Pensieve header: Testing the equivalence KV <-> (R4 and ||Phi_V||)==1.

In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section 1 - Introduction

Initialization

In[ ]:=
```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Initialization

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌐, ⌢}.
```

Initialization

```
FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.
```

Initialization

```
AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA,
  hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.
```

Initialization

```
AwCalculus` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150909.
```

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

In[ ]:=
```
x = LW@"x"; y = LW@"y";
{α, β, γ} = LS /@ {x + b[x, y], y - b[x, b[x, y]], x + y - 2 b[x, y]}
```

Out[ ]=
alphabetagamma

$\{LS[\overline{x}, \overline{xy}, 0, 0, \ldots], LS[\overline{y}, 0, -\overline{x\,\overline{xy}}, 0, \ldots], LS[\overline{x}+\overline{y}, -2\,\overline{xy}, 0, 0, \ldots]\}$

BracketExample

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

BracketExample

$\{LS[0, \overline{xy}, \overline{\overline{xy}\,y}, -\overline{x\,\overline{x\,\overline{xy}}}, \ldots], LS[0, 0, 0, 0, \ldots]\}$

bch

In[∘]:=
```
bch = BCH[x, y]
```

Out[∘]=
bch

$$\mathrm{LS}\left[\overline{x}+\overline{y},\ \frac{\overline{xy}}{2},\ \frac{1}{12}\,\overline{x\,\overline{xy}}+\frac{1}{12}\,\overline{\overline{xy}\,y},\ \frac{1}{24}\,\overline{x\,\overline{\overline{xy}\,y}},\ \ldots\right]$$

bch16

```
Timing@{Length@(bch@16), (bch@16)⟦1090 ;; 1092⟧}
```

bch16

$$\left\{39.5313,\ \left\{2181,\ -\frac{17\,x\,x\,x\,\overline{xy}\,\overline{xy}\,x\,\overline{xy}\,x\,x\,\overline{xy}\,y\,y}{179\,625\,600}+\frac{389\,x\,x\,x\,\overline{xy}\,\overline{xy}\,x\,\overline{xy}\,x\,\overline{xy}\,x\,\overline{xy}}{1\,320\,883\,200}+\frac{53\,x\,x\,x\,\overline{xy}\,\overline{xy}\,x\,\overline{xy}\,x\,\overline{xy}\,\overline{xy}\,y}{1\,089\,728\,640}\right\}\right\}$$

omegas

In[∘]:=
```
{ω1, ω2} = CWS /@ {cw[x] - 3 cw[y, x, x], cw[y] + cw[y, y]}
```

Out[∘]=
omegas

$$\left\{\mathrm{CWS}\left[\widehat{x},\ 0,\ -3\,\widehat{xxy},\ 0,\ \ldots\right],\ \mathrm{CWS}\left[\widehat{y},\ \widehat{yy},\ 0,\ 0,\ \ldots\right]\right\}$$

DegreeScale

```
DegreeScale[h] /@ {ω1, ω2}
```

DegreeScale

$$\left\{\mathrm{CWS}\left[h\,\widehat{x},\ 0,\ -3\,h^3\,\widehat{xxy},\ 0,\ \ldots\right],\ \mathrm{CWS}\left[h\,\widehat{y},\ h^2\,\widehat{yy},\ 0,\ 0,\ \ldots\right]\right\}$$

TangentialDerivative

In[∘]:=
```
{λ = ⟨x → α, y → β⟩, γ // Dλ}
```

Out[∘]=
TangentialDerivative

$$\left\{\left\langle \overline{x}\to \mathrm{LS}\left[\overline{x},\ \overline{xy},\ 0,\ 0,\ \ldots\right],\ \overline{y}\to \mathrm{LS}\left[\overline{y},\ 0,\ -\overline{x\,\overline{xy}},\ 0,\ \ldots\right]\right\rangle,\ \mathrm{LS}\left[0,\ 0,\ \overline{x\,\overline{xy}},\ -\overline{x\,\overline{\overline{xy}\,y}},\ \ldots\right]\right\}$$

tb

In[∘]:=
```
λ1 = λ; λ2 = ⟨x → β, y → γ⟩; tb[λ1, λ2]
```

Out[∘]=
tb

$$\left\langle \overline{x}\to \mathrm{LS}\left[0,\ 0,\ \overline{x\,\overline{xy}},\ -\overline{x\,\overline{\overline{xy}\,y}},\ \ldots\right],\ \overline{y}\to \mathrm{LS}\left[0,\ 0,\ \overline{x\,\overline{xy}},\ -\overline{x\,\overline{\overline{xy}\,y}},\ \ldots\right]\right\rangle$$

tb2

```
lhs = D_{tb[λ1,λ2]}[ω1]; rhs = b[D_{λ1}, D_{λ2}][ω1];
{lhs@{8}, (lhs ≡ rhs)@{8}}
```

tb2

$\{\mathrm{CWS}[0, 0, 0, 0, 0, 0, 0, 18\,\widehat{xxxxyxyy} - 18\,\widehat{xxxxyyxy} - 36\,\widehat{xxxyxxyy} + 36\,\widehat{xxxyyxxy}, \ldots],$
$\mathrm{BS}[9\ \mathrm{True}, \ldots]\}$

TestingGammaODE

$\mathrm{lhs} = \partial_t \Gamma_t[\lambda];\ \mathrm{rhs} = \lambda\ //\ e^{-t D_\lambda}\ //\ \mathrm{adSeries}\left[\frac{\mathrm{ad}}{e^{\mathrm{ad}} - 1}, \Gamma_t[\lambda]\right];$

$\{\Gamma_0[\lambda], \mathrm{lhs}, (\mathrm{lhs} \equiv \mathrm{rhs})@\{6\}\}$

TestingGammaODE

$\{\langle \overline{x} \to \mathrm{LS}[0, 0, 0, 0, \ldots], \overline{y} \to \mathrm{LS}[0, 0, 0, 0, \ldots]\rangle,$
$\langle \overline{x} \to \mathrm{LS}[\overline{x}, \overline{xy}, -t\,\overline{x\overline{xy}}, \frac{1}{4} t^2\,\overline{x\overline{x\overline{xy}}} - t\,\overline{x\overline{\overline{xy}y}}, \ldots], \overline{y} \to \mathrm{LS}[\overline{y}, 0, -\overline{x\overline{xy}}, -t\,\overline{x\overline{\overline{xy}y}}, \ldots]\rangle,$
$\mathrm{BS}[7\ \mathrm{True}, \ldots]\}$

TestingGamma

$\{\gamma\ //\ e^{-t D_\lambda}, \gamma\ //\ \mathrm{CC}[\Gamma_t[\lambda]]\}$

TestingGamma

$\{\mathrm{LS}[\overline{x} + \overline{y}, -2\,\overline{xy}, -t\,\overline{x\overline{xy}}, t\,\overline{x\overline{\overline{xy}y}}, \ldots], \mathrm{LS}[\overline{x} + \overline{y}, -2\,\overline{xy}, -t\,\overline{x\overline{xy}}, t\,\overline{x\overline{\overline{xy}y}}, \ldots]\}$

TestingLambdaODE

$\mathrm{lhs} = \partial_t \Lambda_t[\lambda];\ \mathrm{rhs} = \lambda\ //\ e^{D_{\Lambda_t[\lambda]}}\ //\ \mathrm{adSeries}\left[\frac{\mathrm{ad}}{e^{\mathrm{ad}} - 1}, \Lambda_t[\lambda], \mathrm{tb}\right];$

$\{\Lambda_0[\lambda], \mathrm{lhs}, (\mathrm{lhs} \equiv \mathrm{rhs})@\{6\}\}$

TestingLambdaODE

$\{\langle \overline{x} \to \mathrm{LS}[0, 0, 0, 0, \ldots], \overline{y} \to \mathrm{LS}[0, 0, 0, 0, \ldots]\rangle,$
$\langle \overline{x} \to \mathrm{LS}[\overline{x}, \overline{xy}, t\,\overline{x\overline{xy}}, \frac{1}{2} t^2\,\overline{x\overline{x\overline{xy}}} + t\,\overline{x\overline{\overline{xy}y}}, \ldots], \overline{y} \to \mathrm{LS}[\overline{y}, 0, -\overline{x\overline{xy}}, t\,\overline{x\overline{\overline{xy}y}}, \ldots]\rangle,$
$\mathrm{BS}[7\ \mathrm{True}, \ldots]\}$

TestingLambda

$\{\gamma\ //\ \mathrm{CC}[t\,\lambda], \gamma\ //\ e^{-D_{\Lambda_t[\lambda]}}\}$

TestingLambda

$\{\mathrm{LS}[\overline{x} + \overline{y}, -2\,\overline{xy}, -t\,\overline{x\overline{xy}}, -\frac{1}{2} t^2\,\overline{x\overline{x\overline{xy}}} + t\,\overline{x\overline{\overline{xy}y}}, \ldots],$
$\mathrm{LS}[\overline{x} + \overline{y}, -2\,\overline{xy}, -t\,\overline{x\overline{xy}}, -\frac{1}{2} t^2\,\overline{x\overline{x\overline{xy}}} + t\,\overline{x\overline{\overline{xy}y}}, \ldots]\}$

Unclassified aside: an alternative formulation of Λ (on March 1, 2015, this took 61 Seconds to degree 8):

```
λ3 = ⟨x → RandomLieSeries[{x, y}], y → RandomLieSeries[{x, y}]⟩;
{lhs = λ3 // EulerE // adSeries[(e^ad - 1)/ad, λ3] // RC[-λ3],
  rhs = Λ[λ3] // EulerE // adSeries[(e^ad - 1)/ad, Λ[λ3], tb]; (lhs ≡ rhs)@{7}} // Timing
```

{13.1406,
{⟨$\overline{x}$ → LS[$-2\,\overline{x} - 2\,\overline{y}$, $6\,\overline{xy}$, $-\frac{17}{2}\,\overline{x\,\overline{xy}} - \frac{23}{2}\,\overline{\overline{xy}\,y}$, $\frac{23}{6}\,\overline{x\,\overline{x\,\overline{xy}}} + \frac{89}{2}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{73}{6}\,\overline{\overline{\overline{xy}\,y}\,y}$, ...],
$\overline{y}$ → LS[$-2\,\overline{y}$, $-2\,\overline{xy}$, $\frac{9}{2}\,\overline{x\,\overline{xy}} + 2\,\overline{\overline{xy}\,y}$, $-\frac{47}{6}\,\overline{x\,\overline{x\,\overline{xy}}} - 3\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{5}{6}\,\overline{\overline{\overline{xy}\,y}\,y}$, ...]⟩, BS[
8 True, ...]}}

CCAndRC

```
{α // CC_x[-γ], α // CC_x[-γ] // RC_x[γ], α // CC_x[-γ] // CC_x[γ]}
```

CCAndRC

{LS[$\overline{x}$, $2\,\overline{xy}$, $-\frac{5}{2}\,\overline{x\,\overline{xy}} + \frac{3}{2}\,\overline{\overline{xy}\,y}$, $\frac{7}{6}\,\overline{x\,\overline{x\,\overline{xy}}} - \frac{23}{6}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{2}{3}\,\overline{\overline{\overline{xy}\,y}\,y}$, ...],
LS[$\overline{x}$, $\overline{xy}$, 0, 0, ...], LS[$\overline{x}$, $\overline{xy}$, $-\overline{x\,\overline{xy}}$, $2\,\overline{x\,\overline{x\,\overline{xy}}} + \overline{x\,\overline{\overline{xy}\,y}}$, ...]}

tru

In[∘]:=
```
u = LW@"u"; v = LW@"v";
With[{γ = b[b[v, u], u]}, tr_u[γ]]
```

Out[∘]=
tru

$-\widehat{uv}$

divu

```
With[{γ = u + b[b[v, u], u]}, div_u[γ]]
```

divu

$\widehat{u} - \widehat{uuv}$

Ju

```
J_x[γ]
```

Ju

CWS[$\widehat{x}$, $\frac{5\,\widehat{xy}}{2}$, $-\frac{7\,\widehat{xxy}}{6} + \frac{7\,\widehat{xyy}}{6}$, $\frac{3\,\widehat{xxxy}}{8} - \frac{11\,\widehat{xxyy}}{4} - \frac{3\,\widehat{xyxy}}{4} + \frac{3\,\widehat{xyyy}}{8}$, ...]

j

```
{div[λ]@{5}, j[λ]@{5}}
```

j

{CWS[$\widehat{x} + \widehat{y}$, $-\widehat{xy}$, $-\widehat{xxy}$, 0, 0, ...],
CWS[$\widehat{x} + \widehat{y}$, $-\widehat{xy}$, $-\widehat{xxy}$, $-\widehat{xxyy} + \widehat{xyxy}$, $-\widehat{xxxyy} + \widehat{xxyxy}$, ...]}

cocycle4j

```
lhs = j[BCH_tb[λ1, λ2]]; rhs = j[λ1] + ⅇ^(D_λ1)[j[λ2]];
{lhs, (lhs ≡ rhs)@{8}}
```

cocycle4j

$$\{\text{CWS}[\widehat{x} + 2\,\widehat{y},\ -3\,\widehat{xy},\ 0,\ -9\,\widehat{xxyy} + 9\,\widehat{xyxy},\ \ldots],\ \text{BS}[9\ \text{True},\ \ldots]\}$$

dj

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

dj

$$\{\text{CWS}[\epsilon\,\widehat{x} + \epsilon\,\widehat{y},\ -\epsilon\,\widehat{xy},\ -\epsilon\,\widehat{xxy},\ 0,\ \ldots],\ \text{BS}[5\ \text{True},\ \ldots]\}$$

Section 2.3 - The [AT]-inspired presentation El of $A^{w}{}_{exp}$

ElSetup

In[∘]:=
```
x = LW@"x"; y = LW@"y";
{ℰ_a = El[⟨x → LS[x + b[x, y]], y → LS[y - b[x, b[x, y]]]⟩, CWS[cw[x] - 3 cw[x, y, x]]],
 ℰ_b = El[⟨x → LS[y - b[x, y]], y → LS[x + y + b[y, b[x, y]]]⟩, CWS[cw[y] - 2 cw[x, y]]],
 ℰ_c = El[⟨x → LS[x - b[b[x, y], b[x, y]]], y → LS[y + 3 b[x, b[x, y]]]⟩,
   CWS[cw[x] - 2 cw[x, y] + cw[x, y, x]]]}
```

Out[∘]=
ElSetup

$$\{\text{El}[\langle\overline{x} \to \text{LS}[\overline{x},\ \overline{xy},\ 0,\ 0,\ \ldots],\ \overline{y} \to \text{LS}[\overline{y},\ 0,\ -\overline{x\,\overline{xy}},\ 0,\ \ldots]\rangle],\ \text{CWS}[\widehat{x},\ 0,\ -3\,\widehat{xxy},\ 0,\ \ldots]],$$
$$\text{El}[\langle\overline{x} \to \text{LS}[\overline{y},\ -\overline{xy},\ 0,\ 0,\ \ldots],\ \overline{y} \to \text{LS}[\overline{x} + \overline{y},\ 0,\ -\overline{\overline{xy}\,y},\ 0,\ \ldots]\rangle,$$
$$\text{CWS}[\widehat{y},\ -2\,\widehat{xy},\ 0,\ 0,\ \ldots]],$$
$$\text{El}[\langle\overline{x} \to \text{LS}[\overline{x},\ 0,\ 0,\ 0,\ \ldots],\ \overline{y} \to \text{LS}[\overline{y},\ 0,\ 3\,\overline{x\,\overline{xy}},\ 0,\ \ldots]\rangle],\ \text{CWS}[\widehat{x},\ -2\,\widehat{xy},\ \widehat{xxy},\ 0,\ \ldots]]\}$$

ElAssociativity

```
lhs = ℰ_a ** (ℰ_b ** ℰ_c); rhs = (ℰ_a ** ℰ_b) ** ℰ_c;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

ElAssociativity

$$\{\text{El}[\langle\overline{x} \to \text{LS}[2\,\overline{x} + \overline{y},\ 0,\ \tfrac{1}{2}\,\overline{x\,\overline{xy}},\ \ldots],\ \overline{y} \to \text{LS}[\overline{x} + 3\,\overline{y},\ 0,\ \tfrac{5}{2}\,\overline{x\,\overline{xy}} - \overline{\overline{xy}\,y},\ \ldots]\rangle,$$
$$\text{CWS}[2\,\widehat{x} + \widehat{y},\ -4\,\widehat{xy},\ -2\,\widehat{xxy},\ \ldots]],\ \text{BS}[9\ \text{True},\ \ldots]\}$$

detaExample

```
{ℰ_a // dη^x, ℰ_a // dη^y}
```

detaExample

$$\{\text{El}[\langle\overline{y} \to \text{LS}[\overline{y},\ 0,\ 0,\ 0,\ \ldots]\rangle,\ \text{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]],$$
$$\text{El}[\langle\overline{x} \to \text{LS}[\overline{x},\ 0,\ 0,\ 0,\ \ldots]\rangle,\ \text{CWS}[\widehat{x},\ 0,\ 0,\ 0,\ \ldots]]\}$$

dA1

In[∘]:= `{ξd = El[λ, CWS[0]], ξd // dA}`

Out[∘]=
dA1

$$\left\{\text{El}\left[\left\langle \overline{x}\to \text{LS}\left[\overline{x},\,\overline{xy},\,0,\,0,\,\ldots\right],\,\overline{y}\to \text{LS}\left[\overline{y},\,0,\,-\overline{x\,\overline{xy}},\,0,\,\ldots\right]\right\rangle,\,\text{CWS}[0,\,0,\,0,\,0,\,\ldots]\right],\right.$$
$$\text{El}\left[\left\langle \overline{x}\to \text{LS}\left[-\overline{x},\,-\overline{xy},\,0,\,0,\,\ldots\right],\,\overline{y}\to \text{LS}\left[-\overline{y},\,0,\,\overline{x\,\overline{xy}},\,0,\,\ldots\right]\right\rangle,\right.$$
$$\left.\left.\text{CWS}\left[-\widehat{x}-\widehat{y},\,\widehat{xy},\,\widehat{xxy},\,\widehat{xxyy}-\widehat{xyxy},\,\ldots\right]\right]\right\}$$

dA2

`(ξd ≡ (ξd // dA // dA))@{8}`

dA2

BS[9 True, ...]

dA3

`lhs = (ξa ** ξb) // dA; rhs = (ξb // dA) ** (ξa // dA);`
`{lhs@{3}, (lhs ≡ rhs)@{8}}`

dA3

$$\left\{\text{El}\left[\left\langle \overline{x}\to \text{LS}\left[-\overline{x}-\overline{y},\,0,\,-\frac{1}{2}\,\overline{x\,\overline{xy}},\,\ldots\right],\,\overline{y}\to \text{LS}\left[-\overline{x}-2\,\overline{y},\,0,\,\frac{1}{2}\,\overline{x\,\overline{xy}}+\overline{\overline{xy}\,y},\,\ldots\right]\right\rangle,\right.\right.$$
$$\left.\left.\text{CWS}\left[-\widehat{y},\,-2\,\widehat{xy},\,-2\,\widehat{xxy}-\widehat{xyy},\,\ldots\right]\right],\,\text{BS}[9\ \text{True},\,\ldots]\right\}$$

dS

`ξd // dS`

dS

$$\text{El}\left[\left\langle \overline{x}\to \text{LS}\left[\overline{x},\,-\overline{xy},\,0,\,0,\,\ldots\right],\,\overline{y}\to \text{LS}\left[\overline{y},\,0,\,-\overline{x\,\overline{xy}},\,0,\,\ldots\right]\right\rangle,\right.$$
$$\left.\text{CWS}\left[\widehat{x}+\widehat{y},\,\widehat{xy},\,-\widehat{xxy},\,\widehat{xxyy}-\widehat{xyxy},\,\ldots\right]\right]$$

dD1

`{ξa, ξa // dΔ[y, y, z]}`

dD1

$$\left\{\text{El}\left[\left\langle \overline{x}\to \text{LS}\left[\overline{x},\,\overline{xy},\,0,\,0,\,\ldots\right],\,\overline{y}\to \text{LS}\left[\overline{y},\,0,\,-\overline{x\,\overline{xy}},\,0,\,\ldots\right]\right\rangle,\,\text{CWS}\left[\widehat{x},\,0,\,-3\,\widehat{xxy},\,0,\,\ldots\right]\right],\right.$$
$$\text{El}\left[\left\langle z\to \text{LS}\left[\overline{y}+\overline{z},\,0,\,-\overline{x\,\overline{xy}}-\overline{x\,\overline{xz}},\,0,\,\ldots\right],\,\overline{x}\to \text{LS}\left[\overline{x},\,\overline{xy}+\overline{xz},\,0,\,0,\,\ldots\right],\right.\right.$$
$$\left.\left.\left.\overline{y}\to \text{LS}\left[\overline{y}+\overline{z},\,0,\,-\overline{x\,\overline{xy}}-\overline{x\,\overline{xz}},\,0,\,\ldots\right]\right\rangle,\,\text{CWS}\left[\widehat{x},\,0,\,-3\,\widehat{xxy}-3\,\widehat{xxz},\,0,\,\ldots\right]\right]\right\}$$

dD2

`lhs = (ξa ** ξb) // dΔ[y, y, z]; rhs = (ξa // dΔ[y, y, z]) ** (ξb // dΔ[y, y, z]);`
`{lhs@{3}, (lhs ≡ rhs)@{8}}`

dD2

$$\left\{\text{El}\left[\left\langle z\to \text{LS}\left[\overline{x}+2\,\overline{y}+2\,\overline{z},\,0,\,-\frac{1}{2}\,\overline{x\,\overline{xy}}-\frac{1}{2}\,\overline{x\,\overline{xz}}-\overline{x\,\overline{yz}}-\overline{\overline{xy}\,y}-2\,\overline{\overline{xz}\,y}-\overline{\overline{xz}\,z},\,\ldots\right],\right.\right.\right.$$
$$\overline{x}\to \text{LS}\left[\overline{x}+\overline{y}+\overline{z},\,0,\,\frac{1}{2}\,\overline{x\,\overline{xy}}+\frac{1}{2}\,\overline{x\,\overline{xz}},\,\ldots\right],$$
$$\left.\overline{y}\to \text{LS}\left[\overline{x}+2\,\overline{y}+2\,\overline{z},\,0,\,-\frac{1}{2}\,\overline{x\,\overline{xy}}-\frac{1}{2}\,\overline{x\,\overline{xz}}-\overline{x\,\overline{yz}}-\overline{\overline{xy}\,y}-2\,\overline{\overline{xz}\,y}-\overline{\overline{xz}\,z},\,\ldots\right]\right\rangle,$$
$$\left.\left.\text{CWS}\left[\widehat{x}+\widehat{y}+\widehat{z},\,-2\,\widehat{xy}-2\,\widehat{xz},\,-3\,\widehat{xxy}-3\,\widehat{xxz},\,\ldots\right]\right],\,\text{BS}[9\ \text{True},\,\ldots]\right\}$$

Section 2.4 - The factored presentation Ef of $A^{w}{}_{exp}$ and its stronger precursor Es

EsSetup1

In[∘]:=
```
ξa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
  CWS[cw[u] - 3 cw[u, v, u]]]
```

Out[∘]=
EsSetup1

$$\mathrm{Es}\left[\left\langle 1\to \mathrm{LS}\left[\overline{u},\ \overline{uv},\ 0,\ 0,\ \ldots\right],\ 2\to \mathrm{LS}\left[\overline{v},\ 0,\ -\overline{u\,\overline{uv}},\ 0,\ \ldots\right],\ 3\to \mathrm{LS}\left[\overline{u},\ 0,\ 0,\ 0,\ \ldots\right]\right\rangle,\ \mathrm{CWS}\left[\widehat{u},\ 0,\ -3\,\widehat{uuv},\ 0,\ \ldots\right]\right]$$

EsSetup2

In[∘]:=
```
ξb = RandomEsSeries[0, {1, 2, 3, 4}];
ξb@{2}
```

Out[∘]=
EsSetup2

$$\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[-\overline{1}-2\,\overline{2}+2\,\overline{3}-2\,\overline{4},\ 2\,\overline{12}+\frac{\overline{13}}{2}+\overline{14}-\frac{\overline{23}}{2}-\frac{\overline{24}}{2}+2\,\overline{34},\ \ldots\Big],$$
$$2\to \mathrm{LS}\Big[2\,\overline{1}-\overline{2}-2\,\overline{3}+\overline{4},\ 2\,\overline{12}+\frac{3\,\overline{13}}{2}-2\,\overline{14}-\overline{23}-\overline{24}-\frac{\overline{34}}{2},\ \ldots\Big],$$
$$3\to \mathrm{LS}\Big[-\overline{1}+\overline{2}+2\,\overline{4},\ -2\,\overline{12}+2\,\overline{13}-\overline{14}-\frac{3\,\overline{23}}{2}+2\,\overline{24}-2\,\overline{34},\ \ldots\Big],$$
$$4\to \mathrm{LS}\Big[-2\,\overline{1}+2\,\overline{2}+2\,\overline{3}+\overline{4},\ -\frac{\overline{12}}{2}+\frac{3\,\overline{13}}{2}-2\,\overline{24}+\overline{34},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[\widehat{3}-\widehat{4},\ \frac{3\,\widehat{11}}{2}+\frac{3\,\widehat{12}}{2}-2\,\widehat{13}+\widehat{14}+\widehat{22}+2\,\widehat{23}-\frac{\widehat{24}}{2}-2\,\widehat{33}-\widehat{34}+\widehat{44},\ \ldots\Big]\Big]$$

haction

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4]; rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

haction

$$\Big\{\mathrm{Es}\Big[\Big\langle 3\to \mathrm{LS}\Big[\overline{u},\ -\overline{uv},\ -\overline{u\,\overline{uv}}+\frac{1}{2}\,\overline{\overline{uv}\,v},\ \frac{3}{2}\,\overline{u\,\overline{u\,\overline{uv}}}+\overline{u\,\overline{\overline{uv}\,v}}-\frac{1}{6}\,\overline{\overline{\overline{uv}\,v}\,v},\ \ldots\Big],$$
$$4\to \mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{\overline{uv}}{2},\ -\frac{23}{12}\,\overline{u\,\overline{uv}}-\frac{5}{12}\,\overline{\overline{uv}\,v},\ \overline{u\,\overline{u\,\overline{uv}}}+\frac{13}{24}\,\overline{u\,\overline{\overline{uv}\,v}}+\frac{1}{12}\,\overline{\overline{\overline{uv}\,v}\,v},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[2\,\widehat{u},\ -\widehat{uv},\ -\frac{3\,\widehat{uuv}}{2},\ -\frac{\widehat{uuuv}}{6}+\widehat{uuvv}-\widehat{uvuv},\ \ldots\Big]\Big],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\Big\}$$

metaassoc

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

metaassoc

$$\left\{\text{Es}\left[\left\langle 1 \to \text{LS}\left[-2\,\overline{1}+\overline{4},\ -\frac{3\,\overline{14}}{2},\ 20\,\overline{1\,\overline{14}}-\frac{19}{3}\,\overline{\overline{14}\,4},\ \ldots\right],\ 4 \to \text{LS}\left[2\,\overline{1}+\overline{4},\ \overline{14},\ -\frac{31}{2}\,\overline{1\,\overline{14}}-\frac{13}{6}\,\overline{\overline{14}\,4},\ \ldots\right]\right\rangle,\ \text{CWS}\left[3\,\widehat{1}-\widehat{4},\ -3\,\widehat{11}+\frac{\widehat{14}}{2}+\widehat{44},\ \frac{71\,\widehat{111}}{4}+\frac{19\,\widehat{114}}{4}-\frac{7\,\widehat{144}}{6}-\frac{2\,\widehat{444}}{3},\ \ldots\right]\right],\ \text{BS}[6\ \text{True},\ \ldots]\right\}$$

Section 3.1 - Tangle Invariants

Section 3.1.1 - The General Framework

RDefs

```
In[•]:= Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

R3

```
lhs = Rl[1, 2] ** Rl[1, 3] ** Rl[2, 3]; rhs = Rl[2, 3] ** Rl[1, 3] ** Rl[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

R3

$$\left\{\text{El}\left[\left\langle 1 \to \text{LS}[0,0,0,\ldots],\ 2 \to \text{LS}\left[\overline{1},0,0,\ldots\right],\ 3 \to \text{LS}\left[\overline{1}+\overline{2},0,0,\ldots\right]\right\rangle,\ \text{CWS}[0,0,0,\ldots]\right],\ \text{BS}[6\ \text{True},\ \ldots]\right\}$$

Section 3.1.2 - The Knot 8_{17} and the Borromean Tangle

817

```
t1 = iRs[12, 1] iRs[2, 7] iRs[8, 3] iRs[4, 11] Rs[16, 5] Rs[6, 13] Rs[14, 9] Rs[10, 15];
Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
t1@{6}
```

817

$$\text{Es}\left[\langle 1 \to \text{LS}[0,0,0,0,0,0,\ldots]\rangle,\ \text{CWS}\left[0,\ -\widehat{11},\ 0,\ -\frac{31\,\widehat{1111}}{12},\ 0,\ -\frac{1351\,\widehat{111111}}{360},\ \ldots\right]\right]$$

Borromean

```
t2 = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
 Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

Borromean

```
Es[⟨b → LS[0, gr, 1/2 ggr + brg + 1/2 grr,
    -1/2 bbrg + 1/6 gggr + 1/4 ggrr - 1/2 bgbr - 1/2 brgg - 1/2 brrg + 1/6 grrr, ...],
  g → LS[0, -br, 1/2 bbr - bgr - brg + 1/2 brr, -1/6 bbbr - 1/2 bbgr - 1/2 bggr - 1/2 bbrg -
    1/4 bbrr + 1/2 bgrr + 1/2 bgbr + brgr - bgrg - 1/2 brgg + 1/2 brrg - 1/6 brrr, ...],
  r → LS[0, bg, 1/2 bbg + bgr + 1/2 bgg, 1/6 bbbg + 1/2 bbgr +
    1/2 bggr + 1/4 bbgg + 1/2 bgrr + 1/6 bggg, ...]⟩,
 CWS[0, 0, 2 bgr, bbgr - bgbr + bggr - bgrg + bgrr - brgr, ...]]
```

Section 3.2 - Solutions of the Kashiwara-Vergne Equations

Continues pensieve://2013-10/SolvingWKO.nb.

VSetup

```
In[∘]:= x = LW["x"]; y = LW["y"]; z = LW["z"];
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V0 = Es[⟨x → α, y → β⟩, γ];
```

CapSetup

```
In[∘]:= κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
```

VCapEqns

```
In[∘]:= R4Eqn = V0 ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V0;
UnitarityEqn = V0 ** (V0 // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]];
CapEqn = (V0 ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]);
```

VCapSolution

In[◦]:=

```
βs["x"] = 1 / 2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ}, (ℏ^-1 R4Eqn) ∧ UnitarityEqn ∧ CapEqn];
{V0@{4}, κ@{4}}
```

VCapSolution

SeriesSolve: In degree 1 arbitrarily setting {κs[x] → 0}.

VCapSolution

SeriesSolve: In degree 3 arbitrarily setting {αs[x, y, y] → 0}.

Out[◦]=
VCapSolution

$$\left\{\text{Es}\left[\left\langle \overline{x} \to \text{LS}\left[0, -\frac{\overline{xy}}{24}, 0, \frac{7\,\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440}, \ldots\right],\right.\right.\right.$$

$$\left.\overline{y} \to \text{LS}\left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \frac{\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y}, \ldots\right]\right\rangle,$$

$$\left.\left.\text{CWS}\left[0, -\frac{\widehat{xy}}{48}, 0, \frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880}, \ldots\right]\right], \text{CWS}\left[0, -\frac{\widehat{xx}}{96}, 0, \frac{\widehat{xxxx}}{11\,520}, \ldots\right]\right\}$$

In[•]:= **V_0@{7}**

SeriesSolve: In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

SeriesSolve: In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

Out[•]=

$$\mathrm{Es}\Big[\Big\langle \overline{x} \to \mathrm{LS}\Big[0, -\frac{\overline{xy}}{24}, 0, \frac{7\,\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440}, 0,$$

$$-\frac{31\,\overline{x\,\overline{x\,\overline{x\,\overline{x\,\overline{xy}}}}}}{967680} + \frac{31\,\overline{x\,\overline{x\,\overline{x\,\overline{\overline{xy}\,y}}}}}{483840} - \frac{83\,\overline{x\,\overline{x\,\overline{\overline{\overline{xy}\,y}\,y}}}}{967680} - \frac{31\,\overline{x\,\overline{\overline{xy}\,\overline{\overline{xy}\,y}}}}{725760} - \frac{31\,\overline{x\,\overline{\overline{x\,\overline{xy}}\,\overline{xy}}}}{645120} +$$

$$\frac{13\,\overline{x\,\overline{\overline{\overline{\overline{xy}\,y}\,y}\,y}}}{241920} + \frac{101\,\overline{\overline{xy}\,\overline{\overline{\overline{xy}\,y}\,y}}}{1451520} + \frac{527\,\overline{\overline{x\,\overline{\overline{xy}\,y}}\,\overline{xy}}}{5806080} - \frac{\overline{\overline{\overline{\overline{\overline{xy}\,y}\,y}\,y}\,y}}{60480}, 0, \ldots\Big],$$

$$\overline{y} \to \mathrm{LS}\Big[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \frac{\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y}, -\frac{\overline{x\,\overline{x\,\overline{x\,\overline{xy}}}}}{7680} + \frac{\overline{x\,\overline{x\,\overline{\overline{xy}\,y}}}}{3840} - \frac{\overline{\overline{x\,\overline{xy}}\,\overline{xy}}}{6912},$$

$$-\frac{\overline{x\,\overline{x\,\overline{x\,\overline{x\,\overline{xy}}}}}}{645120} + \frac{23\,\overline{x\,\overline{x\,\overline{x\,\overline{\overline{xy}\,y}}}}}{483840} - \frac{13\,\overline{x\,\overline{x\,\overline{\overline{\overline{xy}\,y}\,y}}}}{161280} - \frac{\overline{x\,\overline{\overline{xy}\,\overline{\overline{xy}\,y}}}}{22680} -$$

$$\frac{41\,\overline{x\,\overline{\overline{x\,\overline{xy}}\,\overline{xy}}}}{580608} + \frac{\overline{x\,\overline{\overline{\overline{\overline{xy}\,y}\,y}\,y}}}{15120} + \frac{\overline{\overline{xy}\,\overline{\overline{\overline{xy}\,y}\,y}}}{12096} + \frac{71\,\overline{\overline{x\,\overline{\overline{xy}\,y}}\,\overline{xy}}}{483840} - \frac{\overline{\overline{\overline{\overline{\overline{xy}\,y}\,y}\,y}\,y}}{30240},$$

$$\frac{\overline{x\,\overline{x\,\overline{x\,\overline{x\,\overline{x\,\overline{xy}}}}}}}{258048} - \frac{5\,\overline{x\,\overline{x\,\overline{x\,\overline{x\,\overline{\overline{xy}\,y}}}}}}{387072} + \frac{\overline{x\,\overline{x\,\overline{x\,\overline{\overline{\overline{xy}\,y}\,y}}}}}{64512} + \frac{\overline{x\,\overline{x\,\overline{\overline{xy}\,\overline{\overline{xy}\,y}}}}}{96768} + \frac{5\,\overline{x\,\overline{x\,\overline{\overline{x\,\overline{xy}}\,\overline{xy}}}}}{290304} - \frac{\overline{x\,\overline{x\,\overline{\overline{\overline{\overline{xy}\,y}\,y}\,y}}}}{96768} -$$

$$\frac{17\,\overline{x\,\overline{\overline{xy}\,\overline{\overline{\overline{xy}\,y}\,y}}}}{1451520} - \frac{\overline{x\,\overline{x\,\overline{\overline{\overline{xy}\,y}\,\overline{xy}}}}}{60480} - \frac{\overline{\overline{x\,\overline{xy}}\,\overline{x\,\overline{\overline{xy}\,y}}}}{207360} - \frac{7\,\overline{\overline{x\,\overline{x\,\overline{xy}}}\,\overline{x\,\overline{xy}}}}{1658880} + \frac{\overline{x\,\overline{\overline{\overline{\overline{xy}\,y}\,y}\,\overline{xy}}}}{207360}, \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0, -\frac{\widehat{xy}}{48}, 0, \frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880}, 0, -\frac{\widehat{xxxxxy}}{120960} - \frac{\widehat{xxxxyy}}{120960} - \frac{\widehat{xxxyxy}}{120960} - \frac{\widehat{xxxyyy}}{120960} -$$

$$\frac{\widehat{xxyxxy}}{241920} - \frac{\widehat{xxyxyy}}{120960} - \frac{\widehat{xxyyxy}}{120960} - \frac{\widehat{xxyyyy}}{120960} - \frac{\widehat{xyxyxy}}{362880} - \frac{\widehat{xyxyyy}}{120960} - \frac{\widehat{xyyxyy}}{241920} - \frac{\widehat{xyyyyy}}{120960}, 0, \ldots\Big]\Big]$$

Sinh

$$\mathbf{Series}\Big[\frac{1}{4}\,\mathbf{Log}\Big[\frac{\hbar/2}{\mathbf{Sinh}[\hbar/2]}\Big], \{\hbar, 0, 12\}\Big]$$

Sinh

$$-\frac{\hbar^2}{96} + \frac{\hbar^4}{11520} - \frac{\hbar^6}{725760} + \frac{\hbar^8}{38707200} - \frac{\hbar^{10}}{1916006400} + \frac{691\,\hbar^{12}}{62768369664000} + O[\hbar]^{13}$$

LambdaV

```
Λ[V0]
```

LambdaV

$$El\Big[\Big\langle \overline{x}\to LS\Big[0, -\frac{\overline{xy}}{24}, \frac{1}{96}\overline{x\overline{xy}}, \frac{\overline{x\overline{x\overline{xy}}}}{2880} - \frac{1}{480}\overline{x\overline{\overline{xy}y}} + \frac{\overline{\overline{\overline{xy}y}y}}{1440}, \dots\Big],$$

$$\overline{y}\to LS\Big[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, \frac{1}{96}\overline{x\overline{xy}}, \frac{1}{960}\overline{x\overline{x\overline{xy}}} - \frac{1}{320}\overline{x\overline{\overline{xy}y}} + \frac{1}{720}\overline{\overline{\overline{xy}y}y}, \dots\Big]\Big\rangle,$$

$$CWS\Big[0, -\frac{\overparen{xy}}{48}, 0, \frac{\overparen{xxxy}}{2880} + \frac{\overparen{xxyy}}{2880} + \frac{\overparen{xyxy}}{5760} + \frac{\overparen{xyyy}}{2880}, \dots\Big]\Big]$$

logF

In[∘]:=
```
logF = Λ[V0]⟦1⟧ // dσ[{x, y} → {y, x}]
```

Out[∘]=
logF

$$\Big\langle \overline{x}\to LS\Big[\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, \frac{1}{96}\overline{\overline{xy}y}, -\frac{1}{720}\overline{x\overline{x\overline{xy}}} + \frac{1}{320}\overline{x\overline{\overline{xy}y}} - \frac{1}{960}\overline{\overline{\overline{xy}y}y}, \dots\Big],$$

$$\overline{y}\to LS\Big[0, \frac{\overline{xy}}{24}, \frac{1}{96}\overline{\overline{xy}y}, -\frac{\overline{x\overline{x\overline{xy}}}}{1440} + \frac{1}{480}\overline{x\overline{\overline{xy}y}} - \frac{\overline{\overline{\overline{xy}y}y}}{2880}, \dots\Big]\Big\rangle$$

atkv

In[∘]:=
```
atkv = logF // EulerE // adSeries[(e^ad - 1)/ad, logF, tb];
{f = atkv_x, g = atkv_y}
```

Out[∘]=
atkv

$$\Big\{LS\Big[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \dots\Big],$$

$$LS\Big[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \dots\Big]\Big\}$$

On March 1, 2015, the following took 379 seconds in degree 8:

KVTest

```
(ħ^-1 (LS[x + y] - BCH[y, x] ≡ f - g - Ad[-x][f] + Ad[y][g]) ∧
   div_x[f] + div_y[g] ≡ 1/2 tr_u[adSeries[ad/(e^ad - 1), x][u] +
       adSeries[ad/(e^ad - 1), y][u] - adSeries[ad/(e^ad - 1), BCH[x, y]][u]]) @ {6} // Timing
```

KVTest

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

KVTest

{13.8281, BS[7 True, ...]}

KVDirect

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1 / 2;
SeriesSolve[{F, G},
  ℏ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ∧ div_x[F] + div_y[G] ≡ 1/2 tr_u[
    adSeries[ad/(e^ad - 1), x][u] + adSeries[ad/(e^ad - 1), y][u] - adSeries[ad/(e^ad - 1), BCH[x, y]][u]]];
{F, G}
```

KVDirect

$$\left\{LS\left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}\,y}, -\frac{1}{180}\overline{x\,\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{360}\overline{\overline{\overline{xy}\,y}\,y}, \ldots\right],\right.$$
$$\left.LS\left[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}\,y}, -\frac{1}{360}\overline{x\,\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{180}\overline{\overline{\overline{xy}\,y}\,y}, \ldots\right]\right\}$$

Section 3.3 - The involution τ and the twist equation

Theta

In[∘]:=
```
Θl[x_, y_, s_] := El[⟨x → LS[s LW@y], y → LS[s LW@x]⟩, CWS[0]];
Θs[x_, y_, s_] := Θl[x, y, s] // Γ;
{Θl[x, y, 1], Θs[x, y, 1]}
```

Out[∘]=
Theta

$$\{El[\langle \overline{x} \to LS[\overline{y}, 0, 0, 0, \ldots], \overline{y} \to LS[\overline{x}, 0, 0, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]],$$
$$Es\left[\left\langle \overline{x} \to LS\left[\overline{y}, \frac{\overline{xy}}{2}, \frac{1}{6}\overline{x\overline{xy}} - \frac{1}{12}\overline{\overline{xy}\,y}, \frac{1}{24}\overline{x\,\overline{x\overline{xy}}} - \frac{1}{24}\overline{x\,\overline{\overline{xy}\,y}}, \ldots\right],\right.\right.$$
$$\left.\left.\overline{y} \to LS\left[\overline{x}, -\frac{\overline{xy}}{2}, -\frac{1}{12}\overline{x\overline{xy}} + \frac{1}{6}\overline{\overline{xy}\,y}, \frac{1}{24}\overline{x\,\overline{\overline{xy}\,y}} - \frac{1}{24}\overline{\overline{\overline{xy}\,y}\,y}, \ldots\right]\right\rangle, CWS[0, 0, 0, 0, \ldots]\right]\}$$

Vtau

```
τV = Rs[x, y] ** (V₀ // dσ[{x, y} → {y, x}]) ** Θs[x, y, -1 / 2];
(V₀ ≡ τV) @ {6}
```

Vtau

BS[7 True, ...]

Linearized

In[∘]:=
```
{A = LS[{x, y}, As], B = LS[{x, y}, Bs]};
msgs = SeriesSolve[{A, B},
   ℏ^-1 (b[x, A] + b[y, B] ≡ LS[0]) ∧ (div_x[A] + div_y[B] ≡ CWS[0])];
{A, B}
```

Linearized

SeriesSolve: In degree 1 arbitrarily setting {As[y] → 0}.

Out[∘]=
Linearized

{LS[0, 0, 0, 0, ...], LS[0, 0, 0, 0, ...]}

msgs

```
Read[msgs]
```

msgs

```
{{ArbitrarilySetting, 1, {Hold[As[y]] → 0}}, {ArbitrarilySetting, 2, {}},
 {ArbitrarilySetting, 3, {}}, {ArbitrarilySetting, 4, {}}}
```

dims

```
A@12; Length[Last[#]] & /@ Read[msgs]
```

dims

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {As[x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 11 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y, y] → 0}.

dims

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

dims

```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

dims1

```
{A1 = LS[{x, y}, A1s], B1 = LS[{x, y}, B1s]};
msgs1 = SeriesSolve[{A1, B1},
   ℏ^-1 (b[x, A1] + b[y, B1] ≡ LS[0]) ∧
    (div_x[A1] + div_y[B1] ≡ CWS[0]) ∧ (A1 ≡ (B1 // LieMorphism[x → y, y → x]))];
A1@12; Length[Last[#]] & /@ Read[msgs1]
```

dims1

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {A1s[y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {A1s[x, x, x, x, y, x, y, y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {A1s[x, x, x, x, x, x, y, x, y, y] → 0}.

dims1

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

dims1

```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

Section 3.4 - Drinfel’d Associators

4T

```
{b[t[1, 3], t[4, 2]], b[t[1, 2] + t[1, 3], t[2, 3]]}
```

4T

```
{0, 0}
```

DKExample

```
b[t[1, 3], t[1, 2]]
```

DKExample

$\mathrm{DK}\left[3, -\overline{12}\right]$

DKSExample

b[t[1, 3], t[1, 2]] // DKS

DKSExample

$\mathrm{DKS}\left[0,\ -\overline{t_{13}\, t_{23}},\ 0,\ 0,\ \ldots\right]$

sigmaExample

$\{t[2,3]^{\sigma[\{2,4\},\{1,5\},\{3,7,8\},\{9\}]}$ // DKS, $t[2,3]^{\sigma[24,15,378,9]}$ // DKS$\}$

sigmaExample

$\left\{\mathrm{DKS}\left[\overline{t_{13}} + \overline{t_{17}} + \overline{t_{18}} + \overline{t_{35}} + \overline{t_{57}} + \overline{t_{58}},\ 0,\ 0,\ 0,\ \ldots\right],\ \mathrm{DKS}\left[\overline{t_{13}} + \overline{t_{17}} + \overline{t_{18}} + \overline{t_{35}} + \overline{t_{57}} + \overline{t_{58}},\ 0,\ 0,\ 0,\ \ldots\right]\right\}$

BCH4DK

R = DKS[t[1, 2] / 2];

$\{R ** R^{\sigma[2,3]},\ R ** R^{\sigma[12,3]}\}$

BCH4DK

$$\left\{\mathrm{DKS}\left[\frac{\overline{t_{12}}}{2} + \frac{\overline{t_{23}}}{2},\ -\frac{1}{8}\overline{t_{13}\, t_{23}},\ -\frac{1}{48}\overline{\overline{t_{13}\, t_{23}}\, t_{23}} + \frac{1}{96}\overline{t_{13}\,\overline{t_{13}\, t_{23}}},\ -\frac{1}{384}\overline{\overline{\overline{t_{13}\, t_{23}}\, t_{23}}\, t_{23}} + \frac{1}{384}\overline{t_{13}\,\overline{\overline{t_{13}\, t_{23}}\, t_{23}}},\ \ldots\right],\ \mathrm{DKS}\left[\frac{\overline{t_{12}}}{2} + \frac{\overline{t_{13}}}{2} + \frac{\overline{t_{23}}}{2},\ 0,\ 0,\ 0,\ \ldots\right]\right\}$$

Phi

In[∘]:=

$\Phi s[2,1] = \Phi s[3,1] = \Phi s[3,2] = 0$; $\Phi s[3,1,2] = 1/24$; Φ_0 = DKS[3, Φs];

SeriesSolve$\left[\Phi_0,\ \left(\Phi_0^{\sigma[3,2,1]} \equiv -\Phi_0\right) \wedge \left(\Phi_0 ** \Phi_0^{\sigma[1,23,4]} ** \Phi_0^{\sigma[2,3,4]} \equiv \Phi_0^{\sigma[12,3,4]} ** \Phi_0^{\sigma[1,2,34]}\right)\right]$;

Φ_0@{6}

Phi

SeriesSolve: In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.

Phi

SeriesSolve: In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

Out[∘]=
Phi

$$\mathrm{DKS}\Bigg[0,\ \frac{1}{24}\overline{t_{13}\, t_{23}},\ 0,\ -\frac{7\,\overline{\overline{\overline{t_{13}\, t_{23}}\, t_{23}}\, t_{23}}}{5760} + \frac{7\,\overline{t_{13}\,\overline{\overline{t_{13}\, t_{23}}\, t_{23}}}}{5760} - \frac{\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\, t_{23}}}}}{1440},$$

$$0,\ \frac{31\,\overline{\overline{\overline{\overline{\overline{t_{13}\, t_{23}}\, t_{23}}\, t_{23}}\, t_{23}}\, t_{23}}}{967\,680} - \frac{157\,\overline{\overline{t_{13}\,\overline{\overline{t_{13}\, t_{23}}\, t_{23}}}\,\overline{t_{13}\, t_{23}}}}{1\,935\,360} - \frac{31\,\overline{\overline{t_{13}\, t_{23}}\,\overline{\overline{\overline{t_{13}\, t_{23}}\, t_{23}}\, t_{23}}}}{387\,072} -$$

$$\frac{31\,\overline{t_{13}\,\overline{\overline{\overline{\overline{t_{13}\, t_{23}}\, t_{23}}\, t_{23}}\, t_{23}}}}{483\,840} + \frac{11\,\overline{t_{13}\,\overline{\overline{t_{13}\,\overline{t_{13}\, t_{23}}}\,\overline{t_{13}\, t_{23}}}}}{290\,304} + \frac{31\,\overline{t_{13}\,\overline{\overline{t_{13}\, t_{23}}\,\overline{\overline{t_{13}\, t_{23}}\, t_{23}}}}}{725\,760} +$$

$$\frac{83\,\overline{t_{13}\,\overline{t_{13}\,\overline{\overline{\overline{t_{13}\, t_{23}}\, t_{23}}\, t_{23}}}}}{967\,680} - \frac{13\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{\overline{t_{13}\, t_{23}}\, t_{23}}}}}}{241\,920} + \frac{\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\, t_{23}}}}}}}{60\,480},\ \ldots\Bigg]$$

Hexagons

```
R = DKS[t[1, 2] / 2];
(R^σ[12,3] ≡ Φ0 ** R^σ[2,3] ** (-Φ0)^σ[1,3,2] ** R^σ[1,3] ** Φ0^σ[3,1,2] ∧
   (-R)^σ[12,3] ≡ Φ0 ** (-R)^σ[2,3] ** (-Φ0)^σ[1,3,2] ** (-R)^σ[1,3] ** Φ0^σ[3,1,2]) @ {6}
```

Hexagons

BS[7 True, ...]

Section 3.5 - Associators in $\mathcal{A}^w$

PhiV

In[∘]:=
```
V12 = V0 // dσ[{x, y} → {1, 2}];
ΦV = (V12 // dA)^σ[12,3] ** (V12 // dA)^σ[1,2] ** V12^σ[2,3] ** V12^σ[1,23]
```

Out[∘]=
PhiV

$$\mathrm{Es}\Big[\Big\langle 1 \to \mathrm{LS}\Big[0,\ \frac{\overline{23}}{24},\ 0,\ -\frac{\overline{1\,\overline{1\,\overline{23}}}}{1440} + \frac{7\,\overline{1\,\overline{2\,\overline{23}}}}{5760} + \frac{\overline{1\,\overline{\overline{23}\,3}}}{5760} - \frac{7\,\overline{2\,\overline{2\,\overline{23}}}}{5760} +$$
$$\frac{7\,\overline{2\,\overline{\overline{23}\,3}}}{5760} + \frac{1}{480}\,\overline{12}\,\overline{13} - \frac{\overline{13}\,\overline{23}}{1920} + \frac{1}{640}\,\overline{\overline{1\,\overline{23}}\,2} - \frac{\overline{\overline{\overline{13}\,2}\,2}}{1152} - \frac{\overline{\overline{\overline{13}\,3}\,2}}{1152} - \frac{\overline{\overline{\overline{23}\,3}\,3}}{1440},\ \ldots\Big],$$
$$2 \to \mathrm{LS}\Big[0,\ -\frac{\overline{13}}{24},\ 0,\ \frac{\overline{1\,\overline{1\,\overline{13}}}}{1440} - \frac{\overline{1\,\overline{1\,\overline{23}}}}{1152} + \frac{7\,\overline{1\,\overline{2\,\overline{23}}}}{1920} - \frac{1}{480}\,\overline{1\,\overline{\overline{13}\,2}} - \frac{\overline{1\,\overline{\overline{13}\,3}}}{5760} + \frac{\overline{1\,\overline{\overline{23}\,3}}}{1152} +$$
$$\frac{7\,\overline{12}\,\overline{13}}{5760} + \frac{19\,\overline{13}\,\overline{23}}{5760} + \frac{7\,\overline{\overline{1\,\overline{23}}\,2}}{1920} + \frac{7\,\overline{\overline{\overline{13}\,2}\,2}}{5760} + \frac{7\,\overline{\overline{\overline{13}\,3}\,2}}{5760} + \frac{\overline{\overline{\overline{13}\,3}\,3}}{1440},\ \ldots\Big],$$
$$3 \to \mathrm{LS}\Big[0,\ \frac{\overline{12}}{24},\ 0,\ -\frac{\overline{1\,\overline{1\,\overline{12}}}}{1440} + \frac{\overline{1\,\overline{1\,\overline{23}}}}{5760} + \frac{7\,\overline{1\,\overline{2\,\overline{23}}}}{5760} + \frac{7\,\overline{1\,\overline{\overline{12}\,2}}}{5760} - \frac{\overline{1\,\overline{\overline{13}\,2}}}{1440} - \frac{\overline{1\,\overline{\overline{23}\,3}}}{1440} + \frac{\overline{12}\,\overline{13}}{5760} +$$
$$\frac{\overline{13}\,\overline{23}}{1440} - \frac{\overline{\overline{1\,\overline{23}}\,2}}{1152} - \frac{7\,\overline{\overline{\overline{12}\,2}\,2}}{5760} - \frac{7\,\overline{\overline{\overline{13}\,3}\,2}}{5760} - \frac{\overline{\overline{\overline{13}\,3}\,2}}{1440},\ \ldots\Big]\Big\rangle,\ \mathrm{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big]$$

In[∘]:= `(ΦV ** dA[ΦV]) @ {7}`

Out[∘]=

Es[⟨1 → LS[0, 0, 0, 0, 0, 0, 0, ...], 2 → LS[0, 0, 0, 0, 0, 0, 0, ...],
3 → LS[0, 0, 0, 0, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, 0, 0, 0, ...]]

PentPhiV

```
ΦV ** ΦV^σ[1,23,4] ** ΦV^σ[2,3,4] ≡ ΦV^σ[12,3,4] ** ΦV^σ[1,2,34]
```

PentPhiV

BS[5 True, ...]

Phi_is_sder

```
ϕ = (ΦV // Λ)〚1〛;
(b[LW@1, ϕ1] + b[LW@2, ϕ2] + b[LW@3, ϕ3])@{6}
```

Phi_is_sder

```
LS[0, 0, 0, 0, 0, 0, ...]
```

DK2Es

In[∘]:=

```
DK2Es[s___][ℰ_] := El[ℰ // αMap[s], CWS[0]] // Γ;
DK2Es[1, 2, 3][Φ0]
```

Out[∘]=
DK2Es

$$\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[0,\ \frac{\overline{23}}{24},\ 0,\ -\frac{\overline{1\overline{1\overline{23}}}}{1440}+\frac{7\,\overline{1\overline{2\overline{23}}}}{5760}+\frac{\overline{1\overline{\overline{23}3}}}{5760}-\frac{7\,\overline{2\overline{2\overline{23}}}}{5760}+\frac{7\,\overline{2\overline{\overline{23}3}}}{5760}+\frac{1}{480}\overline{\overline{12}\,\overline{13}}-\frac{\overline{\overline{13}\,\overline{23}}}{1920}+\frac{1}{640}\overline{1\overline{\overline{23}2}}-\frac{\overline{\overline{\overline{13}2}2}}{1152}-\frac{\overline{\overline{\overline{13}3}2}}{1152}-\frac{\overline{\overline{\overline{23}3}3}}{1440},\ \ldots\Big],$$

$$2\to \mathrm{LS}\Big[0,\ -\frac{\overline{13}}{24},\ 0,\ \frac{\overline{1\overline{1\overline{13}}}}{1440}-\frac{\overline{1\overline{1\overline{23}}}}{1152}+\frac{7\,\overline{1\overline{2\overline{23}}}}{1920}-\frac{1}{480}\overline{1\overline{\overline{13}2}}-\frac{\overline{1\overline{\overline{13}3}}}{5760}+\frac{\overline{1\overline{\overline{23}3}}}{1152}+\frac{7\,\overline{\overline{12}\,\overline{13}}}{5760}+\frac{19\,\overline{\overline{13}\,\overline{23}}}{5760}+\frac{7\,\overline{1\overline{\overline{23}2}}}{1920}+\frac{7\,\overline{\overline{\overline{13}2}2}}{5760}+\frac{7\,\overline{\overline{\overline{13}3}2}}{5760}+\frac{\overline{\overline{\overline{13}3}3}}{1440},\ \ldots\Big],$$

$$3\to \mathrm{LS}\Big[0,\ \frac{\overline{12}}{24},\ 0,\ -\frac{\overline{1\overline{1\overline{12}}}}{1440}+\frac{\overline{1\overline{1\overline{23}}}}{5760}+\frac{7\,\overline{1\overline{2\overline{23}}}}{5760}+\frac{7\,\overline{1\overline{\overline{12}2}}}{5760}-\frac{\overline{1\overline{\overline{13}2}}}{1440}-\frac{\overline{1\overline{\overline{23}3}}}{1440}+\frac{\overline{\overline{12}\,\overline{13}}}{5760}+\frac{\overline{\overline{13}\,\overline{23}}}{1440}-\frac{\overline{\overline{1\overline{23}}2}}{1152}-\frac{7\,\overline{\overline{\overline{12}2}2}}{5760}-\frac{7\,\overline{\overline{\overline{13}2}2}}{5760}-\frac{\overline{\overline{\overline{13}3}2}}{1440},\ \ldots\Big]\Big\rangle,\ \mathrm{CWS}[0,0,0,0,\ldots]\Big]$$

The computation below takes a a couple of hours and yields “BS[8 True,False,...]”:

```
TrueQ[DK2Es[1, 2, 3][Φ0] ≡ ΦV]@{8}
BS[8 True, False, ...]
```

Section 3.6 - Solving the Kashiwara-Vergne Equations Using a Drinfel'd Associator

ZB

In[ ]:=

```
R = DKS[t[1, 2] / 2];
Z_B = (-Φ_0)^σ[13,2,4] ** Φ_0^σ[1,3,2] ** R^σ[2,3] ** (-Φ_0)^σ[1,2,3] ** Φ_0^σ[12,3,4]
```

Out[ ]=
ZB

$$\mathrm{DKS}\Big[\frac{\overline{t_{23}}}{2},\ -\frac{1}{12}\overline{t_{13}t_{23}}-\frac{1}{24}\overline{t_{14}t_{24}}+\frac{1}{24}\overline{t_{14}t_{34}}+\frac{1}{12}\overline{t_{24}t_{34}},\ 0,$$

$$\frac{\overline{\overline{\overline{t_{13}t_{23}}t_{23}}t_{23}}}{5760}+\frac{7\,\overline{\overline{\overline{t_{14}t_{24}}t_{24}}t_{24}}}{5760}+\frac{\overline{\overline{\overline{t_{14}t_{34}}t_{24}}t_{24}}}{1920}-\frac{\overline{\overline{\overline{t_{14}t_{34}}t_{34}}t_{24}}}{1920}-\frac{7\,\overline{\overline{\overline{t_{14}t_{34}}t_{34}}t_{34}}}{5760}-$$

$$\frac{\overline{\overline{\overline{t_{24}t_{34}}t_{34}}t_{34}}}{5760}+\frac{\overline{\overline{t_{14}\overline{t_{24}t_{34}}}t_{24}}}{1920}+\frac{\overline{\overline{t_{14}t_{24}}\,\overline{t_{14}t_{34}}}}{1920}-\frac{\overline{\overline{t_{14}t_{34}}\,\overline{t_{24}t_{34}}}}{1920}-\frac{1}{720}\overline{t_{13}\overline{\overline{t_{13}t_{23}}t_{23}}}+$$

$$\frac{1}{720}\overline{t_{13}\overline{t_{13}\overline{t_{13}t_{23}}}}-\frac{7\,\overline{t_{14}\overline{\overline{t_{14}t_{24}}t_{24}}}}{5760}+\frac{7\,\overline{t_{14}\overline{\overline{t_{14}t_{34}}t_{34}}}}{5760}-\frac{\overline{t_{14}\overline{\overline{t_{24}t_{34}}t_{34}}}}{5760}+\frac{\overline{t_{14}\overline{t_{14}\overline{t_{14}t_{24}}}}}{1440}-$$

$$\frac{\overline{t_{14}\overline{t_{14}\overline{t_{14}t_{34}}}}}{1440}-\frac{1}{960}\overline{t_{14}\overline{t_{14}\overline{t_{24}t_{34}}}}+\frac{\overline{t_{14}\overline{t_{24}\overline{t_{24}t_{34}}}}}{5760}-\frac{1}{960}\overline{t_{24}\overline{\overline{t_{24}t_{34}}t_{34}}}-\frac{\overline{t_{24}\overline{t_{24}\overline{t_{24}t_{34}}}}}{5760},\ \ldots\Big]$$

VfromPhi

```
Z_B // DK2Es[1, 2, 3, 4] // tη^1 // tη^3
```

VfromPhi

$$\mathrm{Es}\Big[\Big\langle 1\to\mathrm{LS}\Big[0,\ -\frac{\overline{2\,4}}{24},\ 0,\ \frac{7\,\overline{2\,\overline{2\,\overline{2\,4}}}}{5760}-\frac{7\,\overline{2\,\overline{\overline{2\,4}\,4}}}{5760}+\frac{\overline{\overline{\overline{2\,4}\,4}\,4}}{1440},\ \ldots\Big],$$

$$2\to\mathrm{LS}[0,0,0,0,\ldots],\ 3\to\mathrm{LS}\Big[\frac{\overline{2}}{2},\ -\frac{\overline{2\,4}}{12},\ 0,\ \frac{\overline{2\,\overline{2\,\overline{2\,4}}}}{5760}-\frac{1}{720}\overline{2\,\overline{\overline{2\,4}\,4}}+\frac{1}{720}\overline{\overline{\overline{2\,4}\,4}\,4},\ \ldots\Big],$$

$$4\to\mathrm{LS}[0,0,0,0,\ldots]\Big\rangle,\ \mathrm{CWS}[0,0,0,0,\ldots]\Big]$$

The computation below takes a few hours and yields "BS[8 True,False,...]":

```
V_B = Z_B // DK2Es[1, 2, 3, 4] // tη^1 // tη^3 // hη^2 // hη^4 // hσ[{1, 3} → {x, y}] //
   tσ[{2, 4} → {x, y}];
TrueQ[V_B⟦1⟧ ≡ V_0⟦1⟧] @ {8}
```

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 1, 2] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. »

BS[8 True, False, ...]

nu

In[∘]:=
```
νinv = Φ0 // DK2Es[1, 2, 3] // dS[2] // dm[3, 2, 2] // dm[2, 1, x]
```

Out[∘]=
nu

$$\text{Es}\left[\langle \overline{x} \to \text{LS}[0, 0, 0, 0, \ldots]\rangle, \text{CWS}\left[0, \frac{\widehat{xx}}{24}, 0, -\frac{\widehat{xxxx}}{2880}, \ldots\right]\right]$$

nucap4

```
(νinv ** Cap ** Cap ** Cap ** Cap) @ {6}
```

nucap4

$$\text{Es}\left[\langle \overline{x} \to \text{LS}[0, 0, 0, 0, 0, 0, \ldots]\rangle, \text{CWS}[0, 0, 0, 0, 0, 0, \ldots]\right]$$

Phi1

In[∘]:=
```
Φ1 = (Φ0)3 // LieMorphism[LW@1 → -x - y, LW@2 → y]
```

Out[∘]=
Phi1

$$\text{LS}\left[0, -\frac{\overline{xy}}{24}, 0, \frac{\overline{x\,\overline{x\,\overline{xy}}}}{1440} - \frac{\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440}, \ldots\right]$$

F

In[∘]:=
```
F = ⟨x → LieMorphism[y → -x - y][-Φ1],
   y → LS[(x + y) / 2] ~BCH~ LieMorphism[x → y, y → -x - y][-Φ1] ~BCH~ LS[-y / 2]⟩
```

Out[∘]=
F

$$\left\langle \overline{x} \to \text{LS}\left[0, -\frac{\overline{xy}}{24}, 0, \frac{7\,\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440}, \ldots\right],\right.$$
$$\left.\overline{y} \to \text{LS}\left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \frac{\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y}, \ldots\right]\right\rangle$$

FV

```
(F ≡ V0[[1]]) @ {7}
```

FV

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 1, 2] → 0}.

FV

BS[8 True, ...]

Section 3.7 - A Potential S_4 Action on Solutions of KV

rho2

```
ρ2[V_] := V // (-1)^deg;
V1 = Es[⟨x → LS[0], y → LS[-x / 2]⟩, CWS[0]] ** V0;
{(V1 ≡ ρ2[V1]) @ {8}, (V0 ≡ Rs[x, y] ** ρ2[V0]) @ {8}}
```

rho2

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.

rho2

{BS[9 True, ...], BS[9 True, ...]}

rho3

In[∘]:=
```
ρ3[ξ_Es] := ξ // dS[y] // dΔ[y, y, z] // dm[x, z, x] // dσ[{x, y} → {y, x}];
ξc = RandomEsSeries[1, {x, y}];
ξc ≡ (ξc // ρ3 // ρ3 // ρ3)
```

Out[∘]=
rho3

```
BS[5 True, ...]
```

V2

```
V2 = V0 ** Θs[x, y, -1 / 4] **
   Es[⟨x → LS@0, y → LS@0⟩, CWS[cw[x] / 12 - cw[y] / 12] - (2 Cap⟦2⟧ // tΔ[x, x, y])];
(V2 ≡ ρ3[V2]) @ {6}
```

V2

```
BS[7 True, ...]
```