Pensieve header: Calculations appearing in the WKO4 paper.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section 1 - Introduction

Initialization

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Initialization

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS,
  CC, Crop, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, EulerE,
  Exp, InvertLieMorphism, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW,
  LyndonFactorization, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve,
  Support, tb, TopBracketForm, tr, UndeterminedCoefficients, Γ, ι, Λ, ℏ, ⌐, ⌢}.
```

Initialization

```
AwCalculus` implements / extends {*, **, E, ≡, dA, deg,
  dm, dS, dΔ, dη, dσ, E1, Es, hA, hm, hS, hη, hσ, tA, tha, tm, tS, tσ, Γ, Λ}.
```

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

```
x1 = LW[1]; x2 = LW[2];
{α, β, γ} = LS /@ {x1 + b[x1, x2], x2 - b[x1, b[x1, x2]], x1 + x2 - 2 b[x1, x2]}
```

alphabetagamma

{LS[$\overline{1}$, $\overline{1\,2}$, 0, 0, ...], LS[$\overline{2}$, 0, $-\overline{1\,\overline{1\,2}}$, 0, ...], LS[$\overline{1}+\overline{2}$, $-2\,\overline{1\,2}$, 0, 0, ...]}

BracketExample

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

BracketExample

{LS[0, $\overline{1\,2}$, $\overline{\overline{1\,2}\,2}$, $-\overline{1\,\overline{1\,\overline{1\,2}}}$, ...], LS[0, 0, 0, 0, ...]}

bch

```
bch = BCH[LW@x, LW@y]
```

bch

LS[$\overline{x}+\overline{y}$, $\frac{\overline{x\,y}}{2}$, $\frac{1}{12}\overline{x\,\overline{x\,y}}+\frac{1}{12}\overline{\overline{x\,y}\,y}$, $\frac{1}{24}\overline{x\,\overline{\overline{x\,y}\,y}}$, ...]

bch16

```
Timing@{Length@(bch@16), (bch@16)[[1090 ;; 1092]] // TopBracketForm}
```

bch16

$$\left\{45.474291, \left\{2181, \frac{53\,x\,x\,x\,x\,y\,x\,y\,x\,x\,y\,x\,x\,y\,x\,y\,y}{1\,089\,728\,640} - \frac{17\,x\,x\,x\,x\,y\,x\,y\,x\,x\,y\,x\,x\,x\,y\,y\,y}{179\,625\,600} + \frac{389\,x\,x\,x\,x\,y\,x\,y\,x\,x\,y\,x\,x\,y\,x\,x\,y}{1\,320\,883\,200}\right\}\right\}$$

omegas

```
{ω1, ω2} = CWS /@ {CW["1"] - 3 CW["211"], CW["2"] + CW["22"]}
```

omegas

$\{\text{CWS}[\widehat{1}, 0, -3\,\widehat{112}, 0, \ldots], \text{CWS}[\widehat{2}, \widehat{22}, 0, 0, \ldots]\}$

DegreeScale

```
DegreeScale[h] /@ {ω1, ω2}
```

DegreeScale

$\{\text{CWS}[h\,\widehat{1}, 0, -3\,h^3\,\widehat{112}, 0, \ldots], \text{CWS}[h\,\widehat{2}, h^2\,\widehat{22}, 0, 0, \ldots]\}$

TangentialDerivative

```
{λ = ⟨1 → α, 2 → β⟩, γ // Dλ}
```

TangentialDerivative

$\{\langle 1 \to \text{LS}[\overline{1}, \overline{1\,2}, 0, 0, \ldots], 2 \to \text{LS}[\overline{2}, 0, -\overline{1\,\overline{1\,2}}, 0, \ldots]\rangle, \text{LS}[0, 0, \overline{1\,\overline{1\,2}}, -\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots]\}$

tb

```
λ1 = λ; λ2 = ⟨1 → β, 2 → γ⟩; tb[λ1, λ2]
```

tb

$\langle 1 \to \text{LS}[0, 0, \overline{1\,\overline{1\,2}}, -\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots], 2 \to \text{LS}[0, 0, \overline{1\,\overline{1\,2}}, -\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots]\rangle$

tb2

```
lhs = D_tb[λ1,λ2][ω1]; rhs = b[Dλ1, Dλ2][ω1];
{lhs@{8}, (lhs ≡ rhs)@{8}}
```

tb2

$\{\text{CWS}[0, 0, 0, 0, 0, 0, 0, 18\,\widehat{11112122} - 18\,\widehat{11112212} - 36\,\widehat{11121122} + 36\,\widehat{11122112}, \ldots],$
$\text{BS}[9\ \text{True}, \ldots]\}$

TestingGammaODE

```
lhs = ∂t Γt[λ]; rhs = λ // e^(-t Dλ) // adSeries[ad/(e^ad - 1), Γt[λ]];
{Γ0[λ], lhs, (lhs ≡ rhs)@{6}}
```

TestingGammaODE

$\{\langle 1 \to \text{LS}[0, 0, 0, 0, \ldots], 2 \to \text{LS}[0, 0, 0, 0, \ldots]\rangle,$
$\langle 1 \to \text{LS}[\overline{1}, \overline{1\,2}, -t\,\overline{1\,\overline{1\,2}}, \frac{1}{4}\,t^2\,\overline{1\,\overline{1\,\overline{1\,2}}} - t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots],$
$2 \to \text{LS}[\overline{2}, 0, -\overline{1\,\overline{1\,2}}, -t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots]\rangle, \text{BS}[7\ \text{True}, \ldots]\}$

TestingGamma

$\{\gamma$ // $e^{-t\,D_\lambda}$, γ // CC[$\Gamma_t[\lambda]$]$\}$

TestingGamma

$\{\mathrm{LS}[\bar 1+\bar 2,\ -2\,\overline{\bar 1 2},\ -t\,\overline{1\overline{\bar 1 2}},\ t\,\overline{1\overline{\overline{\bar 1 2}2}},\ \ldots],\ \mathrm{LS}[\bar 1+\bar 2,\ -2\,\overline{\bar 1 2},\ -t\,\overline{1\overline{\bar 1 2}},\ t\,\overline{1\overline{\overline{\bar 1 2}2}},\ \ldots]\}$

TestingLambdaODE

lhs = $\partial_t \Lambda_t[\lambda]$; rhs = λ // $e^{D_{\Lambda_t[\lambda]}}$ // adSeries$\left[\frac{ad}{e^{ad}-1}, \Lambda_t[\lambda], tb\right]$;

$\{\Lambda_0[\lambda]$, lhs, (lhs ≡ rhs)@{6}$\}$

TestingLambdaODE

$\{\langle 1\to \mathrm{LS}[0,0,0,0,\ldots],\ 2\to \mathrm{LS}[0,0,0,0,\ldots]\rangle,$

$\left\langle 1\to \mathrm{LS}\left[\bar 1,\ \overline{\bar 1 2},\ t\,\overline{1\overline{\bar 1 2}},\ \frac{1}{2}t^2\,\overline{1\overline{1\overline{\bar 1 2}}}+t\,\overline{1\overline{\overline{\bar 1 2}2}},\ \ldots\right],\ 2\to \mathrm{LS}\left[\bar 2,\ 0,\ -\overline{1\overline{\bar 1 2}},\ t\,\overline{1\overline{\overline{\bar 1 2}2}},\ \ldots\right]\right\rangle,$

BS[7 True, ...]$\}$

TestingLambda

$\{\gamma$ // CC[t λ], γ // $e^{-D_{\Lambda_t[\lambda]}}\}$

TestingLambda

$\{\mathrm{LS}\left[\bar 1+\bar 2,\ -2\,\overline{\bar 1 2},\ -t\,\overline{1\overline{\bar 1 2}},\ -\frac{1}{2}t^2\,\overline{1\overline{1\overline{\bar 1 2}}}+t\,\overline{1\overline{\overline{\bar 1 2}2}},\ \ldots\right],$

$\mathrm{LS}\left[\bar 1+\bar 2,\ -2\,\overline{\bar 1 2},\ -t\,\overline{1\overline{\bar 1 2}},\ -\frac{1}{2}t^2\,\overline{1\overline{1\overline{\bar 1 2}}}+t\,\overline{1\overline{\overline{\bar 1 2}2}},\ \ldots\right]\}$

CCAndRC

$\{\alpha$ // $CC_1[-\gamma]$, α // $CC_1[-\gamma]$ // $RC_1[\gamma]$, α // $CC_1[-\gamma]$ // $CC_1[\gamma]\}$

CCAndRC

$\{\mathrm{LS}\left[\bar 1,\ 2\,\overline{\bar 1 2},\ -\frac{5}{2}\,\overline{1\overline{\bar 1 2}}+\frac{3}{2}\,\overline{\overline{\bar 1 2}2},\ \frac{7}{6}\,\overline{1\overline{1\overline{\bar 1 2}}}-\frac{23}{6}\,\overline{1\overline{\overline{\bar 1 2}2}}+\frac{2}{3}\,\overline{\overline{\overline{\bar 1 2}2}2},\ \ldots\right],$

$\mathrm{LS}[\bar 1,\ \overline{\bar 1 2},\ 0,\ 0,\ \ldots],\ \mathrm{LS}\left[\bar 1,\ \overline{\bar 1 2},\ -\overline{1\overline{\bar 1 2}},\ 2\,\overline{1\overline{1\overline{\bar 1 2}}}+\overline{1\overline{\overline{\bar 1 2}2}},\ \ldots\right]\}$

divu

With[{γ = LW@u + b[b[LW@v, LW@u], LW@u]}, $div_u[\gamma]$] // TopBracketForm

divu

$\widehat{u}-\widehat{uuv}$

Ju

$J_1[\gamma]$

Ju

$\mathrm{CWS}\left[\widehat{1},\ \frac{5\,\widehat{12}}{2},\ -\frac{7\,\widehat{112}}{6}+\frac{7\,\widehat{122}}{6},\ \frac{3\,\widehat{1112}}{8}-\frac{11\,\widehat{1122}}{4}-\frac{3\,\widehat{1212}}{4}+\frac{3\,\widehat{1222}}{8},\ \ldots\right]$

j

{div[λ]@{5}, j[λ]@{5}}

j

$\{\mathrm{CWS}[\widehat{1}+\widehat{2},\ -\widehat{12},\ -\widehat{112},\ 0,\ 0,\ \ldots],$

$\mathrm{CWS}[\widehat{1}+\widehat{2},\ -\widehat{12},\ -\widehat{112},\ -\widehat{1122}+\widehat{1212},\ -\widehat{11122}+\widehat{11212},\ \ldots]\}$

cocycle4j

lhs = j[BCH_{tb}[λ1, λ2]]; rhs = j[λ1] + $e^{D_{\lambda 1}}$[j[λ2]];

{lhs, (lhs ≡ rhs)@{8}}

cocycle4j

$\{\mathrm{CWS}[\widehat{1}+2\,\widehat{2},\ -3\,\widehat{12},\ 0,\ -9\,\widehat{1122}+9\,\widehat{1212},\ \ldots],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\}$

```
lhs = j[BCHb[λ1, λ2]]; rhs = j[λ1] + e^(Dλ1)[j[λ2]];
{lhs, (lhs ≡ rhs)}
```

$\{CWS[\widehat{1} + 2\,\widehat{2}, -4\,\widehat{12}, -\frac{5\,\widehat{122}}{12}, \widehat{1112} - \frac{101\,\widehat{1122}}{6} + \frac{53\,\widehat{1212}}{3} - \frac{\widehat{1222}}{24}, \ldots],$

$BS[2\,True, -4\,CW[12] == -3\,CW[12], -4\,CW[12] == -3\,CW[12]\ \&\&\ -\frac{5\,CW[122]}{12} == 0,$

$-4\,CW[12] == -3\,CW[12]\ \&\&\ -\frac{5\,CW[122]}{12} == 0\ \&\&$

$CW[1112] - \frac{101\,CW[1122]}{6} + \frac{53\,CW[1212]}{3} - \frac{CW[1222]}{24} == -9\,CW[1122] + 9\,CW[1212], \ldots]\}$

dj

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

dj

$\{CWS[\epsilon\,\widehat{1} + \epsilon\,\widehat{2}, -\epsilon\,\widehat{12}, -\epsilon\,\widehat{112}, 0, \ldots], BS[5\,True, \ldots]\}$

Section 2.3 - The [AT]-inspired presentation E_l of $A^w{}_{exp}$

ElSetup

```
x1 = LW[1]; x2 = LW[2];
{ζa =
   E1[⟨1 → LS[x1 + b[x1, x2]], 2 → LS[x2 - b[x1, b[x1, x2]]]⟩, CWS[CW["1"] - 3 CW["121"]]],
  ζb = E1[⟨1 → LS[x2 - b[x1, x2]], 2 → LS[x1 + x2 + b[x2, b[x1, x2]]]⟩,
    CWS[CW["2"] - 2 CW["12"]]],
  ζc = E1[⟨1 → LS[x1 - b[b[x1, x2], b[x1, x2]]], 2 → LS[x2 + 3 b[x1, b[x1, x2]]]⟩,
    CWS[CW["1"] - 2 CW["12"] + CW["121"]]]}
```

ElSetup

$\{E_l[\langle 1 \to LS[\overline{1}, \overline{12}, 0, 0, \ldots], 2 \to LS[\overline{2}, 0, -\overline{1\,\overline{12}}, 0, \ldots]\rangle, CWS[\widehat{1}, 0, -3\,\widehat{112}, 0, \ldots]],$

$E_l[\langle 1 \to LS[\overline{2}, -\overline{12}, 0, 0, \ldots], 2 \to LS[\overline{1} + \overline{2}, 0, -\overline{\overline{12}\,2}, 0, \ldots]\rangle,$

$CWS[\widehat{2}, -2\,\widehat{12}, 0, 0, \ldots]],$

$E_l[\langle 1 \to LS[\overline{1}, 0, 0, 0, \ldots], 2 \to LS[\overline{2}, 0, 3\,\overline{1\,\overline{12}}, 0, \ldots]\rangle, CWS[\widehat{1}, -2\,\widehat{12}, \widehat{112}, 0, \ldots]]\}$

ElAssociativity

```
lhs = ζa ** (ζb ** ζc); rhs = (ζa ** ζb) ** ζc;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

ElAssociativity

$\{E_l[\langle 1 \to LS[2\,\overline{1} + \overline{2}, 0, \frac{1}{2}\,\overline{1\,\overline{12}}, \ldots], 2 \to LS[\overline{1} + 3\,\overline{2}, 0, \frac{5}{2}\,\overline{1\,\overline{12}} - \overline{\overline{12}\,2}, \ldots]\rangle,$

$CWS[2\,\widehat{1} + \widehat{2}, -4\,\widehat{12}, -2\,\widehat{112}, \ldots]], BS[9\,True, \ldots]\}$

detaExample

```
{ζa // dη^1, ζa // dη^2}
```

detaExample

$$\{E_1[\langle 2\to LS[\overline{2}, 0, 0, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]],\ E_1[\langle 1\to LS[\overline{1}, 0, 0, 0, \ldots]\rangle, CWS[\wideparen{1}, 0, 0, 0, \ldots]]\}$$

dA1

```
{ζd = E1[λ, CWS[0]], ζd // dA}
```

dA1

$$\{E_1[\langle 1\to LS[\overline{1}, \overline{12}, 0, 0, \ldots], 2\to LS[\overline{2}, 0, -\overline{1\overline{12}}, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]],\ E_1[\langle 1\to LS[-\overline{1}, -\overline{12}, 0, 0, \ldots], 2\to LS[-\overline{2}, 0, \overline{1\overline{12}}, 0, \ldots]\rangle, CWS[-\wideparen{1}-\wideparen{2}, \wideparen{12}, \wideparen{112}, \wideparen{1122}-\wideparen{1212}, \ldots]]\}$$

dA2

```
(ζd ≡ (ζd // dA // dA))@{8}
```

dA2

BS[9 True, ...]

dA3

```
lhs = (ζa ** ζb) // dA; rhs = (ζb // dA) ** (ζa // dA);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dA3

$$\{E_1[\langle 1\to LS[-\overline{1}-\overline{2}, 0, -\frac{1}{2}\overline{1\overline{12}}, \ldots], 2\to LS[-\overline{1}-2\,\overline{2}, 0, \frac{1}{2}\overline{1\overline{12}}+\overline{\overline{12}2}, \ldots]\rangle, CWS[-\wideparen{2}, -2\,\wideparen{12}, -2\,\wideparen{112}-\wideparen{122}, \ldots]], BS[9\ True, \ldots]\}$$

dS

```
ζd // dS
```

dS

$$E_1[\langle 1\to LS[\overline{1}, -\overline{12}, 0, 0, \ldots], 2\to LS[\overline{2}, 0, -\overline{1\overline{12}}, 0, \ldots]\rangle, CWS[\wideparen{1}+\wideparen{2}, \wideparen{12}, -\wideparen{112}, \wideparen{1122}-\wideparen{1212}, \ldots]]$$

dD1

```
{ζa, ζa // dΔ[2, 2, 3]}
```

dD1

$$\{E_1[\langle 1\to LS[\overline{1}, \overline{12}, 0, 0, \ldots], 2\to LS[\overline{2}, 0, -\overline{1\overline{12}}, 0, \ldots]\rangle, CWS[\wideparen{1}, 0, -3\,\wideparen{112}, 0, \ldots]],\ E_1[\langle 1\to LS[\overline{1}, \overline{12}+\overline{13}, 0, 0, \ldots], 2\to LS[\overline{2}+\overline{3}, 0, -\overline{1\overline{12}}-\overline{1\overline{13}}, 0, \ldots], 3\to LS[\overline{2}+\overline{3}, 0, -\overline{1\overline{12}}-\overline{1\overline{13}}, 0, \ldots]\rangle, CWS[\wideparen{1}, 0, -3\,\wideparen{112}-3\,\wideparen{113}, 0, \ldots]]\}$$

dD2

```
lhs = (ξa ** ξb) // dΔ[2, 2, 3]; rhs = (ξa // dΔ[2, 2, 3]) ** (ξb // dΔ[2, 2, 3]);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dD2

$$\{E_1[\langle 1 \to LS[\overline{1}+\overline{2}+\overline{3}, 0, \frac{1}{2}\overline{1\overline{1}\overline{2}} + \frac{1}{2}\overline{1\overline{1}\overline{3}}, \ldots],$$
$$2 \to LS[\overline{1}+2\overline{2}+2\overline{3}, 0, -\frac{1}{2}\overline{1\overline{1}\overline{2}} - \frac{1}{2}\overline{1\overline{1}\overline{3}} - \overline{1\overline{2}\overline{3}} - \overline{\overline{1}\overline{2}2} - 2\overline{\overline{1}\overline{3}2} - \overline{\overline{1}\overline{3}3}, \ldots],$$
$$3 \to LS[\overline{1}+2\overline{2}+2\overline{3}, 0, -\frac{1}{2}\overline{1\overline{1}\overline{2}} - \frac{1}{2}\overline{1\overline{1}\overline{3}} - \overline{1\overline{2}\overline{3}} - \overline{\overline{1}\overline{2}2} - 2\overline{\overline{1}\overline{3}2} - \overline{\overline{1}\overline{3}3}, \ldots]\rangle,$$
$$CWS[\widehat{1}+\widehat{2}+\widehat{3}, -2\widehat{12}-2\widehat{13}, -3\widehat{112}-3\widehat{113}, \ldots]], BS[9\,True, \ldots]\}$$

Section 2.4 - The factored presentation E_f of $A^w{}_{exp}$ and its stronger precursor E_s

EsSetup1

```
u = LW@u; v = LW@v;
ξa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
  CWS[CW["u"] - 3 CW["uvu"]]]
```

EsSetup1

$$E_s[\langle 1 \to LS[\overline{u}, \overline{u\,v}, 0, 0, \ldots], 2 \to LS[\overline{v}, 0, -\overline{u\overline{u\,v}}, 0, \ldots], 3 \to LS[\overline{u}, 0, 0, 0, \ldots]\rangle,$$
$$CWS[\widehat{u}, 0, -3\,\widehat{uuv}, 0, \ldots]]$$

EsSetup2

```
SeedRandom[0]; ξb =
 Es[⟨Table[i → RandomLieSeries[{1, 2, 3, 4}], {i, 4}]⟩, RandomCWSeries[{1, 2, 3, 4}]];
ξb@
 {2}
```

EsSetup2

$$E_s[\langle 1 \to LS[-\overline{1}-2\overline{2}+2\overline{3}-2\overline{4}, 2\overline{1\,2}+\frac{\overline{1\,3}}{2}+\overline{1\,4}-\frac{\overline{2\,3}}{2}-\frac{\overline{2\,4}}{2}+2\overline{3\,4}, \ldots],$$
$$2 \to LS[2\overline{1}-\overline{2}-2\overline{3}+\overline{4}, 2\overline{1\,2}+\frac{3\overline{1\,3}}{2}-2\overline{1\,4}-\overline{2\,3}-\overline{2\,4}-\frac{\overline{3\,4}}{2}, \ldots],$$
$$3 \to LS[-\overline{1}+\overline{2}+2\overline{4}, -2\overline{1\,2}+2\overline{1\,3}-\overline{1\,4}-\frac{3\overline{2\,3}}{2}+2\overline{2\,4}-2\overline{3\,4}, \ldots],$$
$$4 \to LS[-2\overline{1}+2\overline{2}+2\overline{3}+\overline{4}, -\frac{\overline{1\,2}}{2}+\frac{3\overline{1\,3}}{2}-2\overline{2\,4}+\overline{3\,4}, \ldots]\rangle,$$
$$CWS[\widehat{3}-\widehat{4}, \frac{3\widehat{11}}{2}+\frac{3\widehat{12}}{2}-2\widehat{13}+\widehat{14}+\widehat{22}+2\widehat{23}-\frac{\widehat{24}}{2}-2\widehat{33}-\widehat{34}+\widehat{44}, \ldots]]$$

haction

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4];
rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

haction

$\{E_s[\langle 3 \to LS[\overline{u}, -\overline{uv}, -\overline{u\,\overline{uv}} + \frac{1}{2}\overline{\overline{uv}\,v}, \frac{3}{2}\overline{u\,\overline{u\,\overline{uv}}} + \overline{u\,\overline{\overline{uv}\,v}} - \frac{1}{6}\overline{\overline{\overline{uv}\,v}\,v}, \ldots],$
$4 \to LS[\overline{u} + \overline{v}, \frac{\overline{uv}}{2}, -\frac{23}{12}\overline{u\,\overline{uv}} - \frac{5}{12}\overline{\overline{uv}\,v}, \overline{u\,\overline{u\,\overline{uv}}} + \frac{13}{24}\overline{u\,\overline{\overline{uv}\,v}} + \frac{1}{12}\overline{\overline{\overline{uv}\,v}\,v}, \ldots]\rangle,$
$CWS[2\,\widehat{u}, -\widehat{uv}, -\frac{3\,\widehat{uuv}}{2}, -\frac{\widehat{uuuv}}{6} + \widehat{uuvv} - \widehat{uvuv}, \ldots]], BS[9\,True, \ldots]\}$

metaassoc

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

metaassoc

$\{E_s[\langle 1 \to LS[-2\,\overline{1} + \overline{4}, -\frac{3\,\overline{14}}{2}, 20\,\overline{1\,\overline{14}} - \frac{19}{3}\overline{\overline{14}\,4}, \ldots],$
$4 \to LS[2\,\overline{1} + \overline{4}, \overline{14}, -\frac{31}{2}\overline{1\,\overline{14}} - \frac{13}{6}\overline{\overline{14}\,4}, \ldots]\rangle,$
$CWS[3\,\widehat{1} - \widehat{4}, -3\,\widehat{11} + \frac{\widehat{14}}{2} + \widehat{44}, \frac{71\,\widehat{111}}{4} + \frac{19\,\widehat{114}}{4} - \frac{7\,\widehat{144}}{6} - \frac{2\,\widehat{444}}{3}, \ldots]], BS[6\,True, \ldots]\}$

Section 3.1 - Tangle Invariants

Section 3.1.1 - The General Framework

zetaDefinitions

```
R⁺[a_, b_] // ζt:(1|s) := Et[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R⁻[a_, b_] // ζt:(1|s) := Et[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
ζt:(1|s)[K1_ ** K2_] := ζt[K1] ** ζt[K2];
ζt:(1|s)[K1_ K2_] := ζt[K1] ζt[K2];
```

Section 3.1.2 - Verifying Reidemeister 3

R3

```
lhs = R⁺[1, 2] ** R⁺[1, 3] ** R⁺[2, 3] // ζ1; rhs = R⁺[2, 3] ** R⁺[1, 3] ** R⁺[1, 2] // ζ1;
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

R3

$\{E_1[\langle 1 \to LS[0, 0, 0, \ldots], 2 \to LS[\overline{1}, 0, 0, \ldots], 3 \to LS[\overline{1} + \overline{2}, 0, 0, \ldots]\rangle,$
$CWS[0, 0, 0, \ldots]], BS[6\,True, \ldots]\}$

Section 3.1.3 - The Knot 8_{17}

```
t1 = R^-[12, 1] R^-[2, 7] R^-[8, 3] R^-[4, 11] R^+[16, 5] R^+[6, 13] R^+[14, 9] R^+[10, 15] // ξs;
Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
t1@{6}
```

$$E_s\left[\langle 1 \to LS[0, 0, 0, 0, 0, 0, \ldots]\rangle, CWS\left[0, -\overgroup{11}, 0, -\frac{31\,\overgroup{1111}}{12}, 0, -\frac{1351\,\overgroup{111111}}{360}, \ldots\right]\right]$$

Section 3.1.4 - The Borromean Tangle

```
t2 = R^-[r, 6] R^+[2, 4] R^-[g, 9] R^+[5, 7] R^-[b, 3] R^+[8, 1] // ξs;
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
 Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

$$E_s\Big[\Big\langle b \to LS\Big[0, \overline{gr}, \frac{1}{2}\overline{g\overline{gr}} + \overline{\overline{br}g} + \frac{1}{2}\overline{\overline{gr}r},$$
$$-\frac{1}{2}\overline{b\overline{\overline{br}g}} + \frac{1}{6}\overline{g\overline{g\overline{gr}}} + \frac{1}{4}\overline{g\overline{\overline{gr}r}} - \frac{1}{2}\overline{\overline{bg}\,\overline{br}} - \frac{1}{2}\overline{\overline{\overline{br}g}g} - \frac{1}{2}\overline{\overline{\overline{br}r}g} + \frac{1}{6}\overline{\overline{\overline{gr}r}r}, \ldots\Big], g \to$$
$$LS\Big[0, -\overline{br}, \frac{1}{2}\overline{b\overline{br}} - \overline{b\overline{gr}} - \overline{\overline{br}g} + \frac{1}{2}\overline{\overline{br}r}, -\frac{1}{6}\overline{b\overline{b\overline{br}}} - \frac{1}{2}\overline{b\overline{b\overline{gr}}} - \frac{1}{2}\overline{b\overline{g\overline{gr}}} - \frac{1}{2}\overline{b\overline{\overline{br}g}} -$$
$$\frac{1}{4}\overline{b\overline{\overline{br}r}} + \frac{1}{2}\overline{b\overline{\overline{gr}r}} + \frac{1}{2}\overline{\overline{bg}\,\overline{br}} + \overline{\overline{br}\,\overline{gr}} - \overline{\overline{b\overline{gr}}g} - \frac{1}{2}\overline{\overline{\overline{br}g}g} + \frac{1}{2}\overline{\overline{\overline{br}r}g} - \frac{1}{6}\overline{\overline{\overline{br}r}r}, \ldots\Big],$$
$$r \to LS\Big[0, \overline{bg}, \frac{1}{2}\overline{b\overline{bg}} + \overline{b\overline{gr}} + \frac{1}{2}\overline{\overline{bg}g}, \frac{1}{6}\overline{b\overline{b\overline{bg}}} + \frac{1}{2}\overline{b\overline{b\overline{gr}}} +$$
$$\frac{1}{2}\overline{b\overline{g\overline{gr}}} + \frac{1}{4}\overline{b\overline{\overline{bg}g}} + \frac{1}{2}\overline{b\overline{\overline{gr}r}} + \frac{1}{6}\overline{\overline{\overline{bg}g}g}, \ldots\Big]\Big\rangle,$$
$$CWS\left[0, 0, 2\,\overgroup{bgr}, \overgroup{bbgr} - \overgroup{bgbr} + \overgroup{bggr} - \overgroup{bgrg} + \overgroup{bgrr} - \overgroup{brgr}, \ldots\right]\Big]$$

Section 3.1.5 - The "Handshake" 2-Links

It is possible that the caps here are placed on the wrong ends of strands!

```
R^+[1, l] R^+[2, r] // ξs // dm[2, l, l] // dm[1, r, r] // hη[l] // hη[r]
```

$$E_s\left[\langle\rangle, CWS\left[0, -\overgroup{lr}, -\frac{\overgroup{llr}}{2} - \frac{\overgroup{lrr}}{2}, -\frac{\overgroup{lllr}}{6} + \frac{7\,\overgroup{llrr}}{4} - \frac{5\,\overgroup{lrlr}}{2} - \frac{\overgroup{lrrr}}{6}, \ldots\right]\right]$$

```
R^+[1, l] R^-[2, r] // ξs // dm[2, l, l] // dm[1, r, r] // hη[l] // hη[r]
```

$$E_s\left[\langle\rangle, CWS\left[0, \overgroup{lr}, -\frac{\overgroup{llr}}{2} + \frac{\overgroup{lrr}}{2}, \frac{\overgroup{lllr}}{6} + \frac{7\,\overgroup{llrr}}{4} - \frac{5\,\overgroup{lrlr}}{2} + \frac{\overgroup{lrrr}}{6}, \ldots\right]\right]$$

```
R^-[1, l] R^+[2, r] // ξs // dm[2, l, l] // dm[1, r, r] // hη[l] // hη[r]
```

$$E_s\left[\langle\rangle, CWS\left[0, \overgroup{lr}, \frac{\overgroup{llr}}{2} - \frac{\overgroup{lrr}}{2}, \frac{\overgroup{lllr}}{6} + \frac{7\,\overgroup{llrr}}{4} - \frac{5\,\overgroup{lrlr}}{2} + \frac{\overgroup{lrrr}}{6}, \ldots\right]\right]$$

```
R⁻[1, l] R⁻[2, r] // ζs // dm[2, l, l] // dm[1, r, r] // hη[l] // hη[r]
```

$$E_s\left[\langle\rangle,\ CWS\left[0,\ -\overline{lr},\ \frac{\overline{llr}}{2}+\frac{\overline{lrr}}{2},\ -\frac{\overline{lllr}}{6}+\frac{7\,\overline{llrr}}{4}-\frac{5\,\overline{lrlr}}{2}-\frac{\overline{lrrr}}{6},\ \ldots\right]\right]$$

Section 3.1 - Solutions of the Kashiwara-Vergne Equations

Continues pensieve://2013-10/SolvingWKO.nb.

```
α = LS[{"1", "2"}, αs]; β = LS[{"1", "2"}, βs];
γ = CWS[{"1", "2"}, γs]; V = Es[⟨1 → α, 2 → β⟩, γ];
κ = CWS[{"1"}, κs]; Cap = Es[⟨1 → LS[0]⟩, κ];
αs["2"] = -1/2;
SeriesSolve[{α, β, γ, κ},
  ħ^-1 (ζs[R⁺[2, 3] ** R⁺[1, 3]] ** V ≡ V ** (ζs[R⁺[1, 3]] // dΔ[1, 1, 2]))
   && V ** (V // dA[1] // dA[2]) ≡ Es[⟨1 → LS[0], 2 → LS[0]⟩, CWS[0]]
   && (V ** (Cap // dΔ[1, 1, 2]) // dc[1] // dc[2]) ≡
    (Cap (Cap // dσ[1, 2]) // dc[1] // dc[2])
 ];
{V, κ}
```

```
Arbitrator called on {κs[1]}...
Arbitrator called on {αs[122]}...
```

$$\Big\{E_s\Big[\Big\langle 1\to LS\Big[-\frac{\overline{2}}{2},\ \frac{\overline{12}}{12},\ 0,\ -\frac{1}{720}\overline{1\overline{1\overline{12}}}+\frac{1}{720}\overline{1\overline{\overline{12}2}}-\frac{\overline{\overline{\overline{12}2}2}}{5760},\ \ldots\Big],$$
$$2\to LS\Big[0,\ \frac{\overline{12}}{24},\ 0,\ -\frac{\overline{1\overline{1\overline{12}}}}{1440}+\frac{7\,\overline{1\overline{\overline{12}2}}}{5760}-\frac{7\,\overline{\overline{\overline{12}2}2}}{5760},\ \ldots\Big]\Big\rangle,$$
$$CWS\Big[0,\ -\frac{\overline{12}}{48},\ 0,\ \frac{\overline{1112}}{2880}+\frac{\overline{1122}}{2880}+\frac{\overline{1212}}{5760}+\frac{\overline{1222}}{2880},\ \ldots\Big]\Big],\ CWS\Big[0,\ -\frac{\overline{11}}{96},\ 0,\ \frac{\overline{1111}}{11\,520},\ \ldots\Big]\Big\}$$

Conjecture. *For any Lie algebra* $\mathfrak{g}$ *of finite dimension, we can find F and G such that they satisfy*

a) $x+y-\log e^y e^x=(1-e^{-\operatorname{ad}x})F+(e^{\operatorname{ad}y}-1)G.$

b) *F and G give* $\mathfrak{g}$*-valued convergent power series on* $(x,y)\in\mathfrak{g}\times\mathfrak{g}$.

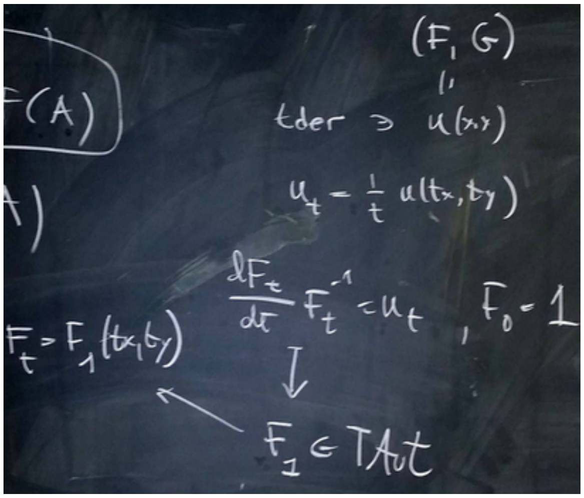
c) $\operatorname{tr}((\operatorname{ad}x)(\partial_x F);\mathfrak{g})+\operatorname{tr}((\operatorname{ad}y)(\partial_y G);\mathfrak{g})$

$$=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1;\mathfrak{g}\right).$$

Here $z=\log e^x e^y$ *and* $\partial_x F$ (resp. $\partial_y G$) *is the* **End**($\mathfrak{g}$)*-valued real analytic function defined by*

$$\mathfrak{g}\ni a\mapsto\frac{d}{dt}F(x+ta,y)|_{t=0}\quad\left(\text{resp. }\mathfrak{g}\ni a\mapsto\frac{d}{dt}G(x,y+ta)|_{t=0}\right),$$

and tr *denotes the trace of an endomorphism of* $\mathfrak{g}$.




```
{F = LS[{x, y}, fs], G = LS[{x, y}, gs]};
SeriesSolve[{F, G},
 ħ^-1 (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ F - G - Ad[-LW@x][F] + Ad[LW@y][G])
  && div_x[F] + div_y[G] ≡ 1/2 tr_LW@u[adSeries[ad/(e^ad - 1), LW@x][LW@u] +
      adSeries[ad/(e^ad - 1), LW@y][LW@u] - adSeries[ad/(e^ad - 1), BCH[LW@x, LW@y]][LW@u]]
]
kv = ⟨1 → F, 2 → G⟩
```

Arbitrator called on {fs[y]}...

$$\left\langle 1 \to \mathrm{LS}\left[0, -\frac{\overline{x\,y}}{12}, -\frac{1}{24}\overline{x\,\overline{x\,y}}, -\frac{1}{180}\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{1}{120}\overline{x\,\overline{\overline{x\,y}\,y}} + \frac{1}{360}\overline{\overline{\overline{x\,y}\,y}\,y}, \ldots\right],\right.$$
$$\left. 2 \to \mathrm{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{x\,y}}{6}, -\frac{1}{24}\overline{x\,\overline{x\,y}}, -\frac{1}{360}\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{1}{80}\overline{x\,\overline{\overline{x\,y}\,y}} + \frac{1}{180}\overline{\overline{\overline{x\,y}\,y}\,y}, \ldots\right]\right\rangle$$

```
at = V[[1]] // dσ[{1, 2} → {2, 1}];
atkv = at // EulerE // adSeries[(e^ad - 1)/ad, at] // RC[-at] //
  LieMorphism[LW@1 → LW@x, LW@2 → LW@y]
```

$$\left\langle 1 \to \mathrm{LS}\left[0, -\frac{\overline{x\,y}}{12}, -\frac{1}{24}\overline{x\,\overline{x\,y}}, -\frac{1}{180}\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{1}{120}\overline{x\,\overline{\overline{x\,y}\,y}} + \frac{1}{360}\overline{\overline{\overline{x\,y}\,y}\,y}, \ldots\right],\right.$$
$$\left. 2 \to \mathrm{LS}\left[-\frac{\overline{x}}{2}, -\frac{\overline{x\,y}}{6}, -\frac{1}{24}\overline{x\,\overline{x\,y}}, -\frac{1}{360}\overline{x\,\overline{x\,\overline{x\,y}}} - \frac{1}{80}\overline{x\,\overline{\overline{x\,y}\,y}} + \frac{1}{180}\overline{\overline{\overline{x\,y}\,y}\,y}, \ldots\right]\right\rangle$$

at1 = at // Λ;

atkv1 = at1 // EulerE // adSeries[$\frac{e^{ad}-1}{ad}$, at1, tb] //
LieMorphism[LW@1 → LW@x, LW@2 → LW@y]

$$\left\langle 1 \to LS\left[0, -\frac{\overline{xy}}{12}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} - \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right],\; 2 \to LS\left[-\frac{\overline{x}}{2}, -\frac{\overline{xy}}{6}, -\frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} - \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\rangle$$

λ2 = ⟨1 → RandomLieSeries[{1, 2}], 2 → RandomLieSeries[{1, 2}]⟩

$$\left\langle 1 \to LS\left[\overline{2}, 0, -\frac{1}{3}\overline{1\overline{12}} - \overline{\overline{12}2}, \frac{41}{24}\overline{1\overline{1\overline{12}}} + \frac{19}{12}\overline{1\overline{\overline{12}2}} + \frac{7}{12}\overline{\overline{\overline{12}2}2}, \ldots\right],\; 2 \to LS\left[2\,\overline{1} + 2\,\overline{2}, -2\,\overline{12}, -\frac{1}{3}\overline{1\overline{12}} + \frac{5}{6}\overline{\overline{12}2}, \frac{29}{24}\overline{1\overline{1\overline{12}}} - \frac{11}{8}\overline{1\overline{\overline{12}2}} - \frac{17}{12}\overline{\overline{\overline{12}2}2}, \ldots\right]\right\rangle$$

{lhs = λ2 // EulerE // adSeries[$\frac{e^{ad}-1}{ad}$, λ2] // RC[-λ2],
rhs = Λ[λ2] // EulerE // adSeries[$\frac{e^{ad}-1}{ad}$, Λ[λ2], tb]; (lhs ≡ rhs)@{8}}

$$\left\{\left\langle 1 \to LS\left[\overline{2}, -2\,\overline{12}, \overline{1\overline{12}} - \overline{\overline{12}2}, \frac{15}{2}\overline{1\overline{1\overline{12}}} + \frac{20}{3}\overline{1\overline{\overline{12}2}} - \frac{5}{6}\overline{\overline{\overline{12}2}2}, \ldots\right],\; 2 \to LS\left[2\,\overline{1} + 2\,\overline{2}, -6\,\overline{12}, 5\,\overline{1\overline{12}} + \frac{11}{2}\overline{\overline{12}2}, \frac{3}{2}\overline{1\overline{1\overline{12}}} - \frac{45}{2}\overline{1\overline{\overline{12}2}} - \frac{49}{6}\overline{\overline{\overline{12}2}2}, \ldots\right]\right\rangle, BS[9\ True, \ldots]\right\}$$

(atkv ≡ atkv1)@{9}

BS[10 True, ...]

(atkv ≡ kv)@{8}

Arbitrator called on {αs[11122]}...

Arbitrator called on {αs[1111122]}...

Arbitrator called on {αs[11112122]}...

Arbitrator called on {fs[xxxxyxyy]}...

$$BS\Big[8\ True, -\frac{LW[xxxxxxxy]}{151200} + \frac{LW[xxxxxxyy]}{37800} - \frac{LW[xxxxxyxy]}{151200} - \frac{LW[xxxxxyyy]}{37800} - \frac{LW[xxxxyxxy]}{50400} - \frac{61\,LW[xxxxyxyy]}{1360800} - \frac{61\,LW[xxxxyyxy]}{680400} - \frac{LW[xxxxyyyy]}{30240} - \frac{829\,LW[xxxyxxyy]}{10886400} + \frac{7\,LW[xxxyxyxy]}{259200} - \frac{653\,LW[xxxyxyyy]}{7257600} + \frac{251\,LW[xxxyyxxy]}{5443200} + \frac{67\,LW[xxxyyxyy]}{2419200} + \frac{467\,LW[xxxyyyxy]}{2419200} + \frac{LW[xxxyyyyy]}{18900} + \frac{19\,LW[xxyxxyxy]}{2721600} + \frac{653\,LW[xxyxxyyy]}{7257600} - \frac{307\,LW[xxyxyxyy]}{2419200} + \frac{173\,LW[xxyxyyxy]}{2419200} + \frac{1313\,LW[xxyxyyyy]}{7257600} - \frac{67\,LW[xxyyxyxy]}{1209600} + \frac{667\,LW[xxyyxyyy]}{7257600} -$$

$$\frac{25\,\text{LW}[\text{xxyyyxyy}]}{290\,304}-\frac{773\,\text{LW}[\text{xxyyyyxy}]}{7\,257\,600}-\frac{\text{LW}[\text{xxyyyyyy}]}{43\,200}+\frac{59\,\text{LW}[\text{xyxyxyyy}]}{259\,200}-$$
$$\frac{173\,\text{LW}[\text{xyxyyxyy}]}{2\,419\,200}-\frac{59\,\text{LW}[\text{xyxyyyyy}]}{777\,600}-\frac{197\,\text{LW}[\text{xyyxyyyy}]}{2\,177\,280}+\frac{\text{LW}[\text{xyyyyyyy}]}{302\,400}==$$
$$-\frac{\text{LW}[\text{xxxxxxxy}]}{151\,200}+\frac{\text{LW}[\text{xxxxxxyy}]}{37\,800}-\frac{\text{LW}[\text{xxxxxyxy}]}{151\,200}-\frac{\text{LW}[\text{xxxxxyyy}]}{37\,800}-$$
$$\frac{\text{LW}[\text{xxxxyxxy}]}{50\,400}-\frac{\text{LW}[\text{xxxxyyyy}]}{30\,240}-\frac{13\,\text{LW}[\text{xxxyxxyy}]}{241\,920}-\frac{13\,\text{LW}[\text{xxxyxyxy}]}{120\,960}-$$
$$\frac{299\,\text{LW}[\text{xxxyxyyy}]}{2\,419\,200}+\frac{11\,\text{LW}[\text{xxxyyxxy}]}{120\,960}-\frac{59\,\text{LW}[\text{xxxyyxyy}]}{806\,400}+\frac{223\,\text{LW}[\text{xxxyyyxy}]}{2\,419\,200}+$$
$$\frac{\text{LW}[\text{xxxyyyyy}]}{18\,900}-\frac{\text{LW}[\text{xxyxxyxy}]}{12\,096}+\frac{299\,\text{LW}[\text{xxyxxyyy}]}{2\,419\,200}-\frac{\text{LW}[\text{xxyxyxyy}]}{38\,400}+$$
$$\frac{139\,\text{LW}[\text{xxyxyyxy}]}{806\,400}+\frac{173\,\text{LW}[\text{xxyxyyyy}]}{806\,400}+\frac{59\,\text{LW}[\text{xxyyxyxy}]}{403\,200}+\frac{47\,\text{LW}[\text{xxyyxyyy}]}{806\,400}-$$
$$\frac{41\,\text{LW}[\text{xxyyyxyy}]}{161\,280}-\frac{113\,\text{LW}[\text{xxyyyyxy}]}{806\,400}-\frac{\text{LW}[\text{xxyyyyyy}]}{43\,200}+\frac{73\,\text{LW}[\text{xyxyxyyy}]}{201\,600}-$$
$$\frac{139\,\text{LW}[\text{xyxyyxyy}]}{806\,400}-\frac{73\,\text{LW}[\text{xyxyyyyy}]}{604\,800}-\frac{7\,\text{LW}[\text{xyyxyyyy}]}{34\,560}+\frac{\text{LW}[\text{xyyyyyyy}]}{302\,400}\&\&$$
$$-\frac{\text{LW}[\text{xxxxxxxy}]}{302\,400}+\frac{\text{LW}[\text{xxxxxxyy}]}{67\,200}+\frac{89\,\text{LW}[\text{xxxxxyxy}]}{5\,443\,200}-\frac{\text{LW}[\text{xxxxxyyy}]}{302\,400}-$$
$$\frac{181\,\text{LW}[\text{xxxxyxxy}]}{10\,886\,400}-\frac{73\,\text{LW}[\text{xxxxyxyy}]}{1\,451\,520}-\frac{461\,\text{LW}[\text{xxxxyyxy}]}{2\,419\,200}-$$
$$\frac{\text{LW}[\text{xxxxyyyy}]}{16\,128}-\frac{643\,\text{LW}[\text{xxxyxxyy}]}{7\,257\,600}+\frac{\text{LW}[\text{xxxyxyxy}]}{12\,600}-\frac{907\,\text{LW}[\text{xxxyxyyy}]}{7\,257\,600}+$$
$$\frac{\text{LW}[\text{xxxyyxxy}]}{25\,200}+\frac{31\,\text{LW}[\text{xxxyyxyy}]}{518\,400}+\frac{1987\,\text{LW}[\text{xxxyyyxy}]}{7\,257\,600}+\frac{23\,\text{LW}[\text{xxxyyyyy}]}{302\,400}+$$
$$\frac{211\,\text{LW}[\text{xxyxxyxy}]}{7\,257\,600}+\frac{\text{LW}[\text{xxyxxyyy}]}{10\,080}-\frac{667\,\text{LW}[\text{xxyxyxyy}]}{3\,628\,800}+\frac{59\,\text{LW}[\text{xxyxyyxy}]}{1\,036\,800}+$$
$$\frac{1207\,\text{LW}[\text{xxyxyyyy}]}{5\,443\,200}-\frac{1133\,\text{LW}[\text{xxyyxyxy}]}{7\,257\,600}+\frac{61\,\text{LW}[\text{xxyyxyyy}]}{518\,400}-$$
$$\frac{\text{LW}[\text{xxyyyxyy}]}{15\,120}-\frac{\text{LW}[\text{xxyyyyxy}]}{7560}-\frac{\text{LW}[\text{xxyyyyyy}]}{28\,800}+\frac{1493\,\text{LW}[\text{xyxyxyyy}]}{5\,443\,200}-$$
$$\frac{667\,\text{LW}[\text{xyxyyxyy}]}{10\,886\,400}-\frac{\text{LW}[\text{xyxyyyyy}]}{10\,080}-\frac{\text{LW}[\text{xyyxyyyy}]}{10\,080}+\frac{\text{LW}[\text{xyyyyyyy}]}{151\,200}==$$
$$-\frac{\text{LW}[\text{xxxxxxxy}]}{302\,400}+\frac{\text{LW}[\text{xxxxxxyy}]}{67\,200}+\frac{37\,\text{LW}[\text{xxxxxyxy}]}{604\,800}-\frac{\text{LW}[\text{xxxxxyyy}]}{302\,400}+\frac{\text{LW}[\text{xxxxyxxy}]}{172\,800}-$$
$$\frac{29\,\text{LW}[\text{xxxxyxyy}]}{345\,600}-\frac{47\,\text{LW}[\text{xxxxyyxy}]}{161\,280}-\frac{\text{LW}[\text{xxxxyyyy}]}{16\,128}-\frac{19\,\text{LW}[\text{xxxyxxyy}]}{345\,600}+$$
$$\frac{\text{LW}[\text{xxxyxyxy}]}{12\,600}-\frac{221\,\text{LW}[\text{xxxyxyyy}]}{2\,419\,200}+\frac{\text{LW}[\text{xxxyyxxy}]}{25\,200}-\frac{\text{LW}[\text{xxxyyxyy}]}{134\,400}+\frac{83\,\text{LW}[\text{xxxyyyxy}]}{345\,600}+$$
$$\frac{23\,\text{LW}[\text{xxxyyyyy}]}{302\,400}-\frac{11\,\text{LW}[\text{xxyxxyxy}]}{2\,419\,200}+\frac{\text{LW}[\text{xxyxxyyy}]}{10\,080}-\frac{47\,\text{LW}[\text{xxyxyxyy}]}{403\,200}+$$
$$\frac{73\,\text{LW}[\text{xxyxyyxy}]}{806\,400}+\frac{107\,\text{LW}[\text{xxyxyyyy}]}{604\,800}-\frac{17\,\text{LW}[\text{xxyyxyxy}]}{89\,600}+\frac{61\,\text{LW}[\text{xxyyxyyy}]}{1\,209\,600}-$$
$$\frac{\text{LW}[\text{xxyyyxyy}]}{15\,120}-\frac{\text{LW}[\text{xxyyyyxy}]}{7560}-\frac{\text{LW}[\text{xxyyyyyy}]}{28\,800}+\frac{193\,\text{LW}[\text{xyxyxyyy}]}{604\,800}-$$
$$\frac{47\,\text{LW}[\text{xyxyyxyy}]}{1\,209\,600}-\frac{\text{LW}[\text{xyxyyyyy}]}{10\,080}-\frac{\text{LW}[\text{xyyxyyyy}]}{10\,080}+\frac{\text{LW}[\text{xyyyyyyy}]}{151\,200},\ \ldots]$$

{A, B} = {$\mathbf{atkv_1}$, $\mathbf{atkv_2}$}

$$\left\{\mathrm{LS}\left[0,\ -\frac{\overline{x\,y}}{12},\ -\frac{1}{24}\,\overline{x\,\overline{x\,y}},\ -\frac{1}{180}\,\overline{x\,\overline{x\,\overline{x\,y}}}-\frac{1}{120}\,\overline{x\,\overline{\overline{x\,y}\,y}}+\frac{1}{360}\,\overline{\overline{\overline{x\,y}\,y}\,y},\ \ldots\right],\right.$$
$$\left.\mathrm{LS}\left[-\frac{\overline{x}}{2},\ -\frac{\overline{x\,y}}{6},\ -\frac{1}{24}\,\overline{x\,\overline{x\,y}},\ -\frac{1}{360}\,\overline{x\,\overline{x\,\overline{x\,y}}}-\frac{1}{80}\,\overline{x\,\overline{\overline{x\,y}\,y}}+\frac{1}{180}\,\overline{\overline{\overline{x\,y}\,y}\,y},\ \ldots\right]\right\}$$

(ℏ^-1 (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ A - B - Ad[-LW@x][A] + Ad[LW@y][B]) &&
div_x[A] + div_y[B] ≡ $\frac{1}{2}$ $\mathrm{tr}_{LW@u}$[adSeries[$\frac{ad}{e^{ad}-1}$, LW@x][LW@u] + adSeries[$\frac{ad}{e^{ad}-1}$, LW@y][LW@u] - adSeries[$\frac{ad}{e^{ad}-1}$, BCH[LW@x, LW@y]][LW@u]]) @ {9}

BS[10 True, ...]

---

f = (V // Λ)⟦1⟧

$$\left\langle 1\to\mathrm{LS}\left[0,\ -\frac{\overline{1\,2}}{24},\ \frac{1}{96}\,\overline{1\,\overline{1\,2}},\ \frac{\overline{1\,\overline{1\,\overline{1\,2}}}}{2880}-\frac{1}{480}\,\overline{1\,\overline{\overline{1\,2}\,2}}+\frac{\overline{\overline{\overline{1\,2}\,2}\,2}}{1440},\ \ldots\right],\right.$$
$$\left.2\to\mathrm{LS}\left[\frac{\overline{1}}{2},\ -\frac{\overline{1\,2}}{12},\ \frac{1}{96}\,\overline{1\,\overline{1\,2}},\ \frac{1}{960}\,\overline{1\,\overline{1\,\overline{1\,2}}}-\frac{1}{320}\,\overline{1\,\overline{\overline{1\,2}\,2}}+\frac{1}{720}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

f // RC[-f]

$$\left\langle 1\to\mathrm{LS}\left[0,\ -\frac{\overline{1\,2}}{24},\ \frac{1}{32}\,\overline{1\,\overline{1\,2}},\ -\frac{29\,\overline{1\,\overline{1\,\overline{1\,2}}}}{2880}-\frac{11\,\overline{1\,\overline{\overline{1\,2}\,2}}}{2880}+\frac{\overline{\overline{\overline{1\,2}\,2}\,2}}{1440},\ \ldots\right],\right.$$
$$\left.2\to\mathrm{LS}\left[\frac{\overline{1}}{2},\ -\frac{\overline{1\,2}}{12},\ \frac{1}{32}\,\overline{1\,\overline{1\,2}},\ \frac{1}{960}\,\overline{1\,\overline{1\,\overline{1\,2}}}-\frac{19\,\overline{1\,\overline{\overline{1\,2}\,2}}}{2880}+\frac{1}{720}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

V⟦1⟧ // RC[-V⟦1⟧]

$$\left\langle 1\to\mathrm{LS}\left[0,\ -\frac{\overline{1\,2}}{24},\ \frac{1}{48}\,\overline{1\,\overline{1\,2}},\ -\frac{23\,\overline{1\,\overline{1\,\overline{1\,2}}}}{5760}-\frac{17\,\overline{1\,\overline{\overline{1\,2}\,2}}}{5760}+\frac{\overline{\overline{\overline{1\,2}\,2}\,2}}{1440},\ \ldots\right],\right.$$
$$\left.2\to\mathrm{LS}\left[\frac{\overline{1}}{2},\ -\frac{\overline{1\,2}}{12},\ \frac{1}{48}\,\overline{1\,\overline{1\,2}},\ \frac{\overline{1\,\overline{1\,\overline{1\,2}}}}{5760}-\frac{7\,\overline{1\,\overline{\overline{1\,2}\,2}}}{1440}+\frac{1}{720}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

f - (f // EulerE)

$$\left\langle 1\to\mathrm{LS}\left[0,\ \frac{\overline{1\,2}}{24},\ -\frac{1}{48}\,\overline{1\,\overline{1\,2}},\ -\frac{1}{960}\,\overline{1\,\overline{1\,\overline{1\,2}}}+\frac{1}{160}\,\overline{1\,\overline{\overline{1\,2}\,2}}-\frac{1}{480}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right],\right.$$
$$\left.2\to\mathrm{LS}\left[0,\ \frac{\overline{1\,2}}{12},\ -\frac{1}{48}\,\overline{1\,\overline{1\,2}},\ -\frac{1}{320}\,\overline{1\,\overline{1\,\overline{1\,2}}}+\frac{3}{320}\,\overline{1\,\overline{\overline{1\,2}\,2}}-\frac{1}{240}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

f // EulerE // e^{-D_f}

$$\left\langle 1\to\mathrm{LS}\left[0,\ -\frac{\overline{1\,2}}{12},\ -\frac{1}{96}\,\overline{1\,\overline{1\,2}},\ \frac{19\,\overline{1\,\overline{1\,\overline{1\,2}}}}{2880}-\frac{7\,\overline{1\,\overline{\overline{1\,2}\,2}}}{1440}+\frac{1}{360}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right],\right.$$
$$\left.2\to\mathrm{LS}\left[\frac{\overline{1}}{2},\ -\frac{\overline{1\,2}}{6},\ -\frac{1}{32}\,\overline{1\,\overline{1\,2}},\ -\frac{1}{960}\,\overline{1\,\overline{1\,\overline{1\,2}}}-\frac{1}{180}\,\overline{1\,\overline{\overline{1\,2}\,2}}+\frac{1}{180}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right]\right\rangle$$

{A = LS[{x, y}, as], B = LS[{x, y}, bs], ϕ = CWS[{x}, ϕs]}

$\{\mathrm{LS}[\mathrm{as}[x]\,\overline{x} + \mathrm{as}[y]\,\overline{y},\ \mathrm{as}[xy]\,\overline{xy},\ \mathrm{as}[xxy]\,\overline{x\overline{xy}} + \mathrm{as}[xyy]\,\overline{\overline{xy}y},$
$\mathrm{as}[xxxy]\,\overline{x\overline{x\overline{xy}}} + \mathrm{as}[xxyy]\,\overline{x\overline{\overline{xy}y}} + \mathrm{as}[xyyy]\,\overline{\overline{\overline{xy}y}y},\ \ldots],$
$\mathrm{LS}[\mathrm{bs}[x]\,\overline{x} + \mathrm{bs}[y]\,\overline{y},\ \mathrm{bs}[xy]\,\overline{xy},\ \mathrm{bs}[xxy]\,\overline{x\overline{xy}} + \mathrm{bs}[xyy]\,\overline{\overline{xy}y},$
$\mathrm{bs}[xxxy]\,\overline{x\overline{x\overline{xy}}} + \mathrm{bs}[xxyy]\,\overline{x\overline{\overline{xy}y}} + \mathrm{bs}[xyyy]\,\overline{\overline{\overline{xy}y}y},\ \ldots],$
$\mathrm{CWS}[\widehat{x}\,\phi s[x],\ \widehat{xx}\,\phi s[xx],\ \widehat{xxx}\,\phi s[xxx],\ \widehat{xxxx}\,\phi s[xxxx],\ \ldots]\}$

SeriesSolve[{A, B, ϕ},
$\hbar^{-1}$ (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ A - B - Ad[-LW@x][A] + Ad[LW@y][B])
&& (div_x[A] + div_y[B] ≡
ϕ + LieMorphism[LW@x → LW@y][ϕ] - LieMorphism[LW@x → BCH[LW@x, LW@y]][ϕ])
]

A[1]

Arbitrator called on {as[y], ϕs[x]}...

0

B[1]

$-\frac{\mathrm{LW}[x]}{2}$

ϕ[1]

0

{A, B, ϕ}

Arbitrator called on {as[xyy]}...

$\{\mathrm{LS}[0,\ -\frac{\overline{xy}}{12},\ -\frac{1}{24}\,\overline{x\overline{xy}},\ -\frac{1}{180}\,\overline{x\overline{x\overline{xy}}} - \frac{1}{120}\,\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\,\overline{\overline{\overline{xy}y}y},\ \ldots],$
$\mathrm{LS}[-\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{6},\ -\frac{1}{24}\,\overline{x\overline{xy}},\ -\frac{1}{360}\,\overline{x\overline{x\overline{xy}}} - \frac{1}{80}\,\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\,\overline{\overline{\overline{xy}y}y},\ \ldots],$
$\mathrm{CWS}[0,\ \frac{\widehat{xx}}{24},\ 0,\ -\frac{\widehat{xxxx}}{1440},\ \ldots]\}$

ϕ@{11}

Arbitrator called on {as[xxxxxxxxxyy], as[xxxxxxyxyyy]}...

$\mathrm{CWS}[0,\ \frac{\widehat{xx}}{24},\ 0,\ -\frac{\widehat{xxxx}}{1440},\ -\frac{\widehat{xxxxx}}{1200},\ \frac{\widehat{xxxxxx}}{60\,480},\ \frac{\widehat{xxxxxxx}}{14\,112},$
$-\frac{\widehat{xxxxxxxx}}{2\,419\,200},\ -\frac{\widehat{xxxxxxxxx}}{172\,800},\ \frac{\widehat{xxxxxxxxxx}}{95\,800\,320},\ \frac{17\,\widehat{xxxxxxxxxxx}}{35\,126\,784},\ \ldots]$