

## Cheat Sheet J - Verification

Pensieve header: Cheat Sheet Free Lie Verification; continues CheatSheetJ-Verification @ pensieve://2014-01/.

```

SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
<< FreeLie.m;

tm[u_, v_, w_] := LieMorphism[LW@u -> LW@w, LW@v -> LW@w];
e /: e^2 = 0;
Print /@ {{t, u, v, w} = LW /@ {"t", "u", "v", "w"},
  alpha = RandomLieSeries[{t, u, v}], delta alpha = RandomLieSeries[{t, u, v}],
  beta = RandomLieSeries[{t, u, v}], delta beta = RandomLieSeries[{t, u, v}],
  gamma = RandomLieSeries[{t, u, v}], delta gamma = RandomLieSeries[{t, u, v}]
};
$SeriesShowDegree = 3; $SeriesCompareDegree = 6;

{LW[t], LW[u], LW[v], LW[w]}

LS[overline{t} + 2 overline{u} + overline{v}, -frac{3 overline{tu}}{2} + overline{tv} - overline{uv},
  -frac{1}{2} overline{ttu} - frac{11}{6} overline{ttv} - frac{3}{2} overline{tuv} - 2 overline{uuv} - overline{tuu} + frac{5}{3} overline{tvu} - frac{1}{6} overline{tvv} + frac{4}{3} overline{uvv}]

LS[-overline{t} + 2 overline{u} + overline{v}, -frac{overline{tu}}{2} - overline{tv} - 2 overline{uv},
  -frac{7}{6} overline{ttu} + frac{3}{2} overline{ttv} + frac{4}{3} overline{tuv} + 2 overline{uuv} - 2 overline{tuu} - overline{tvu} + frac{11}{6} overline{tvv} - frac{5}{6} overline{uvv}]

LS[-2 overline{t} + 2 overline{u} - 2 overline{v}, -frac{3 overline{tu}}{2} + 2 overline{tv} - frac{3 overline{uv}}{2},
  -frac{7}{6} overline{ttu} + overline{ttv} + 2 overline{tuv} - frac{3}{2} overline{uuv} + frac{1}{3} overline{tuu} - overline{tvu} + frac{1}{3} overline{tvv} - frac{4}{3} overline{uvv}]

LS[2 overline{t}, overline{tu} - overline{tv} - frac{3 overline{uv}}{2}, -frac{5}{6} overline{ttu} + frac{1}{6} overline{ttv} + overline{tuv} - frac{1}{3} overline{tuu} + frac{1}{2} overline{tvu} - frac{1}{6} overline{tvv} - frac{7}{6} overline{uvv}]

LS[2 overline{u}, -frac{3 overline{tu}}{2} + 2 overline{tv} + overline{uv}, -frac{2}{3} overline{ttu} - frac{1}{2} overline{ttv} + frac{2}{3} overline{tuv} - frac{1}{2} overline{uuv} - 2 overline{tuu} + frac{2}{3} overline{tvu} - frac{5}{6} overline{uvv}]

LS[-2 overline{t} - overline{u} - 2 overline{v}, -frac{overline{tu}}{2} - frac{3 overline{tv}}{2} + overline{uv},
  -frac{5}{3} overline{ttu} - frac{1}{6} overline{ttv} + frac{7}{6} overline{tuv} - frac{5}{3} overline{uuv} + frac{4}{3} overline{tuu} - overline{tvu} + frac{1}{2} overline{tvv} + frac{1}{6} overline{uvv}]

■ Some preliminary testing

(alpha // RC_u[gamma] // CC_u[-gamma]) == alpha
True

(alpha // RC_u[gamma] // RC_u[-gamma // RC_u[gamma]]) == alpha
True

```

## 1. The meaning(s) of RC

**Print** /@ {

```

1 → α,
2 → (t1 = α // CCu[γ] // RCu[-γ]),
3 → α ≡ t1
};

```

$$1 \rightarrow \text{LS} \left[ -2\bar{t} - 2\bar{u}, 2\bar{t}\bar{v} + \frac{\bar{u}\bar{v}}{2}, \right. \\ \left. \frac{7}{6}\overline{t\bar{t}u} + \frac{5}{6}\overline{t\bar{t}v} - \frac{1}{6}\overline{t\bar{u}\bar{v}} + 2\overline{u\bar{u}\bar{v}} - \frac{5}{6}\overline{t\bar{u}u} + \frac{2}{3}\overline{t\bar{v}u} - \frac{2}{3}\overline{t\bar{v}v} - \frac{11}{6}\overline{u\bar{v}v} \right]$$

$$2 \rightarrow \text{LS} \left[ -2\bar{t} - 2\bar{u}, 2\bar{t}\bar{v} + \frac{\bar{u}\bar{v}}{2}, \right. \\ \left. \frac{7}{6}\overline{t\bar{t}u} + \frac{5}{6}\overline{t\bar{t}v} - \frac{1}{6}\overline{t\bar{u}\bar{v}} + 2\overline{u\bar{u}\bar{v}} - \frac{5}{6}\overline{t\bar{u}u} + \frac{2}{3}\overline{t\bar{v}u} - \frac{2}{3}\overline{t\bar{v}v} - \frac{11}{6}\overline{u\bar{v}v} \right]$$

3 → True

**Print** /@ {

```

1 → α,
2 → (t1 = α // CCu[γ // RCu[γ]]),
3 → (t2 = α // RCu[γ]),
4 → t1 ≡ t2
};

```

$$1 \rightarrow \text{LS} \left[ -2\bar{t} - 2\bar{u}, 2\bar{t}\bar{v} + \frac{\bar{u}\bar{v}}{2}, \right. \\ \left. \frac{7}{6}\overline{t\bar{t}u} + \frac{5}{6}\overline{t\bar{t}v} - \frac{1}{6}\overline{t\bar{u}\bar{v}} + 2\overline{u\bar{u}\bar{v}} - \frac{5}{6}\overline{t\bar{u}u} + \frac{2}{3}\overline{t\bar{v}u} - \frac{2}{3}\overline{t\bar{v}v} - \frac{11}{6}\overline{u\bar{v}v} \right]$$

$$2 \rightarrow \text{LS} \left[ -2\bar{t} - 2\bar{u}, 2\bar{t}\bar{v} - \frac{3\bar{u}\bar{v}}{2}, \right. \\ \left. \frac{7}{6}\overline{t\bar{t}u} + \frac{5}{6}\overline{t\bar{t}v} - \frac{1}{6}\overline{t\bar{u}\bar{v}} + 2\overline{u\bar{u}\bar{v}} - \frac{29}{6}\overline{t\bar{u}u} - \frac{10}{3}\overline{t\bar{v}u} - \frac{2}{3}\overline{t\bar{v}v} - \frac{7}{3}\overline{u\bar{v}v} \right]$$

$$3 \rightarrow \text{LS} \left[ -2\bar{t} - 2\bar{u}, 2\bar{t}\bar{v} - \frac{3\bar{u}\bar{v}}{2}, \right. \\ \left. \frac{7}{6}\overline{t\bar{t}u} + \frac{5}{6}\overline{t\bar{t}v} - \frac{1}{6}\overline{t\bar{u}\bar{v}} + 2\overline{u\bar{u}\bar{v}} - \frac{29}{6}\overline{t\bar{u}u} - \frac{10}{3}\overline{t\bar{v}u} - \frac{2}{3}\overline{t\bar{v}v} - \frac{7}{3}\overline{u\bar{v}v} \right]$$

4 → True

## 2. $C_u C_v$ and $RC_u RC_v$

**Print** /@ {

```

1 → {α, β, γ},
2 → (t1 = γ // CCu[α // RCv[-β]] // CCv[β]),
3 → (t2 = γ // CCv[β // RCu[-α]] // CCu[α]),
4 → t1 ≡ t2
};

```

$$1 \rightarrow \left\{ \text{LS} \left[ -2 \overline{t} - 2 \overline{u}, 2 \overline{tv} + \frac{\overline{uv}}{2}, \right. \right. \\ \left. \left. \frac{7}{6} \overline{ttu} + \frac{5}{6} \overline{ttv} - \frac{1}{6} \overline{tuv} + 2 \overline{uuv} - \frac{5}{6} \overline{tuu} + \frac{2}{3} \overline{tvu} - \frac{2}{3} \overline{tvv} - \frac{11}{6} \overline{uvv} \right], \right. \\ \left. \text{LS} \left[ \overline{t} - \overline{v}, -\frac{3 \overline{tu}}{2} + \frac{3 \overline{tv}}{2} + \frac{\overline{uv}}{2}, -2 \overline{ttu} - \frac{11}{6} \overline{ttv} - \frac{5}{3} \overline{tuv} - \frac{5}{6} \overline{uuv} - \right. \right. \\ \left. \left. 2 \overline{tuu} + \frac{1}{2} \overline{tvu} - 2 \overline{tvv} - \overline{uvv} \right], \text{LS} \left[ -2 \overline{u} - \overline{v}, 2 \overline{tu} + 2 \overline{tv} + \overline{uv}, \right. \right. \\ \left. \left. \frac{11}{6} \overline{ttu} + \frac{1}{2} \overline{ttv} - \frac{1}{2} \overline{tuv} - \frac{5}{6} \overline{uuv} - \frac{1}{2} \overline{tuu} + \frac{1}{6} \overline{tvu} + \overline{tvv} - \frac{3}{2} \overline{uvv} \right] \right\}$$

$$2 \rightarrow \text{LS} \left[ -2 \overline{u} - \overline{v}, 6 \overline{tu} + \overline{tv} + \overline{uv}, \right. \\ \left. -\frac{37}{6} \overline{ttu} + 2 \overline{ttv} - \overline{tuv} + \frac{1}{6} \overline{uuv} + \frac{7}{2} \overline{tuu} - \frac{16}{3} \overline{tvu} - \overline{tvv} - 2 \overline{uvv} \right]$$

$$3 \rightarrow \text{LS} \left[ -2 \overline{u} - \overline{v}, 6 \overline{tu} + \overline{tv} + \overline{uv}, \right. \\ \left. -\frac{37}{6} \overline{ttu} + 2 \overline{ttv} - \overline{tuv} + \frac{1}{6} \overline{uuv} + \frac{7}{2} \overline{tuu} - \frac{16}{3} \overline{tvu} - \overline{tvv} - 2 \overline{uvv} \right]$$

4 → True

```
Print /@ {
  1 → {α, β, γ},
  2 → (t1 = γ // RCu[α] // RCv[β // RCu[α]]),
  3 → (t2 = γ // RCv[β] // RCu[α // RCv[β]]),
  4 → t1 ≡ t2
};
```

$$1 \rightarrow \left\{ \text{LS} \left[ -2 \overline{t} - 2 \overline{u}, 2 \overline{tv} + \frac{\overline{uv}}{2}, \right. \right. \\ \left. \left. \frac{7}{6} \overline{ttu} + \frac{5}{6} \overline{ttv} - \frac{1}{6} \overline{tuv} + 2 \overline{uuv} - \frac{5}{6} \overline{tuu} + \frac{2}{3} \overline{tvu} - \frac{2}{3} \overline{tvv} - \frac{11}{6} \overline{uvv} \right], \right. \\ \left. \text{LS} \left[ \overline{t} - \overline{v}, -\frac{3 \overline{tu}}{2} + \frac{3 \overline{tv}}{2} + \frac{\overline{uv}}{2}, -2 \overline{ttu} - \frac{11}{6} \overline{ttv} - \frac{5}{3} \overline{tuv} - \frac{5}{6} \overline{uuv} - \right. \right. \\ \left. \left. 2 \overline{tuu} + \frac{1}{2} \overline{tvu} - 2 \overline{tvv} - \overline{uvv} \right], \text{LS} \left[ -2 \overline{u} - \overline{v}, 2 \overline{tu} + 2 \overline{tv} + \overline{uv}, \right. \right. \\ \left. \left. \frac{11}{6} \overline{ttu} + \frac{1}{2} \overline{ttv} - \frac{1}{2} \overline{tuv} - \frac{5}{6} \overline{uuv} - \frac{1}{2} \overline{tuu} + \frac{1}{6} \overline{tvu} + \overline{tvv} - \frac{3}{2} \overline{uvv} \right] \right\}$$

$$2 \rightarrow \text{LS} \left[ -2 \overline{u} - \overline{v}, 6 \overline{tu} + \overline{tv} + \overline{uv}, -\frac{37}{6} \overline{ttu} + 2 \overline{ttv} - \overline{tuv} + \frac{1}{6} \overline{uuv} - \frac{9}{2} \overline{tuu} - \frac{16}{3} \overline{tvu} - 2 \overline{uvv} \right]$$

$$3 \rightarrow \text{LS} \left[ -2 \overline{u} - \overline{v}, 6 \overline{tu} + \overline{tv} + \overline{uv}, -\frac{37}{6} \overline{ttu} + 2 \overline{ttv} - \overline{tuv} + \frac{1}{6} \overline{uuv} - \frac{9}{2} \overline{tuu} - \frac{16}{3} \overline{tvu} - 2 \overline{uvv} \right]$$

4 → True

3.

4.

5. The map  $\Gamma$

```
Print /@ {  
  0 → { $\lambda = \langle u \rightarrow \alpha, v \rightarrow \beta \rangle, \gamma$ },  
  1 → (t1 =  $\Gamma_{\tau}[\lambda]$ ),  
  2 → (t2 =  $\gamma$  // DerivationExp[- $\tau D_{\lambda}$ ]),  
  3 → (t3 =  $\gamma$  // CC[ $\Gamma_{\tau}[\lambda]$ ]),  
  4 → t2 ≡ t3  
};
```

$$0 \rightarrow \left\{ \left( \text{LW}[u] \rightarrow \text{LS} \left[ -\bar{t} + \bar{u} - 2\bar{v}, -2\bar{t}\bar{u} - \frac{\bar{t}\bar{v}}{2} - \bar{u}\bar{v}, \right. \right. \right. \\ \left. \left. \left. - \frac{11}{6} \overline{t\bar{t}\bar{u}} + \frac{5}{6} \overline{t\bar{t}\bar{v}} + \frac{1}{6} \overline{t\bar{u}\bar{v}} - \frac{5}{6} \overline{u\bar{u}\bar{v}} + \frac{5}{6} \overline{t\bar{u}\bar{u}} + \frac{2}{3} \overline{t\bar{v}\bar{u}} + \frac{11}{6} \overline{t\bar{v}\bar{v}} - \frac{4}{3} \overline{u\bar{v}\bar{v}} \right], \right. \\ \left. \text{LW}[v] \rightarrow \text{LS} \left[ \bar{t} + \bar{u} - 2\bar{v}, \frac{\bar{t}\bar{u}}{2} - 2\bar{t}\bar{v} + \frac{\bar{u}\bar{v}}{2}, -2\bar{t}\bar{t}\bar{u} + \frac{1}{2} \overline{t\bar{t}\bar{v}} - \overline{t\bar{u}\bar{v}} + \frac{11}{6} \overline{u\bar{u}\bar{v}} - \right. \right. \\ \left. \left. \frac{5}{6} \overline{t\bar{u}\bar{u}} + \overline{t\bar{v}\bar{u}} - 2\overline{t\bar{v}\bar{v}} - \frac{5}{3} \overline{u\bar{v}\bar{v}} \right], \text{LS} \left[ \bar{t} + 2\bar{u} + 2\bar{v}, -\frac{3\bar{t}\bar{u}}{2} + \frac{\bar{t}\bar{v}}{2} + \frac{\bar{u}\bar{v}}{2}, \right. \right. \\ \left. \left. \frac{1}{2} \overline{t\bar{t}\bar{u}} - \frac{5}{3} \overline{t\bar{t}\bar{v}} + \frac{1}{3} \overline{t\bar{u}\bar{v}} - \frac{1}{6} \overline{u\bar{u}\bar{v}} + \frac{3}{2} \overline{t\bar{u}\bar{u}} + \frac{5}{3} \overline{t\bar{v}\bar{u}} - \frac{5}{3} \overline{t\bar{v}\bar{v}} - \frac{11}{6} \overline{u\bar{v}\bar{v}} \right] \right\}$$

$$1 \rightarrow \left( \text{LW}[u] \rightarrow \text{LS} \left[ -\tau\bar{t} + \tau\bar{u} - 2\tau\bar{v}, -2\tau\bar{t}\bar{u} - \frac{1}{2}\tau^2\bar{t}\bar{u} - \frac{1}{2}\tau\bar{t}\bar{v} - \tau^2\bar{t}\bar{v} - \tau\bar{u}\bar{v}, \right. \right. \\ \left. \left. - \frac{11}{6}\tau\overline{t\bar{t}\bar{u}} + \tau^2\overline{t\bar{t}\bar{u}} + \frac{1}{12}\tau^3\overline{t\bar{t}\bar{u}} + \frac{5}{6}\tau\overline{t\bar{t}\bar{v}} - \frac{1}{4}\tau^2\overline{t\bar{t}\bar{v}} - \frac{1}{2}\tau^3\overline{t\bar{t}\bar{v}} + \frac{1}{6}\tau\overline{t\bar{u}\bar{v}} - \right. \right. \\ \left. \left. \frac{9}{4}\tau^2\overline{t\bar{u}\bar{v}} - \frac{1}{2}\tau^3\overline{t\bar{u}\bar{v}} - \frac{5}{6}\tau\overline{u\bar{u}\bar{v}} + \frac{5}{6}\tau\overline{t\bar{u}\bar{u}} - \tau^2\overline{t\bar{u}\bar{u}} - \frac{1}{12}\tau^3\overline{t\bar{u}\bar{u}} + \frac{2}{3}\tau\overline{t\bar{v}\bar{u}} + \right. \right. \\ \left. \left. \frac{1}{4}\tau^2\overline{t\bar{v}\bar{u}} + \frac{11}{6}\tau\overline{t\bar{v}\bar{v}} + 2\tau^2\overline{t\bar{v}\bar{v}} + \frac{1}{3}\tau^3\overline{t\bar{v}\bar{v}} - \frac{4}{3}\tau\overline{u\bar{v}\bar{v}} - \frac{3}{2}\tau^2\overline{u\bar{v}\bar{v}} \right], \right. \\ \left. \text{LW}[v] \rightarrow \text{LS} \left[ \tau\bar{t} + \tau\bar{u} - 2\tau\bar{v}, \frac{1}{2}\tau\bar{t}\bar{u} - \frac{1}{2}\tau^2\bar{t}\bar{u} - 2\tau\bar{t}\bar{v} - \tau^2\bar{t}\bar{v} + \frac{1}{2}\tau\bar{u}\bar{v}, \right. \right. \\ \left. \left. -2\tau\overline{t\bar{t}\bar{u}} - \frac{1}{4}\tau^2\overline{t\bar{t}\bar{u}} + \frac{1}{4}\tau^3\overline{t\bar{t}\bar{u}} + \frac{1}{2}\tau\overline{t\bar{t}\bar{v}} - \tau^2\overline{t\bar{t}\bar{v}} - \frac{1}{6}\tau^3\overline{t\bar{t}\bar{v}} - \tau\overline{t\bar{u}\bar{v}} - \right. \right. \\ \left. \left. \frac{5}{4}\tau^2\overline{t\bar{u}\bar{v}} - \frac{1}{2}\tau^3\overline{t\bar{u}\bar{v}} + \frac{11}{6}\tau\overline{u\bar{u}\bar{v}} + \frac{3}{4}\tau^2\overline{u\bar{u}\bar{v}} - \frac{5}{6}\tau\overline{t\bar{u}\bar{u}} - \tau^2\overline{t\bar{u}\bar{u}} - \right. \right. \\ \left. \left. \frac{1}{12}\tau^3\overline{t\bar{u}\bar{u}} + \tau\overline{t\bar{v}\bar{u}} - \frac{5}{4}\tau^2\overline{t\bar{v}\bar{u}} - 2\tau\overline{t\bar{v}\bar{v}} + 2\tau^2\overline{t\bar{v}\bar{v}} + \frac{1}{3}\tau^3\overline{t\bar{v}\bar{v}} - \frac{5}{3}\tau\overline{u\bar{v}\bar{v}} \right] \right)$$

$$2 \rightarrow \text{LS} \left[ \bar{t} + 2\bar{u} + 2\bar{v}, -\frac{3\bar{t}\bar{u}}{2} - 2\tau\bar{t}\bar{u} + \frac{\bar{t}\bar{v}}{2} + 2\tau\bar{t}\bar{v} + \frac{\bar{u}\bar{v}}{2} + 6\tau\bar{u}\bar{v}, \right. \\ \left. \frac{1}{2}\overline{t\bar{t}\bar{u}} + \frac{3}{2}\tau\overline{t\bar{t}\bar{u}} + \tau^2\overline{t\bar{t}\bar{u}} - \frac{5}{3}\overline{t\bar{t}\bar{v}} + \frac{1}{2}\tau\overline{t\bar{t}\bar{v}} + \tau^2\overline{t\bar{t}\bar{v}} + \frac{1}{3}\overline{t\bar{u}\bar{v}} - 2\tau\overline{t\bar{u}\bar{v}} - \right. \\ \left. 4\tau^2\overline{t\bar{u}\bar{v}} - \frac{1}{6}\overline{u\bar{u}\bar{v}} + \frac{5}{2}\tau\overline{u\bar{u}\bar{v}} + 3\tau^2\overline{u\bar{u}\bar{v}} + \frac{3}{2}\overline{t\bar{u}\bar{u}} - 4\tau\overline{t\bar{u}\bar{u}} + \frac{5}{3}\overline{t\bar{v}\bar{u}} - \right. \\ \left. \tau\overline{t\bar{v}\bar{u}} - 6\tau^2\overline{t\bar{v}\bar{u}} - \frac{5}{3}\overline{t\bar{v}\bar{v}} - 4\tau\overline{t\bar{v}\bar{v}} - \frac{11}{6}\overline{u\bar{v}\bar{v}} + 2\tau\overline{u\bar{v}\bar{v}} + 6\tau^2\overline{u\bar{v}\bar{v}} \right]$$

$$3 \rightarrow \text{LS} \left[ \bar{t} + 2\bar{u} + 2\bar{v}, -\frac{3\bar{t}\bar{u}}{2} - 2\tau\bar{t}\bar{u} + \frac{\bar{t}\bar{v}}{2} + 2\tau\bar{t}\bar{v} + \frac{\bar{u}\bar{v}}{2} + 6\tau\bar{u}\bar{v}, \right. \\ \left. \frac{1}{2}\overline{t\bar{t}\bar{u}} + \frac{3}{2}\tau\overline{t\bar{t}\bar{u}} + \tau^2\overline{t\bar{t}\bar{u}} - \frac{5}{3}\overline{t\bar{t}\bar{v}} + \frac{1}{2}\tau\overline{t\bar{t}\bar{v}} + \tau^2\overline{t\bar{t}\bar{v}} + \frac{1}{3}\overline{t\bar{u}\bar{v}} - 2\tau\overline{t\bar{u}\bar{v}} - \right. \\ \left. 4\tau^2\overline{t\bar{u}\bar{v}} - \frac{1}{6}\overline{u\bar{u}\bar{v}} + \frac{5}{2}\tau\overline{u\bar{u}\bar{v}} + 3\tau^2\overline{u\bar{u}\bar{v}} + \frac{3}{2}\overline{t\bar{u}\bar{u}} - 4\tau\overline{t\bar{u}\bar{u}} + \frac{5}{3}\overline{t\bar{v}\bar{u}} - \right. \\ \left. \tau\overline{t\bar{v}\bar{u}} - 6\tau^2\overline{t\bar{v}\bar{u}} - \frac{5}{3}\overline{t\bar{v}\bar{v}} - 4\tau\overline{t\bar{v}\bar{v}} - \frac{11}{6}\overline{u\bar{v}\bar{v}} + 2\tau\overline{u\bar{v}\bar{v}} + 6\tau^2\overline{u\bar{v}\bar{v}} \right]$$

4 → True

## 6. The map $\Lambda$

```
Print /@ {  
  0 → { $\lambda = \langle u \rightarrow \alpha, v \rightarrow \beta \rangle, \gamma$ },  
  1 → (t1 =  $\Lambda_{\tau}[\lambda]$ ),  
  2 → (t2 =  $\gamma$  // DerivationExp[-D $_{\Lambda_{\tau}[\lambda]}$ ]),  
  3 → (t3 =  $\gamma$  // CC[ $\tau \lambda$ ]),  
  4 → t2 ≡ t3  
};
```

$$0 \rightarrow \left\{ \begin{aligned} & \text{LW}[u] \rightarrow \text{LS} \left[ -\bar{t} + \bar{u} - 2\bar{v}, -2\bar{t}\bar{u} - \frac{\bar{t}\bar{v}}{2} - \bar{u}\bar{v}, \right. \\ & \quad \left. -\frac{11}{6}\overline{t\bar{t}\bar{u}} + \frac{5}{6}\overline{t\bar{t}\bar{v}} + \frac{1}{6}\overline{t\bar{u}\bar{v}} - \frac{5}{6}\overline{u\bar{u}\bar{v}} + \frac{5}{6}\overline{t\bar{u}\bar{u}} + \frac{2}{3}\overline{t\bar{v}\bar{u}} + \frac{11}{6}\overline{t\bar{v}\bar{v}} - \frac{4}{3}\overline{u\bar{v}\bar{v}} \right], \\ & \text{LW}[v] \rightarrow \text{LS} \left[ \bar{t} + \bar{u} - 2\bar{v}, \frac{\bar{t}\bar{u}}{2} - 2\bar{t}\bar{v} + \frac{\bar{u}\bar{v}}{2}, -2\overline{t\bar{t}\bar{u}} + \frac{1}{2}\overline{t\bar{t}\bar{v}} - \overline{t\bar{u}\bar{v}} + \frac{11}{6}\overline{u\bar{u}\bar{v}} - \right. \\ & \quad \left. \frac{5}{6}\overline{t\bar{u}\bar{u}} + \overline{t\bar{v}\bar{u}} - 2\overline{t\bar{v}\bar{v}} - \frac{5}{3}\overline{u\bar{v}\bar{v}} \right], \text{LS} \left[ \bar{t} + 2\bar{u} + 2\bar{v}, -\frac{3\bar{t}\bar{u}}{2} + \frac{\bar{t}\bar{v}}{2} + \frac{\bar{u}\bar{v}}{2}, \right. \\ & \quad \left. \frac{1}{2}\overline{t\bar{t}\bar{u}} - \frac{5}{3}\overline{t\bar{t}\bar{v}} + \frac{1}{3}\overline{t\bar{u}\bar{v}} - \frac{1}{6}\overline{u\bar{u}\bar{v}} + \frac{3}{2}\overline{t\bar{u}\bar{u}} + \frac{5}{3}\overline{t\bar{v}\bar{u}} - \frac{5}{3}\overline{t\bar{v}\bar{v}} - \frac{11}{6}\overline{u\bar{v}\bar{v}} \right] \end{aligned} \right\}$$

$$1 \rightarrow \left( \text{LW}[u] \rightarrow \right.$$

$$\text{LS} \left[ -\tau\bar{t} + \tau\bar{u} - 2\tau\bar{v}, -2\tau\bar{t}\bar{u} + \frac{1}{2}\tau^2\bar{t}\bar{u} - \frac{1}{2}\tau\bar{t}\bar{v} + \tau^2\bar{t}\bar{v} - \tau\bar{u}\bar{v}, -\frac{11}{6}\tau\overline{t\bar{t}\bar{u}} - \tau^2\overline{t\bar{t}\bar{u}} + \frac{1}{6}\tau^3\overline{t\bar{t}\bar{u}} + \right. \\ \left. \frac{5}{6}\tau\overline{t\bar{t}\bar{v}} + \frac{1}{4}\tau^2\overline{t\bar{t}\bar{v}} + \frac{1}{6}\tau\overline{t\bar{u}\bar{v}} + \frac{9}{4}\tau^2\overline{t\bar{u}\bar{v}} - \frac{5}{6}\tau\overline{u\bar{u}\bar{v}} + \frac{5}{6}\tau\overline{t\bar{u}\bar{u}} + \tau^2\overline{t\bar{u}\bar{u}} - \frac{1}{6}\tau^3\overline{t\bar{u}\bar{u}} + \right. \\ \left. \frac{2}{3}\tau\overline{t\bar{v}\bar{u}} - \frac{1}{4}\tau^2\overline{t\bar{v}\bar{u}} + \frac{11}{6}\tau\overline{t\bar{v}\bar{v}} - 2\tau^2\overline{t\bar{v}\bar{v}} + \frac{2}{3}\tau^3\overline{t\bar{v}\bar{v}} - \frac{4}{3}\tau\overline{u\bar{v}\bar{v}} + \frac{3}{2}\tau^2\overline{u\bar{v}\bar{v}} \right],$$

$$\text{LW}[v] \rightarrow \text{LS} \left[ \tau\bar{t} + \tau\bar{u} - 2\tau\bar{v}, \frac{1}{2}\tau\bar{t}\bar{u} + \frac{1}{2}\tau^2\bar{t}\bar{u} - 2\tau\bar{t}\bar{v} + \tau^2\bar{t}\bar{v} + \frac{1}{2}\tau\bar{u}\bar{v}, -2\tau\overline{t\bar{t}\bar{u}} + \frac{1}{4}\tau^2\overline{t\bar{t}\bar{u}} + \right. \\ \left. \frac{1}{2}\tau\overline{t\bar{t}\bar{v}} + \tau^2\overline{t\bar{t}\bar{v}} - \frac{1}{3}\tau^3\overline{t\bar{t}\bar{v}} - \tau\overline{t\bar{u}\bar{v}} + \frac{5}{4}\tau^2\overline{t\bar{u}\bar{v}} + \frac{11}{6}\tau\overline{u\bar{u}\bar{v}} - \frac{3}{4}\tau^2\overline{u\bar{u}\bar{v}} - \frac{5}{6}\tau\overline{t\bar{u}\bar{u}} + \right. \\ \left. \tau^2\overline{t\bar{u}\bar{u}} - \frac{1}{6}\tau^3\overline{t\bar{u}\bar{u}} + \tau\overline{t\bar{v}\bar{u}} + \frac{5}{4}\tau^2\overline{t\bar{v}\bar{u}} - 2\tau\overline{t\bar{v}\bar{v}} - 2\tau^2\overline{t\bar{v}\bar{v}} + \frac{2}{3}\tau^3\overline{t\bar{v}\bar{v}} - \frac{5}{3}\tau\overline{u\bar{v}\bar{v}} \right]$$

$$2 \rightarrow \text{LS} \left[ \bar{t} + 2\bar{u} + 2\bar{v}, -\frac{3\bar{t}\bar{u}}{2} - 2\tau\bar{t}\bar{u} + \frac{\bar{t}\bar{v}}{2} + 2\tau\bar{t}\bar{v} + \frac{\bar{u}\bar{v}}{2} + 6\tau\bar{u}\bar{v}, \right.$$

$$\frac{1}{2}\overline{t\bar{t}\bar{u}} + \frac{3}{2}\tau\overline{t\bar{t}\bar{u}} + \tau^2\overline{t\bar{t}\bar{u}} - \frac{5}{3}\overline{t\bar{t}\bar{v}} + \frac{1}{2}\tau\overline{t\bar{t}\bar{v}} + \tau^2\overline{t\bar{t}\bar{v}} + \frac{1}{3}\overline{t\bar{u}\bar{v}} - 2\tau\overline{t\bar{u}\bar{v}} - \\ 3\tau^2\overline{t\bar{u}\bar{v}} - \frac{1}{6}\overline{u\bar{u}\bar{v}} + \frac{5}{2}\tau\overline{u\bar{u}\bar{v}} + 3\tau^2\overline{u\bar{u}\bar{v}} + \frac{3}{2}\overline{t\bar{u}\bar{u}} - 4\tau\overline{t\bar{u}\bar{u}} + \tau^2\overline{t\bar{u}\bar{u}} + \frac{5}{3}\overline{t\bar{v}\bar{u}} - \\ \tau\overline{t\bar{v}\bar{u}} - 3\tau^2\overline{t\bar{v}\bar{u}} - \frac{5}{3}\overline{t\bar{v}\bar{v}} - 4\tau\overline{t\bar{v}\bar{v}} + 2\tau^2\overline{t\bar{v}\bar{v}} - \frac{11}{6}\overline{u\bar{v}\bar{v}} + 2\tau\overline{u\bar{v}\bar{v}} + 6\tau^2\overline{u\bar{v}\bar{v}} \left. \right]$$

$$3 \rightarrow \text{LS} \left[ \bar{t} + 2\bar{u} + 2\bar{v}, -\frac{3\bar{t}\bar{u}}{2} - 2\tau\bar{t}\bar{u} + \frac{\bar{t}\bar{v}}{2} + 2\tau\bar{t}\bar{v} + \frac{\bar{u}\bar{v}}{2} + 6\tau\bar{u}\bar{v}, \right.$$

$$\frac{1}{2}\overline{t\bar{t}\bar{u}} + \frac{3}{2}\tau\overline{t\bar{t}\bar{u}} + \tau^2\overline{t\bar{t}\bar{u}} - \frac{5}{3}\overline{t\bar{t}\bar{v}} + \frac{1}{2}\tau\overline{t\bar{t}\bar{v}} + \tau^2\overline{t\bar{t}\bar{v}} + \frac{1}{3}\overline{t\bar{u}\bar{v}} - 2\tau\overline{t\bar{u}\bar{v}} - \\ 3\tau^2\overline{t\bar{u}\bar{v}} - \frac{1}{6}\overline{u\bar{u}\bar{v}} + \frac{5}{2}\tau\overline{u\bar{u}\bar{v}} + 3\tau^2\overline{u\bar{u}\bar{v}} + \frac{3}{2}\overline{t\bar{u}\bar{u}} - 4\tau\overline{t\bar{u}\bar{u}} + \tau^2\overline{t\bar{u}\bar{u}} + \frac{5}{3}\overline{t\bar{v}\bar{u}} - \\ \tau\overline{t\bar{v}\bar{u}} - 3\tau^2\overline{t\bar{v}\bar{u}} - \frac{5}{3}\overline{t\bar{v}\bar{v}} - 4\tau\overline{t\bar{v}\bar{v}} + 2\tau^2\overline{t\bar{v}\bar{v}} - \frac{11}{6}\overline{u\bar{v}\bar{v}} + 2\tau\overline{u\bar{v}\bar{v}} + 6\tau^2\overline{u\bar{v}\bar{v}} \left. \right]$$

$$4 \rightarrow \text{True}$$

```

Print /@ {
  0 → {λ = ⟨u → α, v → β⟩},
  1 → (t1 = Λτ[λ] / τ),
  2 → (t2 = Γτ[t1] / τ),
  4 → t2 ≡ λ
};
0 →
{
  {LW[u] → LS[τ + 2ū + v̄, - $\frac{3\overline{tu}}{2}$  + τv̄ - uv̄, - $\frac{1}{2}$   $\overline{ttu}$  -  $\frac{11}{6}$   $\overline{ttv}$  -  $\frac{3}{2}$   $\overline{tuv}$  - 2  $\overline{uuv}$  -  $\overline{tuu}$  +  $\frac{5}{3}$   $\overline{tvu}$  -
     $\frac{1}{6}$   $\overline{tvv}$  +  $\frac{4}{3}$   $\overline{uvv}$ ], LW[v] → LS[-2τ + 2ū - 2v̄, - $\frac{3\overline{tu}}{2}$  + 2τv̄ -  $\frac{3\overline{uv}}{2}$ ,
    - $\frac{7}{6}$   $\overline{ttu}$  +  $\overline{ttv}$  + 2  $\overline{tuv}$  -  $\frac{3}{2}$   $\overline{uuv}$  +  $\frac{1}{3}$   $\overline{tuu}$  -  $\overline{tvu}$  +  $\frac{1}{3}$   $\overline{tvv}$  -  $\frac{4}{3}$   $\overline{uvv}$ ]}},
  1 →
  {
    {LW[u] → LS[τ + 2ū + v̄, - $\frac{3\overline{tu}}{2}$  - τ  $\overline{tu}$  + τv̄ + τ  $\overline{tv}$  - uv̄, - $\frac{1}{2}$   $\overline{ttu}$  +  $\frac{3}{4}$  τ  $\overline{ttu}$  +  $\frac{1}{3}$  τ2  $\overline{ttu}$  -  $\frac{11}{6}$   $\overline{ttv}$  +
      τ  $\overline{ttv}$  +  $\frac{1}{6}$  τ2  $\overline{ttv}$  -  $\frac{3}{2}$   $\overline{tuv}$  -  $\frac{1}{2}$  τ  $\overline{tuv}$  -  $\frac{1}{6}$  τ2  $\overline{tuv}$  - 2  $\overline{uuv}$  -  $\overline{tuu}$  +  $\frac{3}{2}$  τ  $\overline{tuu}$  +  $\frac{2}{3}$  τ2  $\overline{tuu}$  +
       $\frac{5}{3}$   $\overline{tvu}$  +  $\frac{5}{4}$  τ  $\overline{tvu}$  -  $\frac{1}{3}$  τ2  $\overline{tvu}$  -  $\frac{1}{6}$   $\overline{tvv}$  - τ  $\overline{tvv}$  +  $\frac{7}{6}$  τ2  $\overline{tvv}$  +  $\frac{4}{3}$   $\overline{uvv}$  +  $\frac{1}{4}$  τ  $\overline{uvv}$  -  $\frac{3}{2}$  τ2  $\overline{uvv}$ ],
      LW[v] → LS[-2τ + 2ū - 2v̄, - $\frac{3\overline{tu}}{2}$  - τ  $\overline{tu}$  + 2τv̄ - 2τ  $\overline{tv}$  -  $\frac{3\overline{uv}}{2}$  + 3τ  $\overline{uv}$ ,
      - $\frac{7}{6}$   $\overline{ttu}$  +  $\frac{3}{4}$  τ  $\overline{ttu}$  -  $\frac{1}{6}$  τ2  $\overline{ttu}$  +  $\overline{ttv}$  + 2τ  $\overline{ttv}$  -  $\frac{4}{3}$  τ2  $\overline{ttv}$  + 2  $\overline{tuv}$  -  $\frac{7}{2}$  τ  $\overline{tuv}$  +
       $\frac{1}{3}$  τ2  $\overline{tuv}$  -  $\frac{3}{2}$   $\overline{uuv}$  +  $\frac{1}{2}$  τ  $\overline{uuv}$  - 2τ2  $\overline{uuv}$  +  $\frac{1}{3}$   $\overline{tuu}$  +  $\frac{3}{2}$  τ  $\overline{tuu}$  +  $\frac{2}{3}$  τ2  $\overline{tuu}$  -  $\overline{tvu}$  -
       $\frac{1}{4}$  τ  $\overline{tvu}$  -  $\frac{23}{6}$  τ2  $\overline{tvu}$  +  $\frac{1}{3}$   $\overline{tvv}$  + 2τ  $\overline{tvv}$  -  $\frac{4}{3}$  τ2  $\overline{tvv}$  -  $\frac{4}{3}$   $\overline{uvv}$  -  $\frac{9}{4}$  τ  $\overline{uvv}$  +  $\frac{5}{2}$  τ2  $\overline{uvv}$ ]}},
    2 → {LW[u] → LS[τ + 2ū + v̄, - $\frac{3\overline{tu}}{2}$  + τv̄ - uv̄,
      - $\frac{1}{2}$   $\overline{ttu}$  -  $\frac{11}{6}$   $\overline{ttv}$  -  $\frac{3}{2}$   $\overline{tuv}$  - 2  $\overline{uuv}$  -  $\overline{tuu}$  +  $\frac{5}{3}$   $\overline{tvu}$  -  $\frac{1}{6}$   $\overline{tvv}$  +  $\frac{4}{3}$   $\overline{uvv}$ ],
      LW[v] → LS[-2τ + 2ū - 2v̄, - $\frac{3\overline{tu}}{2}$  + 2τv̄ -  $\frac{3\overline{uv}}{2}$ ,
      - $\frac{7}{6}$   $\overline{ttu}$  +  $\overline{ttv}$  + 2  $\overline{tuv}$  -  $\frac{3}{2}$   $\overline{uuv}$  +  $\frac{1}{3}$   $\overline{tuu}$  -  $\overline{tvu}$  +  $\frac{1}{3}$   $\overline{tvv}$  -  $\frac{4}{3}$   $\overline{uvv}$ ]}},
    4 → ⟨LW[u] → True, LW[v] → True⟩
  }
}

```

7.

8. div property uv



```

Print /@ {
  0 → {α, β},
  1 → (t1 = divu[α] // adv[β]),
  2 → (t2 = divv[β] // adu[α]),
  3 → (t3 = MakeCWSeries[0]),
  4 → (t4 = divu[α // adv[β]]),
  5 → (t5 = divv[β // adu[α]]),
  6 → t1 - t2 ≡ t3 + t4 - t5
};

0 → {LS[-2 t̄ - 2 ū, 2 tv̄ +  $\frac{uv̄}{2}$ ,
 $\frac{7}{6} \overline{ttu} + \frac{5}{6} \overline{ttv} - \frac{1}{6} \overline{tuv} + 2 \overline{uuv} - \frac{5}{6} \overline{tuu} + \frac{2}{3} \overline{tvu} - \frac{2}{3} \overline{tvv} - \frac{11}{6} \overline{uvv}$ ], LS[t̄ - v̄,
 $-\frac{3}{2} \overline{tu} + \frac{3}{2} \overline{tv} + \frac{uv̄}{2}$ , -2  $\overline{ttu} - \frac{11}{6} \overline{ttv} - \frac{5}{3} \overline{tuv} - \frac{5}{6} \overline{uuv} - 2 \overline{tuu} + \frac{1}{2} \overline{tvu} - 2 \overline{tvv} - \overline{uvv}$ ]}

1 → CWS[0, 0,  $\frac{\overline{tuv}}{2} - \frac{\overline{tvu}}{2}$ ]
2 → CWS[0, 0, - $\overline{tuv} + \overline{tvu}$ ]
3 → CWS[0, 0, 0]
4 → CWS[0, 0,  $\frac{\overline{tuv}}{2} - \frac{\overline{tvu}}{2}$ ]
5 → CWS[0, 0, - $\overline{tuv} + \overline{tvu}$ ]
6 → True

```

### 9. div property uu

```

Print /@ {
  0 → {α, β},
  1 → (t1 = divu[α] // adu[β]),
  2 → (t2 = divu[β] // adu[α]),
  3 → (t3 = divu[b[α, β]]),
  4 → (t4 = divu[α // adu[β]]),
  5 → (t5 = divu[β // adu[α]]),
  6 → t1 - t2 ≡ t3 + t4 - t5
};

```

```

0 -> {LS[-2 t - 2 u, 2 tv + (uv)/2,
  7 (ttu)/6 + 5 (ttv)/6 - 1 (tuv)/6 + 2 (uuv)/6 - 5 (tuu)/6 + 2 (tvu)/3 - 2 (tvv)/3 - 11 (uvv)/6], LS[t - v,
  - 3 (tu)/2 + 3 (tv)/2 + (uv)/2, -2 (ttu)/6 - 11 (ttv)/6 - 5 (tuv)/3 - 5 (uuv)/6 - 2 (tuu) + 1 (tvu) - 2 (tvv) - (uvv)]}
1 -> CWS[0, 0, - (tuv)/2 + (tvu)/2]
2 -> CWS[0, 0, (tuv) - (tvu)]
3 -> CWS[0, 2 (tu) - 2 (uv), 3 (ttu) + 3 (tuu) - 3 (tuv) + (9 (tvu))/2 + (uuv) - (uvv)/2]
4 -> CWS[0, -2 (tu) + 2 (uv), -3 (tuu) + (5 (tuv))/2 - 3 (tvu) - (uuv) + (uvv)/2]
5 -> CWS[0, 0, 3 (ttu) + (tuv)]
6 -> True

```

10.

11. The Definition of J

? J

```

Global`J

```

```

J[u_, v_] := J[u, v] =
Module[{s}, Integrate[Subscript[CC, u][s v][Subscript[div, u][Subscript[RC, u][s v][v]]] ds]

```

12. The J<sub>uv</sub> equation

```

Print /@ {
  0 -> {alpha, beta},
  1 -> (t1 = Ju[alpha] + (Jv[beta // RCu[alpha]] // CCu[-alpha])),
  2 -> (t2 = Jv[beta] + (Ju[alpha // RCv[beta]] // CCv[-beta])),
  3 -> t1 == t2
};

```

```

0 -> {LS[-2 t - 2 u, 2 tv + (uv)/2,
  7 (ttu)/6 + 5 (ttv)/6 - 1 (tuv)/6 + 2 (uuv)/6 - 5 (tuu)/6 + 2 (tvu)/3 - 2 (tvv)/3 - 11 (uvv)/6], LS[t - v,
  - 3 (tu)/2 + 3 (tv)/2 + (uv)/2, -2 (ttu)/6 - 11 (ttv)/6 - 5 (tuv)/3 - 5 (uuv)/6 - 2 (tuu) + 1 (tvu) - 2 (tvv) - (uvv)]}
1 -> CWS[-2 u - v, 2 (tu) + (tv), - (ttu)/6 - 5 (ttv)/4 + (13 (tuu))/6 + (2 (tuv))/3 - (23 (tvu))/12 + (31 (tvv))/12 - (10 (uuv))/3 - (7 (uvv))/12]
2 -> CWS[-2 u - v, 2 (tu) + (tv), - (ttu)/6 - 5 (ttv)/4 + (13 (tuu))/6 + (2 (tuv))/3 - (23 (tvu))/12 + (31 (tvv))/12 - (10 (uuv))/3 - (7 (uvv))/12]
3 -> True

```

### 13. The t equation

```

Print /@ {
  0 ->  $\gamma$ ,
  1 -> (t1 = Jw[ $\gamma$  // tm[u, v, w]]),
  2 -> (t2 = Ju[ $\gamma$ ] // tm[u, v, w]),
  3 -> (t3 = Jv[ $\gamma$  // RCu[ $\gamma$ ]] // CCu[- $\gamma$ ] // tm[u, v, w]),
  4 -> t1 == t2 + t3
};

0 -> LS[-2  $\bar{u}$  -  $\bar{v}$ , 2  $\bar{tu}$  + 2  $\bar{tv}$  +  $\bar{uv}$ ,
   $\frac{11}{6} \overline{ttu} + \frac{1}{2} \overline{ttv} - \frac{1}{2} \overline{tuv} - \frac{5}{6} \overline{uuv} - \frac{1}{2} \overline{tuu} + \frac{1}{6} \overline{tvu} + \overline{tvv} - \frac{3}{2} \overline{uvv}$ ]

1 -> CWS[-3  $\widehat{w}$ , 4  $\widehat{tw}$ ,  $\frac{7}{3} \widehat{ttw} + \frac{16}{3} \widehat{tww}$ ]

2 -> CWS[-2  $\widehat{w}$ , 2  $\widehat{tw}$ ,  $\frac{11}{6} \widehat{ttw} + 2 \widehat{tww} - \frac{5}{6} \widehat{www}$ ]

3 -> CWS[- $\widehat{w}$ , 2  $\widehat{tw}$ ,  $\frac{\widehat{ttw}}{2} + \frac{10}{3} \widehat{tww} + \frac{5}{6} \widehat{www}$ ]

4 -> True

Print /@ {
  0 -> { $\gamma$ ,  $\gamma w = \gamma$  // tm[u, v, w]},
  1 -> (t1 = J[w,  $\gamma w$ ] // RCw[ $\gamma w$ ]),
  2 -> (t2 = Ju[ $\gamma$ ] // tm[u, v, w] // RCw[ $\gamma w$ ]),
  3 -> (t3 = Jv[ $\gamma$  // RCu[ $\gamma$ ]] // RCv[ $\gamma$  // RCu[ $\gamma$ ]] // tm[u, v, w]),
  4 -> t1 == t2 + t3
};

0 -> {LS[-2  $\bar{u}$  -  $\bar{v}$ , 2  $\bar{tu}$  + 2  $\bar{tv}$  +  $\bar{uv}$ ,  $\frac{11}{6} \overline{ttu} + \frac{1}{2} \overline{ttv} - \frac{1}{2} \overline{tuv} -$ 
   $\frac{5}{6} \overline{uuv} - \frac{1}{2} \overline{tuu} + \frac{1}{6} \overline{tvu} + \overline{tvv} - \frac{3}{2} \overline{uvv}$ ], LS[-3  $\bar{w}$ , 4  $\bar{tw}$ ,  $\frac{7}{3} \overline{ttw} + \frac{2}{3} \overline{tww}$ ]}

1 -> CWS[-3  $\widehat{w}$ , 4  $\widehat{tw}$ ,  $\frac{7}{3} \widehat{ttw} + \frac{16}{3} \widehat{tww}$ ]

2 -> CWS[-2  $\widehat{w}$ , 2  $\widehat{tw}$ ,  $\frac{11}{6} \widehat{ttw} + 2 \widehat{tww} - \frac{5}{6} \widehat{www}$ ]

3 -> CWS[- $\widehat{w}$ , 2  $\widehat{tw}$ ,  $\frac{\widehat{ttw}}{2} + \frac{10}{3} \widehat{tww} + \frac{5}{6} \widehat{www}$ ]

4 -> True

```

### 14. The h equation

```
Print /@ {
  1 → (t1 = Ju[BCH[α, β]]),
  2 → (t2 = Ju[α]),
  3 → (t3 = Ju[β // RCu[α]] // CCu[-α]),
  4 → t1 ≡ t2 + t3
};
```

$$1 \rightarrow \text{CWS} \left[ -2 \widehat{u}, \frac{\widehat{tu}}{2} - \widehat{uv}, \frac{\widehat{ttu}}{12} + \frac{25 \widehat{tuu}}{6} + \frac{19 \widehat{tuv}}{12} + \frac{9 \widehat{tvu}}{4} - \frac{5 \widehat{uuv}}{3} - \frac{31 \widehat{uvv}}{12} \right]$$

$$2 \rightarrow \text{CWS} \left[ -2 \widehat{u}, 2 \widehat{tu} - \frac{\widehat{uv}}{2}, -\frac{\widehat{ttu}}{6} + \frac{13 \widehat{tuu}}{6} + \frac{4 \widehat{tuv}}{3} - \frac{2 \widehat{tvu}}{3} - \frac{5 \widehat{uuv}}{2} - \frac{11 \widehat{uvv}}{6} \right]$$

$$3 \rightarrow \text{CWS} \left[ 0, -\frac{3 \widehat{tu}}{2} - \frac{\widehat{uv}}{2}, \frac{\widehat{ttu}}{4} + 2 \widehat{tuu} + \frac{\widehat{tuv}}{4} + \frac{35 \widehat{tvu}}{12} + \frac{5 \widehat{uuv}}{6} - \frac{3 \widehat{uvv}}{4} \right]$$

4 → True

■ h and S

```
(Plus[
  Ju[γ] // RCu[γ],
  Ju[-γ] // RCu[γ]]
 // RCu[-γ // RCu[γ]]) @ {6}
CWS[0, 0, 0, 0, 0, 0]
```

- 15.
- 16.
- 17.
- 18.
- 19.

20. The differential of BCH

```
Print /@ {
  1 → (bch = BCH[u, v]),
  2 →  $\frac{\text{BCH}[u + \epsilon t, v + \epsilon w] - \text{bch}}{\epsilon}$ ,
  3 →  $\left( t1 = \frac{\text{BCH}[u + \epsilon t, v + \epsilon w] - \text{bch}}{\epsilon} // \text{adSeries} \left[ \frac{1 - e^{-\text{ad}}}{\text{ad}}, \text{bch} \right] \right)$ ,
  4 →  $\left( t2 = t // \text{adSeries} \left[ \frac{1 - e^{-\text{ad}}}{\text{ad}}, u \right] // \text{Ad}[-v] \right)$ ,
  5 →  $\left( t3 = w // \text{adSeries} \left[ \frac{1 - e^{-\text{ad}}}{\text{ad}}, v \right] \right)$ 
};
t1 ≡ t2 + t3
```

$$\begin{aligned}
1 &\rightarrow \text{LS} \left[ \overline{u} + \overline{v}, \frac{\overline{uv}}{2}, \frac{1}{12} \overline{u\overline{uv}} + \frac{1}{12} \overline{u\overline{v}} \right] \\
2 &\rightarrow \text{LS} \left[ \overline{t} + \overline{w}, \frac{\overline{tv}}{2} + \frac{\overline{uw}}{2}, \frac{1}{12} \overline{t\overline{uv}} + \frac{1}{12} \overline{u\overline{uw}} + \frac{1}{12} \overline{u\overline{vw}} - \frac{1}{12} \overline{t\overline{vu}} + \frac{1}{12} \overline{t\overline{vv}} + \frac{1}{6} \overline{u\overline{wv}} \right] \\
3 &\rightarrow \text{LS} \left[ \overline{t} + \overline{w}, \frac{\overline{tu}}{2} + \overline{tv} - \frac{\overline{vw}}{2}, \frac{1}{2} \overline{t\overline{uv}} + \frac{1}{6} \overline{v\overline{vw}} + \frac{1}{6} \overline{t\overline{uu}} + \frac{1}{2} \overline{t\overline{vu}} + \frac{1}{2} \overline{t\overline{vv}} \right] \\
4 &\rightarrow \text{LS} \left[ \overline{t}, \frac{\overline{tu}}{2} + \overline{tv}, \frac{1}{2} \overline{t\overline{uv}} + \frac{1}{6} \overline{t\overline{uu}} + \frac{1}{2} \overline{t\overline{vu}} + \frac{1}{2} \overline{t\overline{vv}} \right] \\
5 &\rightarrow \text{LS} \left[ \overline{w}, -\frac{\overline{vw}}{2}, \frac{1}{6} \overline{v\overline{vw}} \right]
\end{aligned}$$

True

**21.** The differential of C

```

Print /@ {
  0 → {α, δα, γ},
  1 → (t1 = (γ // CC[u, α + ε δα] - (γ // CC[u, α])) / ε),
  2 → (t2 = γ // ad[u, δα // adSeries[e^ad - 1, α] // RC[u, -α]] // CC[u, α]),
  t1 ≡ t2
};

```

$$\begin{aligned}
0 &\rightarrow \left\{ \text{LS} \left[ -2\overline{t} - 2\overline{u}, 2\overline{tv} + \frac{\overline{uv}}{2}, \right. \right. \\
&\quad \left. \frac{7}{6} \overline{t\overline{tu}} + \frac{5}{6} \overline{t\overline{tv}} - \frac{1}{6} \overline{t\overline{uv}} + 2\overline{u\overline{uv}} - \frac{5}{6} \overline{t\overline{uu}} + \frac{2}{3} \overline{t\overline{vu}} - \frac{2}{3} \overline{t\overline{vv}} - \frac{11}{6} \overline{u\overline{v}} \right], \\
&\quad \text{LS} \left[ \overline{t} + \overline{u} - \overline{v}, -\overline{tu} + \frac{\overline{tv}}{2} + 2\overline{uv}, -2\overline{t\overline{tu}} + \frac{4}{3} \overline{t\overline{tv}} + \frac{7}{6} \overline{t\overline{uv}} + \frac{2}{3} \overline{u\overline{uv}} + \right. \\
&\quad \left. \frac{7}{6} \overline{t\overline{uu}} + \frac{4}{3} \overline{t\overline{vu}} - \frac{1}{6} \overline{t\overline{vv}} - \frac{11}{6} \overline{u\overline{v}} \right], \text{LS} \left[ -2\overline{u} - \overline{v}, 2\overline{tu} + 2\overline{tv} + \overline{uv}, \right. \\
&\quad \left. \frac{11}{6} \overline{t\overline{tu}} + \frac{1}{2} \overline{t\overline{tv}} - \frac{1}{2} \overline{t\overline{uv}} - \frac{5}{6} \overline{u\overline{uv}} - \frac{1}{2} \overline{t\overline{uu}} + \frac{1}{6} \overline{t\overline{vu}} + \overline{t\overline{vv}} - \frac{3}{2} \overline{u\overline{v}} \right] \left. \right\} \\
1 &\rightarrow \text{LS} \left[ 0, -2\overline{tu} - 2\overline{uv}, 6\overline{t\overline{tu}} + 7\overline{t\overline{uv}} + 6\overline{u\overline{uv}} - 2\overline{t\overline{uu}} + 2\overline{t\overline{vu}} + \overline{u\overline{v}} \right] \\
2 &\rightarrow \text{LS} \left[ 0, -2\overline{tu} - 2\overline{uv}, 6\overline{t\overline{tu}} + 7\overline{t\overline{uv}} + 6\overline{u\overline{uv}} - 2\overline{t\overline{uu}} + 2\overline{t\overline{vu}} + \overline{u\overline{v}} \right]
\end{aligned}$$

True

**22.** The differential of C[...]

$$\{\lambda = \langle \mathbf{u} \rightarrow \alpha, \mathbf{v} \rightarrow \beta \rangle, \delta\lambda = \langle \mathbf{u} \rightarrow \delta\alpha, \mathbf{v} \rightarrow \delta\beta \rangle, \gamma\}$$

$$\left\{ \begin{aligned} & \text{LW}[\mathbf{u}] \rightarrow \text{LS} \left[ -2 \overline{t} - 2 \overline{u} + 2 \overline{v}, -\frac{\overline{tu}}{2} + \frac{3 \overline{tv}}{2} - \overline{uv}, \right. \\ & \quad \left. -\frac{7}{6} \overline{ttu} + \frac{7}{6} \overline{ttv} + \frac{5}{3} \overline{tuv} - \frac{7}{6} \overline{uu\overline{v}} - \frac{11}{6} \overline{tuu} + \frac{1}{6} \overline{tvu} - \frac{2}{3} \overline{tvv} - \frac{2}{3} \overline{uv\overline{v}} \right], \\ & \text{LW}[\mathbf{v}] \rightarrow \text{LS} \left[ \overline{v}, -\frac{3 \overline{tu}}{2} - 2 \overline{tv} + \overline{uv}, \frac{4}{3} \overline{ttu} - \frac{2}{3} \overline{ttv} + \overline{tuv} - \right. \\ & \quad \left. \frac{5}{6} \overline{uu\overline{v}} - 2 \overline{tuu} + \overline{tvu} - \frac{4}{3} \overline{tvv} + \frac{3}{2} \overline{uv\overline{v}} \right], \\ & \left( \text{LW}[\mathbf{u}] \rightarrow \text{LS} \left[ \overline{t} - \overline{u} + 2 \overline{v}, -2 \overline{tv} + \frac{\overline{uv}}{2}, 2 \overline{ttu} - \frac{7}{6} \overline{ttv} - \frac{3}{2} \overline{tuv} - \overline{uu\overline{v}} - \frac{4}{3} \overline{tuu} - \right. \right. \\ & \quad \left. \frac{7}{6} \overline{tvu} + \frac{11}{6} \overline{tvv} + \overline{uv\overline{v}} \right], \text{LW}[\mathbf{v}] \rightarrow \text{LS} \left[ -\overline{t} + 2 \overline{u} - 2 \overline{v}, -\overline{tu} + 2 \overline{tv} + \overline{uv}, \right. \\ & \quad \left. -\frac{1}{6} \overline{ttv} - \frac{1}{6} \overline{tuv} + \frac{4}{3} \overline{uu\overline{v}} - \frac{11}{6} \overline{tuu} + \frac{2}{3} \overline{tvu} + \frac{5}{6} \overline{tvv} + \frac{2}{3} \overline{uv\overline{v}} \right], \\ & \text{LS} \left[ \overline{t} - 2 \overline{u} + \overline{v}, 2 \overline{tu} - \frac{\overline{tv}}{2} - \frac{\overline{uv}}{2}, \frac{11}{6} \overline{ttu} - \frac{11}{6} \overline{ttv} + \overline{tuv} + \right. \\ & \quad \left. \frac{3}{2} \overline{uu\overline{v}} - \frac{1}{3} \overline{tuu} - \frac{1}{2} \overline{tvu} - \frac{11}{6} \overline{tvv} - \frac{7}{6} \overline{uv\overline{v}} \right] \end{aligned} \right\}$$

```
Print /@ {
  0 → {λ = ⟨u → α, v → β⟩, δλ = ⟨u → δα, v → δβ⟩, γ},
  1 → (t1 = (γ // CC[λ + ε δλ]) - (γ // CC[λ]) / ε),
  2 → (t2 = γ // -D[δλ // adSeries[e^ad - 1 / ad, λ] // RC[-λ]] // CC[λ]),
  t1 ≡ t2
};
```

0 →

$$\left\{ \left( \text{LW}[u] \rightarrow \text{LS} \left[ -2\overline{t} - 2\overline{u} + 2\overline{v}, -\frac{\overline{tu}}{2} + \frac{3\overline{tv}}{2} - \overline{uv}, -\frac{7}{6}\overline{ttu} + \frac{7}{6}\overline{ttv} + \frac{5}{3}\overline{tuv} - \frac{7}{6}\overline{uuv} - \frac{11}{6}\overline{tuu} + \frac{1}{6}\overline{tvu} - \frac{2}{3}\overline{tvv} - \frac{2}{3}\overline{uvv} \right], \text{LW}[v] \rightarrow \text{LS} \left[ \overline{v}, -\frac{3\overline{tu}}{2} - 2\overline{tv} + \overline{uv}, \frac{4}{3}\overline{ttu} - \frac{2}{3}\overline{ttv} + \overline{tuv} - \frac{5}{6}\overline{uuv} - 2\overline{tuu} + \overline{tvu} - \frac{4}{3}\overline{tvv} + \frac{3}{2}\overline{uvv} \right] \right), \right. \\ \left. \left( \text{LW}[u] \rightarrow \text{LS} \left[ \overline{t} - \overline{u} + 2\overline{v}, -2\overline{tv} + \frac{\overline{uv}}{2}, 2\overline{ttu} - \frac{7}{6}\overline{ttv} - \frac{3}{2}\overline{tuv} - \overline{uuv} - \frac{4}{3}\overline{tuu} - \frac{7}{6}\overline{tvu} + \frac{11}{6}\overline{tvv} + \overline{uvv} \right], \text{LW}[v] \rightarrow \text{LS} \left[ -\overline{t} + 2\overline{u} - 2\overline{v}, -\overline{tu} + 2\overline{tv} + \overline{uv}, -\frac{1}{6}\overline{ttv} - \frac{1}{6}\overline{tuv} + \frac{4}{3}\overline{uuv} - \frac{11}{6}\overline{tuu} + \frac{2}{3}\overline{tvu} + \frac{5}{6}\overline{tvv} + \frac{2}{3}\overline{uvv} \right] \right), \text{LS} \left[ \overline{t} - 2\overline{u} + \overline{v}, 2\overline{tu} - \frac{\overline{tv}}{2} - \frac{\overline{uv}}{2}, \frac{11}{6}\overline{ttu} - \frac{11}{6}\overline{ttv} + \overline{tuv} + \frac{3}{2}\overline{uuv} - \frac{1}{3}\overline{tuu} - \frac{1}{2}\overline{tvu} - \frac{11}{6}\overline{tvv} - \frac{7}{6}\overline{uvv} \right] \right\}$$

$$1 \rightarrow \text{LS} \left[ 0, -2\overline{tu} - \overline{tv} + 6\overline{uv}, 6\overline{ttu} + \frac{1}{2}\overline{ttv} - \frac{21}{2}\overline{tuv} - 6\overline{uuv} + \frac{5}{2}\overline{tvv} - 7\overline{uvv} \right]$$

$$2 \rightarrow \text{LS} \left[ 0, -2\overline{tu} - \overline{tv} + 6\overline{uv}, 6\overline{ttu} + \frac{1}{2}\overline{ttv} - \frac{21}{2}\overline{tuv} - 6\overline{uuv} + \frac{5}{2}\overline{tvv} - 7\overline{uvv} \right]$$

True

### 23. The differential of RC

```
Print /@ {
  0 → {α, β, γ},
  1 → (t1 = (γ // RC_u[α + ε β]) - (γ // RC_u[α])) / ε,
  2 → (t2 = γ // RC_u[α] // ad[u, adSeries[1 - e^-ad / ad, α][β] // RC_u[α]]),
  t1 ≡ t2
};
```

$$\begin{aligned}
 0 \rightarrow & \left\{ \text{LS} \left[ -2 \overline{t} - 2 \overline{u}, 2 \overline{tv} + \frac{\overline{uv}}{2}, \right. \right. \\
 & \left. \frac{7}{6} \overline{ttu} + \frac{5}{6} \overline{ttv} - \frac{1}{6} \overline{tuv} + 2 \overline{uuv} - \frac{5}{6} \overline{tuu} + \frac{2}{3} \overline{tvu} - \frac{2}{3} \overline{tvv} - \frac{11}{6} \overline{uvv} \right], \\
 \text{LS} & \left[ \overline{t} - \overline{v}, -\frac{3 \overline{tu}}{2} + \frac{3 \overline{tv}}{2} + \frac{\overline{uv}}{2}, -2 \overline{ttu} - \frac{11}{6} \overline{ttv} - \frac{5}{3} \overline{tuv} - \frac{5}{6} \overline{uuv} - \right. \\
 & \left. 2 \overline{tuu} + \frac{1}{2} \overline{tvu} - 2 \overline{tvv} - \overline{uvv} \right], \text{LS} \left[ -2 \overline{u} - \overline{v}, 2 \overline{tu} + 2 \overline{tv} + \overline{uv}, \right. \\
 & \left. \frac{11}{6} \overline{ttu} + \frac{1}{2} \overline{ttv} - \frac{1}{2} \overline{tuv} - \frac{5}{6} \overline{uuv} - \frac{1}{2} \overline{tuu} + \frac{1}{6} \overline{tvu} + \overline{tvv} - \frac{3}{2} \overline{uvv} \right] \left. \right\} \\
 1 \rightarrow & \text{LS} \left[ 0, -2 \overline{tu} - 2 \overline{uv}, 6 \overline{ttu} + 7 \overline{tuv} - \overline{uuv} + 5 \overline{tuu} + \overline{uvv} \right] \\
 2 \rightarrow & \text{LS} \left[ 0, -2 \overline{tu} - 2 \overline{uv}, 6 \overline{ttu} + 7 \overline{tuv} - \overline{uuv} + 5 \overline{tuu} + \overline{uvv} \right]
 \end{aligned}$$

True

### 24. The differential of J

```

Print /@ {
  0 -> {α, β};
  1 -> (t0 = (Ju[α + ε β] - Ju[α]) / ε),
  2 -> (t1 = Divu[β // adSeries[1 - e^-ad / ad, α] // RCu[α] // CCu[-α]]),
  t0 ≡ t1
};

```

$$\begin{aligned}
 1 \rightarrow & \text{CWS} \left[ 0, -\frac{5 \overline{tu}}{2} + \frac{\overline{uv}}{2}, \frac{5 \overline{ttu}}{6} - \frac{\overline{tuu}}{6} + \frac{\overline{tuv}}{3} + \frac{\overline{tvu}}{4} + \overline{uuv} - \frac{3 \overline{uvv}}{4} \right] \\
 2 \rightarrow & \text{CWS} \left[ 0, -\frac{5 \overline{tu}}{2} + \frac{\overline{uv}}{2}, \frac{5 \overline{ttu}}{6} - \frac{\overline{tuu}}{6} + \frac{\overline{tuv}}{3} + \frac{\overline{tvu}}{4} + \overline{uuv} - \frac{3 \overline{uvv}}{4} \right]
 \end{aligned}$$

True