-20.m>, <bch-to-22.m>, <bch-to-23.m>.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

```
<< FreeLie.m;
```

```
bch = BCH[LW@x, LW@y]
```

FreeLie` implements / extends

```
{*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Λ, σ, ħ, −, −}.
```

FreeLie` is in the public domain. Dror Bar-Natan is committed to

support it within reason until July 15, 2022. This is version 150814.

$$LS[\overline{x+y}, \frac{\overline{xy}}{2}, \frac{1}{12} \overline{xx\overline{xy}} + \frac{1}{12} \overline{x\overline{yy}}, \dots]$$

bch@{10} // Timing

$$\{0.34375, LS[\overline{x+y}, \frac{\overline{xy}}{2}, \frac{1}{12} \overline{xx\overline{xy}} + \frac{1}{12} \overline{x\overline{yy}}, \frac{1}{24} \overline{xx\overline{xyy}},$$

$$- \frac{1}{720} \overline{xxx\overline{xyy}} + \frac{1}{180} \overline{xx\overline{xyy}} + \frac{1}{180} \overline{x\overline{xyy}y} + \frac{1}{120} \overline{x\overline{xy}y} + \frac{1}{360} \overline{xx\overline{xy}y} - \frac{1}{720} \overline{x\overline{yy}yy},$$

$$- \frac{\overline{xxx\overline{xyy}}}{1440} + \frac{1}{360} \overline{xx\overline{xyy}y} + \frac{1}{240} \overline{x\overline{xy}y} + \frac{1}{720} \overline{xx\overline{xy}y} - \frac{\overline{x\overline{xyy}yy}}{1440},$$

$$\frac{\overline{xxxx\overline{xyy}}}{30240} - \frac{\overline{xxxx\overline{xyy}}}{5040} + \frac{\overline{xxx\overline{xyy}y}}{3780} + \frac{\overline{xx\overline{xy}y}}{1680} + \frac{\overline{xx\overline{xy}y}}{10080} + \frac{\overline{xx\overline{xyy}yy}}{3780} +$$

$$\frac{13 \overline{xx\overline{xy}y}}{15120} + \frac{\overline{xx\overline{xyy}y}}{1260} - \frac{\overline{x\overline{xyy}yy}}{5040} + \frac{\overline{x\overline{xy}y}}{1260} - \frac{\overline{x\overline{xy}y}}{2016} + \frac{\overline{xx\overline{xy}y}}{2016} +$$

$$\frac{\overline{xx\overline{xy}y}}{10080} + \frac{\overline{x\overline{xyy}y}}{10080} - \frac{\overline{x\overline{xyy}y}}{1512} - \frac{\overline{x\overline{xy}y}}{5040} - \frac{\overline{xx\overline{xy}y}}{5040} + \frac{\overline{x\overline{xyy}yy}}{30240},$$

$$\frac{\overline{xxxx\overline{xyy}}}{60480} - \frac{\overline{xxxx\overline{xyy}y}}{10080} - \frac{\overline{xxx\overline{xy}y}}{20160} - \frac{\overline{xxx\overline{xy}y}}{15120} + \frac{23 \overline{xxx\overline{xyy}yy}}{120960} +$$

$$\frac{13 \overline{xx\overline{xy}y}}{30240} + \frac{\overline{xxx\overline{xyy}y}}{2520} - \frac{\overline{xx\overline{xyy}yy}}{10080} + \frac{\overline{x\overline{xy}y}}{2520} -$$

$$\frac{\overline{x\overline{xy}y}}{4032} + \frac{\overline{xx\overline{xy}y}}{4032} + \frac{\overline{xx\overline{xy}y}}{20160} + \frac{\overline{xx\overline{xyy}y}}{20160} -$$

$\frac{\overbrace{\text{x x x x x x x x} \overbrace{\text{x y y}}}{}}{2\ 419\ 200} + \frac{\overbrace{\text{x x x x x x} \overbrace{\text{x y y y}}}{}}{302\ 400} - \frac{\overbrace{\text{x x x x x x y} \overbrace{\text{x y y}}}{}}{604\ 800} + \frac{\overbrace{\text{x x x x x} \overbrace{\text{x y} \overbrace{\text{x y}}}}}{403\ 200} -$
$\frac{37 \overbrace{\text{x x x x x} \overbrace{\text{x y y y y}}}{}}{3\ 628\ 800} - \frac{\overbrace{\text{x x x x x y} \overbrace{\text{x y y y}}}{}}{56\ 700} - \frac{\overbrace{\text{x x x x x} \overbrace{\text{x y y} \overbrace{\text{x y}}}}}{37\ 800} + \frac{\overbrace{\text{x x x x} \overbrace{\text{x y y y y y}}}{}}{67\ 200} -$
$\frac{17 \overbrace{\text{x x x x y} \overbrace{\text{x y x y}}}{}}{604\ 800} + \frac{11 \overbrace{\text{x x x x y} \overbrace{\text{x y y y y}}}{}}{241\ 920} - \frac{\overbrace{\text{x x x x x y} \overbrace{\text{x x y y}}}{}}{75\ 600} -$
$\frac{\overbrace{\text{x x x x} \overbrace{\text{x y} \overbrace{\text{x x y}}}}}{241\ 920} + \frac{\overbrace{\text{x x x x} \overbrace{\text{x y y} \overbrace{\text{x y y}}}}}{43\ 200} + \frac{29 \overbrace{\text{x x x x} \overbrace{\text{x y y y} \overbrace{\text{x y}}}}}{453\ 600} + \frac{\overbrace{\text{x x x} \overbrace{\text{x y y} \overbrace{\text{x y y y}}}}}{40\ 320} +$
$\frac{\overbrace{\text{x x x} \overbrace{\text{x y} \overbrace{\text{x y x y}}}}}{50\ 400} - \frac{37 \overbrace{\text{x x x} \overbrace{\text{x y y y y y y}}}{}}{3\ 628\ 800} + \frac{\overbrace{\text{x x x y} \overbrace{\text{x y} \overbrace{\text{x y y y}}}}}{12\ 096} - \frac{23 \overbrace{\text{x x x y} \overbrace{\text{x y y y y y}}}{}}{604\ 800} +$
$\frac{23 \overbrace{\text{x x x} \overbrace{\text{x y} \overbrace{\text{x x y y y}}}}}{604\ 800} + \frac{\overbrace{\text{x x x x y} \overbrace{\text{x x y y}}}{}}{75\ 600} + \frac{\overbrace{\text{x x x x} \overbrace{\text{x y y} \overbrace{\text{x x y}}}}}{50\ 400} + \frac{\overbrace{\text{x x x x y} \overbrace{\text{x y y} \overbrace{\text{x y}}}}}{67\ 200} -$
$\frac{\overbrace{\text{x x x} \overbrace{\text{x y y y} \overbrace{\text{x y}}}}}{30\ 240} - \frac{\overbrace{\text{x x x} \overbrace{\text{x y y y y} \overbrace{\text{x y}}}}}{15\ 120} - \frac{11 \overbrace{\text{x x} \overbrace{\text{x y y} \overbrace{\text{x y y y y}}}}}{241\ 920} - \frac{\overbrace{\text{x x x y} \overbrace{\text{x y y} \overbrace{\text{x y y}}}}}{40\ 320} -$
$\frac{17 \overbrace{\text{x x x} \overbrace{\text{x y y} \overbrace{\text{x y x y}}}}}{201\ 600} + \frac{\overbrace{\text{x x} \overbrace{\text{x y y y y y y y}}}{}}{302\ 400} + \frac{\overbrace{\text{x x y} \overbrace{\text{x y x y} \overbrace{\text{x y}}}}}{20\ 160} - \frac{\overbrace{\text{x x y} \overbrace{\text{x y} \overbrace{\text{x y y y}}}}}{15\ 120} -$
$\frac{\overbrace{\text{x x y} \overbrace{\text{x y y} \overbrace{\text{x y y y}}}}}{30\ 240} + \frac{\overbrace{\text{x x y} \overbrace{\text{x y y y y y y}}}{}}{86\ 400} + \frac{\overbrace{\text{x x x y} \overbrace{\text{x x y} \overbrace{\text{x y y}}}}}{50\ 400} - \frac{\overbrace{\text{x x x y} \overbrace{\text{x x y y y y}}}{}}{34\ 560} -$
$\frac{\overbrace{\text{x x x y} \overbrace{\text{x x y y} \overbrace{\text{x y}}}}}{16\ 800} + \frac{\overbrace{\text{x x x x y} \overbrace{\text{x x} \overbrace{\text{x y y}}}}}{86\ 400} + \frac{\overbrace{\text{x x x x} \overbrace{\text{x x y} \overbrace{\text{x x y}}}}}{604\ 800} + \frac{\overbrace{\text{x x x} \overbrace{\text{x y y} \overbrace{\text{x x y y}}}}}{134\ 400} -$
$\frac{\overbrace{\text{x x x} \overbrace{\text{x y y y} \overbrace{\text{x x y}}}}}{43\ 200} + \frac{\overbrace{\text{x x} \overbrace{\text{x y y} \overbrace{\text{x x y y y}}}}}{120\ 960} + \frac{\overbrace{\text{x x} \overbrace{\text{x y y} \overbrace{\text{x y} \overbrace{\text{x y y}}}}}{40\ 320} - \frac{\overbrace{\text{x x x y} \overbrace{\text{x y y y} \overbrace{\text{x y}}}}}{20\ 160} -$
$\frac{\overbrace{\text{x x} \overbrace{\text{x x y} \overbrace{\text{x y} \overbrace{\text{x x y}}}}}{60\ 480} + \frac{\overbrace{\text{x x} \overbrace{\text{x y y y} \overbrace{\text{x y y y}}}}}{181\ 440} + \frac{\overbrace{\text{x x} \overbrace{\text{x y y y y} \overbrace{\text{x y y}}}}}{120\ 960} + \frac{\overbrace{\text{x x} \overbrace{\text{x y y y y y} \overbrace{\text{x y}}}}}{43\ 200} +$
$\frac{\overbrace{\text{x} \overbrace{\text{x y y} \overbrace{\text{x y y y y y}}}}}{67\ 200} + \frac{\overbrace{\text{x x y} \overbrace{\text{x y y y y} \overbrace{\text{x y}}}}}{30\ 240} - \frac{\overbrace{\text{x x x y} \overbrace{\text{x y x} \overbrace{\text{x y y}}}}}{50\ 400} + \frac{\overbrace{\text{x x} \overbrace{\text{x y y} \overbrace{\text{x y y} \overbrace{\text{x y}}}}}{40\ 320} +$

$$\left. \begin{aligned} & \frac{\overline{\overline{\overline{\overline{\overline{\overline{\overline{xx \ x\bar{y}y \ y\bar{x}\bar{y}\bar{x}\bar{y}}}}}}}}}}}{20160} + \frac{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{x \ x\bar{y}y \ y \ \bar{x}\bar{y}\bar{y}y}}}}}}}}}}}}}{120960} + \frac{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{x \ \bar{x}\bar{y} \ \bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}}}}}}}}}}}}}}}{100800} - \frac{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{x \ \bar{x}\bar{y}y \ y\bar{y}y\bar{y}y}}}}}}}}}}}}}}}{2419200}, \dots \end{aligned} \right\}$$

The same thing, copy-paste ready and machine readable:

Sum[bch[k], {k, 10}] // InputForm

```
LW[x] + LW[y] + LW[x, y]/2 + LW[x, x, y]/12 + LW[x, y, y]/12 + LW[x, x, y, y]/24 -
LW[x, x, x, x, y]/720 + LW[x, x, x, y, y]/180 + LW[x, x, y, x, y]/360 +
LW[x, x, y, y, y]/180 + LW[x, y, x, y, y]/120 - LW[x, y, y, y, y]/720 -
LW[x, x, x, x, y, y]/1440 + LW[x, x, x, y, x, y]/720 + LW[x, x, x, y, y, y]/360 +
LW[x, x, y, x, y, y]/240 - LW[x, x, y, y, y, y]/1440 + LW[x, x, x, x, x, x, y]/3024
LW[x, x, x, x, x, y, y]/5040 + LW[x, x, x, x, y, x, y]/10080 + LW[x, x, x, x, y, y,
LW[x, x, x, y, x, x, y]/10080 + LW[x, x, x, y, x, y, y]/1680 + LW[x, x, x, y, y, x,
LW[x, x, x, y, y, y, y]/3780 + LW[x, x, y, x, x, y, y]/2016 - LW[x, x, y, x, y, x,
(13*LW[x, x, y, x, y, y, y])/15120 + LW[x, x, y, y, x, y, y]/10080 -
LW[x, x, y, y, y, x, y]/1512 - LW[x, x, y, y, y, y, y]/5040 + LW[x, y, x, y, x, y,
LW[x, y, x, y, y, y, y]/2016 - LW[x, y, y, x, y, y, y]/5040 + LW[x, y, y, y, y, y,
LW[x, x, x, x, x, x, y, y]/60480 - LW[x, x, x, x, x, y, x, y]/15120 -
LW[x, x, x, x, x, y, y, y]/10080 + LW[x, x, x, x, y, x, x, y]/20160 -
LW[x, x, x, x, y, x, y, y]/20160 + LW[x, x, x, x, y, y, x, y]/2520 +
(23*LW[x, x, x, x, y, y, y, y])/120960 + LW[x, x, x, y, x, x, y, y]/4032 -
LW[x, x, x, y, x, y, x, y]/10080 + (13*LW[x, x, x, y, x, y, y, y])/30240 +
LW[x, x, x, y, y, x, y, y]/20160 - LW[x, x, x, y, y, y, x, y]/3024 -
LW[x, x, x, y, y, y, y, y]/10080 + LW[x, x, y, x, y, x, y, y]/2520 -
LW[x, x, y, x, y, y, y, y]/4032 - LW[x, x, y, y, x, y, y, y]/10080 +
LW[x, x, y, y, y, y, y, y]/60480 - LW[x, x, x, x, x, x, x, x, y]/1209600 +
LW[x, x, x, x, x, x, x, y, y]/151200 - LW[x, x, x, x, x, x, y, x, y]/100800 -
LW[x, x, x, x, x, x, y, y, y]/56700 - LW[x, x, x, x, x, y, x, y, y]/43200 +
LW[x, x, x, x, x, y, y, x, y]/75600 + LW[x, x, x, x, x, y, y, y, y]/75600 +
LW[x, x, x, x, y, x, x, x, y]/302400 + LW[x, x, x, x, y, x, x, y, y]/67200 +
LW[x, x, x, x, y, x, y, x, y]/43200 + (11*LW[x, x, x, x, y, x, y, y, y])/302400 +
LW[x, x, x, x, y, y, x, x, y]/25200 + (11*LW[x, x, x, x, y, y, x, y, y])/201600 +
(11*LW[x, x, x, x, y, y, y, x, y])/151200 + LW[x, x, x, x, y, y, y, y, y]/75600 +
LW[x, x, x, y, x, x, x, y, y]/43200 + LW[x, x, x, y, x, x, y, x, y]/37800 +
(23*LW[x, x, x, y, x, x, y, y, y])/302400 - LW[x, x, x, y, x, y, x, x, y]/30240 +
LW[x, x, x, y, x, y, x, y, y]/100800 + LW[x, x, x, y, x, y, y, x, y]/33600 +
LW[x, x, x, y, x, y, y, y, y]/20160 + LW[x, x, x, y, y, x, x, y, y]/67200 -
(17*LW[x, x, x, y, y, x, y, x, y])/100800 + LW[x, x, x, y, y, x, y, y, y]/30240 -
LW[x, x, x, y, y, y, x, x, y]/21600 - LW[x, x, x, y, y, y, x, y, y]/15120 -
LW[x, x, x, y, y, y, y, x, y]/7560 - LW[x, x, x, y, y, y, y, y, y]/56700 +
LW[x, x, y, x, x, y, x, y, y]/25200 - LW[x, x, y, x, x, y, y, x, y]/8400 -
LW[x, x, y, x, x, y, y, y, y]/17280 - LW[x, x, y, x, y, x, x, y, y]/25200 +
LW[x, x, y, x, y, x, y, x, y]/50400 + LW[x, x, y, x, y, x, y, y, y]/6048 -
LW[x, x, y, x, y, y, x, y, y]/20160 - LW[x, x, y, x, y, y, y, x, y]/10080 -
(23*LW[x, x, y, x, y, y, y, y, y])/302400 + LW[x, x, y, y, x, x, y, y, y]/60480 +
LW[x, x, y, y, x, y, x, y, y]/20160 + LW[x, x, y, y, x, y, y, x, y]/20160 -
(11*LW[x, x, y, y, x, y, y, y, y])/120960 + LW[x, x, y, y, y, x, y, x, y]/10080 +
LW[x, x, y, y, y, x, y, y, y]/90720 + LW[x, x, y, y, y, y, x, y, y]/60480 +
LW[x, x, y, y, y, y, y, x, y]/21600 + LW[x, x, y, y, y, y, y, y, y]/151200 +
```

```

LW[x, y, x, y, x, y, x, y, y]/10080 - LW[x, y, x, y, x, y, y, y, y]/7560 -
LW[x, y, x, y, y, x, y, y, y]/15120 + LW[x, y, x, y, y, y, x, y, y]/15120 +
LW[x, y, x, y, y, y, y, y, y]/43200 + LW[x, y, y, x, y, y, y, y, y]/33600 +
LW[x, y, y, y, x, y, y, y, y]/60480 - LW[x, y, y, y, y, y, y, y, y]/1209600 -
LW[x, x, x, x, x, x, x, x, y, y]/2419200 + LW[x, x, x, x, x, x, x, y, x, y]/403200
LW[x, x, x, x, x, x, x, y, y, y]/302400 - LW[x, x, x, x, x, x, y, x, x, y]/241920 -
LW[x, x, x, x, x, x, y, x, y, y]/604800 - LW[x, x, x, x, x, x, y, y, x, y]/37800 -
(37*LW[x, x, x, x, x, x, y, y, y, y])/3628800 + LW[x, x, x, x, x, y, x, x, x, y]/60
LW[x, x, x, x, x, y, x, x, y, y]/75600 + LW[x, x, x, x, x, y, x, y, x, y]/50400 -
LW[x, x, x, x, x, y, x, y, y, y]/56700 + LW[x, x, x, x, x, y, y, x, x, y]/50400 +
LW[x, x, x, x, x, y, y, x, y, y]/43200 + (29*LW[x, x, x, x, x, y, y, y, x, y])/4536
LW[x, x, x, x, x, y, y, y, y, y]/67200 + LW[x, x, x, x, y, x, x, x, y, y]/86400 +
LW[x, x, x, x, y, x, x, y, x, y]/75600 + (23*LW[x, x, x, x, y, x, x, y, y, y])/6048
LW[x, x, x, x, y, x, y, x, x, y]/60480 - (17*LW[x, x, x, x, y, x, y, x, y, y])/6048
LW[x, x, x, x, y, x, y, y, x, y]/67200 + (11*LW[x, x, x, x, y, x, y, y, y, y])/2419
LW[x, x, x, x, y, y, x, x, y, y]/134400 - (17*LW[x, x, x, x, y, y, x, y, x, y])/201
LW[x, x, x, x, y, y, x, y, y, y]/40320 - LW[x, x, x, x, y, y, y, x, x, y]/43200 -
LW[x, x, x, x, y, y, y, x, y, y]/30240 - LW[x, x, x, x, y, y, y, y, x, y]/15120 -
(37*LW[x, x, x, x, y, y, y, y, y, y])/3628800 + LW[x, x, x, y, x, x, y, x, y, y]/50
LW[x, x, x, y, x, x, y, y, x, y]/16800 - LW[x, x, x, y, x, x, y, y, y, y]/34560 -
LW[x, x, x, y, x, y, x, x, y, y]/50400 + LW[x, x, x, y, x, y, x, y, x, y]/100800 +
LW[x, x, x, y, x, y, x, y, y, y]/12096 - LW[x, x, x, y, x, y, y, x, y, y]/40320 -
LW[x, x, x, y, x, y, y, y, x, y]/20160 - (23*LW[x, x, x, y, x, y, y, y, y, y])/6048
LW[x, x, x, y, y, x, x, y, y, y]/120960 + LW[x, x, x, y, y, x, y, x, y, y]/40320 +
LW[x, x, x, y, y, x, y, y, x, y]/40320 - (11*LW[x, x, x, y, y, x, y, y, y, y])/2419
LW[x, x, x, y, y, y, x, y, x, y]/20160 + LW[x, x, x, y, y, y, x, y, y, y]/181440 +
LW[x, x, x, y, y, y, y, x, y, y]/120960 + LW[x, x, x, y, y, y, y, y, x, y]/43200 +
LW[x, x, x, y, y, y, y, y, y, y]/302400 + LW[x, x, y, x, y, x, y, x, y, y]/20160 -
LW[x, x, y, x, y, x, y, y, y, y]/15120 - LW[x, x, y, x, y, y, x, y, y, y]/30240 +
LW[x, x, y, x, y, y, y, x, y, y]/30240 + LW[x, x, y, x, y, y, y, y, y, y]/86400 +
LW[x, x, y, y, x, y, y, y, y, y]/67200 + LW[x, x, y, y, y, x, y, y, y, y]/120960 -
LW[x, x, y, y, y, y, y, y, y, y]/2419200

```

The same thing, machine readable and to degree 20, is in the file quoted below, in the same folder:

```
Put[Sum[bch[k], {k, 20}], "bch-to-20.m"]
```

```
TimeUsed[]
```

```
670.749
```

The same thing, machine readable and to degree 22, is in the file quoted below, in the same folder:

```
Put[Sum[bch[k], {k, 22}], "bch-to-22.m"]
```

```
TimeUsed[]
```

```
3332.34
```

The same thing, machine readable and to degree 23, is in the file quoted below, in the same folder:

```
Put[Sum[bch[k], {k, 23}], "bch-to-23.m"]
```

TimeUsed[]

17 133.2