Cheat Sheet Free Lie

With alphabet T and with $u,v,w \in T$, $\alpha,\beta,\gamma \in FL(T)$, $\lambda \in FL(T)^T$, $D \in \mathfrak{tder}(T)$, $g,h \in \exp(\mathfrak{tder}(T)) = \mathrm{TAut}(T)$.

Checkmarks (✓) as in `CheatSheetFreeLie-Verification.nb`.

Definition. $\mathrm{ad}_u^\gamma = \mathrm{ad}_u\{\gamma\} := \mathrm{der}(u \mapsto [\gamma,u])$ and $\partial_\lambda = \partial\{\lambda\} := -\sum_{s\in S}\mathrm{ad}_s^{\lambda_s}$.

Definition. $C^\lambda := \mathrm{mor}\left(u \mapsto (\mathrm{Ad}\,\lambda_u)(u) = e^{\lambda_u}ue^{-\lambda_u} = e^{\mathrm{ad}\,\lambda_u}u\right)$ and $C_u^\gamma := C^{(u\to\gamma)}$.

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1. ✓ The meaning(s) of RC: $C_u^\gamma /\!/ RC_u^{-\gamma} = 1$, $\quad C_u^{\gamma/\!/RC_u^\gamma} = RC_u^\gamma$

2. ✓ C_uC_v and RC_uRC_v: $C_u^{\alpha/\!/RC_v^{-\beta}} /\!/ C_v^\beta = C_v^{\beta/\!/RC_u^{-\alpha}} /\!/ C_u^\alpha$, $RC_u^\alpha /\!/ RC_v^{\beta/\!/RC_u^\alpha} = RC_v^\beta /\!/ RC_u^{\alpha/\!/RC_v^\beta}$

3. RC equation t: $tm_w^{uv} /\!/ RC_w^{\gamma/\!/tm_w^{uv}} = RC_u^\gamma /\!/ RC_v^{\gamma/\!/RC_u^\gamma} /\!/ tm_w^{uv}$

4. RC equation h: $RC_u^{\mathrm{bch}(\alpha,\beta)} = RC_u^\alpha /\!/ RC_u^{\beta/\!/RC_u^\alpha}$

5. ✓ Γ: With $\Gamma(t) \in FL(T)^T$ solving $\Gamma(0) = 0$, $\Gamma'(s) = \lambda /\!/ e^{-\partial_{s\lambda}} /\!/ \frac{\mathrm{ad}\,\Gamma(s)}{e^{\mathrm{ad}\,\Gamma(s)}-1}$, $\quad e^{-\partial_\lambda} = C^{\Gamma(1)}$

6. ✓ Λ: With $\Lambda(t) \in FL(T)^T$ solving $\Lambda(0) = 0$, $\Lambda'(s) = \lambda /\!/ e^{\partial_{\Lambda(s)}} /\!/ \frac{\mathrm{ad}_{tb}\,\Lambda(s)}{e^{\mathrm{ad}_{tb}\,\Lambda(s)}-1}$, $\quad e^{-\partial_{\Lambda(1)}} = C^\lambda$

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7. div property t: $\mathrm{div}_w(\gamma /\!/ tm_w^{uv}) = (\mathrm{div}_u(\gamma) + \mathrm{div}_v(\gamma)) /\!/ tm_w^{uv}$

8. ✓ div property uv:
$$(\mathrm{div}_u\,\alpha) /\!/ \mathrm{ad}_v^\beta - (\mathrm{div}_v\,\beta) /\!/ \mathrm{ad}_u^\alpha = \mathrm{div}_u(\alpha /\!/ \mathrm{ad}_v^\beta) - \mathrm{div}_v(\beta /\!/ \mathrm{ad}_u^\alpha)$$

9. ✓ div property uu:
$$(\mathrm{div}_u\,\alpha) /\!/ \mathrm{ad}_u\{\beta\} - (\mathrm{div}_u\,\beta) /\!/ \mathrm{ad}_u\{\alpha\} = \mathrm{div}_u\left([\alpha,\beta] + \alpha /\!/ \mathrm{ad}_u\{\beta\} - \beta /\!/ \mathrm{ad}_u\{\alpha\}\right)$$

10. C–div–RC eqns: $\mathrm{div}_u(\alpha /\!/ RC_u^\gamma) /\!/ C_u^\gamma = ?$; $\mathrm{div}_u(\alpha /\!/ C_u^\gamma) /\!/ RC_u^\gamma = ?$

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11. The definition of J: $\quad J_u(\gamma) := \int_0^1 ds\ \mathrm{div}_u\left(\gamma /\!/ RC_u^{s\gamma}\right) /\!/ C_u^{-s\gamma}$

12. ✓ J_{uv} eqn: $J_u(\alpha) + J_v(\beta /\!/ RC_u^\alpha) /\!/ C_u^{-\alpha} = J_v(\beta) + J_u(\alpha /\!/ RC_v^\beta) /\!/ C_v^{-\beta}$

13. ✓ t eqn: $\quad J_w(\gamma /\!/ tm_w^{uv}) = \left(J_u(\gamma) + J_v(\gamma /\!/ RC_u^\gamma) /\!/ C_u^{-\gamma}\right) /\!/ tm_w^{uv}$

14. ✓ The h equation: $\quad J_u(\mathrm{bch}(\alpha,\beta)) = J_u(\alpha) + J_u(\beta /\!/ RC_u^\alpha) /\!/ C_u^{-\alpha}$

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15. The definition of JA: $\quad JA_u(\gamma) := J_u(\gamma) /\!/ RC_u^\gamma$

16. ODE for JA: with $\gamma_s = \gamma /\!/ RC_u^{s\gamma}$,
$$JA(0) = 0, \qquad \frac{dJA(s)}{ds} = JA(s) /\!/ \mathrm{ad}_u\{\gamma_s\} + \mathrm{div}_u\,\gamma_s, \qquad JA(1) = JA_u(\gamma)$$

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17. j following AT: $\quad j(e^D) = \int_0^1 ds\, e^{sD}(\mathrm{div}\,D) = \frac{e^D-1}{D}(\mathrm{div}\,D)$

18. j's cocycle property: $\quad j(gh) = j(g) + g\cdot j(h)$

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19. $d\exp$: $\quad \delta e^\gamma = e^\gamma \cdot \left(\delta\gamma /\!/ \frac{1-e^{-\mathrm{ad}\,\gamma}}{\mathrm{ad}\,\gamma}\right) = \left(\delta\gamma /\!/ \frac{e^{\mathrm{ad}\,\gamma}-1}{\mathrm{ad}\,\gamma}\right)\cdot e^\gamma$

20. ✓ The differential of $\gamma = \mathrm{bch}(\alpha,\beta)$:
$$\delta\gamma /\!/ \frac{1-e^{-\mathrm{ad}\,\gamma}}{\mathrm{ad}\,\gamma} = \left(\delta\alpha /\!/ \frac{1-e^{-\mathrm{ad}\,\alpha}}{\mathrm{ad}\,\alpha} /\!/ e^{-\mathrm{ad}\,\beta}\right) + \left(\delta\beta /\!/ \frac{1-e^{-\mathrm{ad}\,\beta}}{\mathrm{ad}\,\beta}\right)$$

21. ✓ dC: $\quad \delta C_u^\gamma = \mathrm{ad}_u\left\{\delta\gamma /\!/ \frac{e^{\mathrm{ad}\,\gamma}-1}{\mathrm{ad}\,\gamma} /\!/ RC_u^{-\gamma}\right\} /\!/ C_u^\gamma$

22. ✓ dC^λ: $\quad \delta C^\lambda = -\partial\left\{\delta\lambda /\!/ \frac{e^{\mathrm{ad}\,\lambda}-1}{\mathrm{ad}\,\lambda} /\!/ RC^{-\lambda}\right\} /\!/ C^\lambda$

23. ✓ dRC: $\quad \delta RC_u^\gamma = RC_u^\gamma /\!/ \mathrm{ad}_u\left\{\delta\gamma /\!/ \frac{1-e^{-\mathrm{ad}\,\gamma}}{\mathrm{ad}\,\gamma} /\!/ RC_u^\gamma\right\}$

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24. ✓ dJ: $\quad \delta J_u(\gamma) = \delta\gamma /\!/ \frac{1-e^{-\mathrm{ad}\,\gamma}}{\mathrm{ad}\,\gamma} /\!/ RC_u^\gamma /\!/ \mathrm{div}_u /\!/ C_u^{-\gamma}$

The β quotient $[u,v] = c_uv - c_vu$**.** Let $R = R(T) := \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$, $L = L(T) := R \otimes \mathbb{Q}T$. For $\gamma = \sum_u \gamma_u u \in L$ set $c_\gamma := \sum_u \gamma_u c_u \in R$. With this,
$$v /\!/ C_u^{-\gamma} = v /\!/ RC_u^\gamma = v \qquad \text{for } u \neq v \in T,$$
$$\rho /\!/ C_u^{-\gamma} = \rho /\!/ RC_u^\gamma = \rho \qquad \text{for } \rho \in R,$$
$$u /\!/ C_u^{-\gamma} = e^{-c_\gamma}\left(u + c_u\frac{e^{c_\gamma}-1}{c_\gamma}\gamma\right) = e^{-c_\gamma}\left(\left(1 + c_u\gamma_u\frac{e^{c_\gamma}-1}{c_\gamma}\right)u + c_u\frac{e^{c_\gamma}-1}{c_\gamma}\sum_{v\neq u}\gamma_v v\right),$$
$$u /\!/ RC_u^\gamma = \left(1 + c_u\gamma_u\frac{e^{c_\gamma}-1}{c_\gamma}\right)^{-1}\left(e^{c_\gamma}u - c_u\frac{e^{c_\gamma}-1}{c_\gamma}\sum_{v\neq u}\gamma_v v\right),$$
$$\mathrm{bch}(\alpha,\beta) = \frac{c_\alpha + c_\beta}{e^{c_\alpha+c_\beta}-1}\left(\frac{e^{c_\alpha}-1}{c_\alpha}\alpha + e^{c_\alpha}\frac{e^{c_\beta}-1}{c_\beta}\beta\right)$$
$$\mathrm{div}_u\,\gamma = c_u\gamma_u$$
$$J_u(\gamma) = \log\left(1 + \frac{e^{c_\gamma}-1}{c_\gamma}c_u\gamma_u\right).$$

Further include: j, C^λ, RC^λ, Γ, Λ, ∂, $[\,]_{tb}$.

Implement $\pi\colon FL/CW \to L/R$ (under `LSeries`/`RSeries`?), commutativity verifications.

Verifications of 1–24 in β.