Pensieve header: Calculations appearing in the WKO4 paper.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section 1 - Introduction

Initialization

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Initialization

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, ASeries, AW, b, BCH, BooleanSequence, BracketForm,
  BS, CC, Crop, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries,
  div, EulerE, Exp, InvertLieMorphism, j, J, JA, LieDerivation, LieMorphism,
  LieSeries, LS, LW, LyndonFactorization, New, RandomCWSeries, Randomizer,
  RandomLieSeries, RC, Support, tb, TopBracketForm, tr, Γ, ι, Λ, ⌐, ⌢}.
```

Initialization

```
AwCalculus` implements / extends
 {**, ⋃, ≡, dA, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hσ, tA, tha, tm, tS, tσ, Γ, Λ}.
```

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

```
x1 = LW[1]; x2 = LW[2];
{α, β, γ} = LS /@ {x1 + b[x1, x2], x2 - b[x1, b[x1, x2]], x1 + x2 - 2 b[x1, x2]}
```

alphabetagamma

$$\left\{\text{LS}\left[\overline{1},\ \overline{1\,2},\ 0,\ 0,\ \ldots\right],\ \text{LS}\left[\overline{2},\ 0,\ -\overline{1\,\overline{1\,2}},\ 0,\ \ldots\right],\ \text{LS}\left[\overline{1}+\overline{2},\ -2\,\overline{1\,\overline{2}},\ 0,\ 0,\ \ldots\right]\right\}$$

BracketExample

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

BracketExample

$$\left\{\text{LS}\left[0,\ \overline{1\,2},\ \overline{\overline{1\,2}\,2},\ -\overline{1\,\overline{1\,\overline{\overline{1\,2}}}},\ \ldots\right],\ \text{LS}[0,\ 0,\ 0,\ 0,\ \ldots]\right\}$$

bch

```
bch = BCH[LW@x, LW@y]
```

bch

$$\text{LS}\left[\overline{x}+\overline{y},\ \frac{\overline{x\,y}}{2},\ \frac{1}{12}\,\overline{x\,\overline{x\,y}}+\frac{1}{12}\,\overline{\overline{x\,y}\,y},\ \frac{1}{24}\,\overline{x\,\overline{\overline{x\,y}\,y}},\ \ldots\right]$$

bch16

```
Timing@{Length@(bch@16), (bch@16)[[1090 ;; 1092]] // TopBracketForm}
```

bch16

$$\left\{39.936256, \left\{2181, \frac{53\,x\,x\,x\overline{x\,y}\;\overline{x\,y}\;x\overline{x\,y}\;x\overline{x\,y}\;\overline{x\,y}\,y}{1\,089\,728\,640} - \frac{17\,x\,x\,x\overline{x\,y}\;\overline{x\,y}\;x\overline{x\,y}\;x\,x\;\overline{x\,y}\,y\,y}{179\,625\,600} + \frac{389\,x\,x\,x\overline{x\,y}\;\overline{x\,y}\;x\overline{x\,y}\;x\overline{x\,y}\;x\overline{x\,y}}{1\,320\,883\,200}\right\}\right\}$$

omegas

```
{ω1, ω2} = CWS /@ {CW["1"] - 3 CW["211"], CW["2"] + CW["22"]}
```

omegas

$$\{\mathrm{CWS}[\widehat{1}, 0, -3\,\widehat{112}, 0, \ldots], \mathrm{CWS}[\widehat{2}, \widehat{22}, 0, 0, \ldots]\}$$

DegreeScale

```
DegreeScale[h] /@ {ω1, ω2}
```

DegreeScale

$$\{\mathrm{CWS}[h\,\widehat{1}, 0, -3\,h^3\,\widehat{112}, 0, \ldots], \mathrm{CWS}[h\,\widehat{2}, h^2\,\widehat{22}, 0, 0, \ldots]\}$$

TangentialDerivative

```
{λ = ⟨1 → α, 2 → β⟩, γ // Dλ}
```

TangentialDerivative

$$\left\{\left\langle 1 \to \mathrm{LS}[\overline{1}, \overline{1\,2}, 0, 0, \ldots], 2 \to \mathrm{LS}\left[\overline{2}, 0, -\overline{1\,\overline{1\,2}}, 0, \ldots\right]\right\rangle, \mathrm{LS}\left[0, 0, \overline{1\,\overline{1\,\overline{1\,2}}}, -\overline{1\,\overline{1\,\overline{1\,2}}\,2}, \ldots\right]\right\}$$

tb

```
λ1 = λ; λ2 = ⟨1 → β, 2 → γ⟩; tb[λ1, λ2]
```

tb

$$\left\langle 1 \to \mathrm{LS}\left[0, 0, \overline{1\,\overline{1\,2}}, -\overline{1\,\overline{1\,\overline{2\,2}}}, \ldots\right], 2 \to \mathrm{LS}\left[0, 0, \overline{1\,\overline{1\,2}}, -\overline{1\,\overline{1\,\overline{2\,2}}}, \ldots\right]\right\rangle$$

tb2

```
lhs = D_tb[λ1,λ2][ω1]; rhs = b[Dλ1, Dλ2][ω1];
{lhs@{8}, (lhs ≡ rhs)@{8}}
```

tb2

$$\{\mathrm{CWS}[0, 0, 0, 0, 0, 0, 0, 18\,\widehat{11112122} - 18\,\widehat{11112212} - 36\,\widehat{11121122} + 36\,\widehat{11122112}, \ldots],\ \mathrm{BS}[9\ \mathrm{True}, \ldots]\}$$

TestingGammaODE

```
lhs = ∂t Γt[λ]; rhs = λ // e^(-t Dλ) // adSeries[ad/(e^ad - 1), Γt[λ]];
{Γ0[λ], lhs, (lhs ≡ rhs)@{6}}
```

TestingGammaODE

$$\left\{\langle 1 \to \text{LS}[0, 0, 0, 0, \ldots], 2 \to \text{LS}[0, 0, 0, 0, \ldots]\rangle,\right.$$
$$\left\langle 1 \to \text{LS}\left[\overline{1}, \overline{1\,2}, -t\,\overline{1\,\overline{1\,2}}, \frac{1}{4}t^2\,\overline{1\,\overline{1\,\overline{1\,2}}} - t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots\right],\right.$$
$$\left.\left. 2 \to \text{LS}\left[\overline{2}, 0, -\overline{1\,\overline{1\,2}}, -t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots\right]\right\rangle, \text{BS}[7\ \text{True}, \ldots]\right\}$$

TestingGamma

```
{γ // e^(-t Dλ), γ // CC[Γt[λ]]}
```

TestingGamma

$$\left\{\text{LS}\left[\overline{1} + \overline{2}, -2\,\overline{1\,2}, -t\,\overline{1\,\overline{1\,2}}, t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots\right], \text{LS}\left[\overline{1} + \overline{2}, -2\,\overline{1\,2}, -t\,\overline{1\,\overline{1\,2}}, t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots\right]\right\}$$

TestingLambdaODE

```
lhs = ∂t Λt[λ]; rhs = λ // e^(Dλt[λ]) // adSeries[ad/(e^ad - 1), Λt[λ], tb];
{Λ0[λ], lhs, (lhs ≡ rhs)@{6}}
```

TestingLambdaODE

$$\left\{\langle 1 \to \text{LS}[0, 0, 0, 0, \ldots], 2 \to \text{LS}[0, 0, 0, 0, \ldots]\rangle,\right.$$
$$\left\langle 1 \to \text{LS}\left[\overline{1}, \overline{1\,2}, t\,\overline{1\,\overline{1\,2}}, \frac{1}{2}t^2\,\overline{1\,\overline{1\,\overline{1\,2}}} + t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots\right], 2 \to \text{LS}\left[\overline{2}, 0, -\overline{1\,\overline{1\,2}}, t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots\right]\right\rangle,$$
$$\left.\text{BS}[7\ \text{True}, \ldots]\right\}$$

TestingLambda

```
{γ // CC[t λ], γ // e^(-Dλt[λ])}
```

TestingLambda

$$\left\{\text{LS}\left[\overline{1} + \overline{2}, -2\,\overline{1\,2}, -t\,\overline{1\,\overline{1\,2}}, -\frac{1}{2}t^2\,\overline{1\,\overline{1\,\overline{1\,2}}} + t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots\right],\right.$$
$$\left.\text{LS}\left[\overline{1} + \overline{2}, -2\,\overline{1\,2}, -t\,\overline{1\,\overline{1\,2}}, -\frac{1}{2}t^2\,\overline{1\,\overline{1\,\overline{1\,2}}} + t\,\overline{1\,\overline{\overline{1\,2}\,2}}, \ldots\right]\right\}$$

CCAndRC

```
{α // CC1[-γ], α // CC1[-γ] // RC1[γ], α // CC1[-γ] // CC1[γ]}
```

CCAndRC

$$\left\{\text{LS}\left[\overline{1}, 2\,\overline{1\,2}, -\frac{5}{2}\,\overline{1\,\overline{1\,2}} + \frac{3}{2}\,\overline{\overline{1\,2}\,2}, \frac{7}{6}\,\overline{1\,\overline{1\,\overline{1\,2}}} - \frac{23}{6}\,\overline{1\,\overline{\overline{1\,2}\,2}} + \frac{2}{3}\,\overline{\overline{1\,2}\,\overline{2\,2}}, \ldots\right],\right.$$
$$\left.\text{LS}\left[\overline{1}, \overline{1\,2}, 0, 0, \ldots\right], \text{LS}\left[\overline{1}, \overline{1\,2}, -\overline{1\,\overline{1\,2}}, 2\,\overline{1\,\overline{1\,\overline{1\,2}}} + \overline{1\,\overline{\overline{1\,2}\,2}}, \ldots\right]\right\}$$

divu

```
With[{γ = LW@u + b[b[LW@v, LW@u], LW@u]}, divu[γ]] // TopBracketForm
```

divu

$$\widehat{u} - \widehat{uuv}$$

Ju

```
J1[γ]
```

Ju

$$\text{CWS}\left[\widehat{1},\ \frac{5\,\widehat{12}}{2},\ -\frac{7\,\widehat{112}}{6}+\frac{7\,\widehat{122}}{6},\ \frac{3\,\widehat{1112}}{8}-\frac{11\,\widehat{1122}}{4}-\frac{3\,\widehat{1212}}{4}+\frac{3\,\widehat{1222}}{8},\ \ldots\right]$$

j

```
{div[λ]@{5}, j[λ]@{5}}
```

j

$$\left\{\text{CWS}\left[\widehat{1}+\widehat{2},\ -\widehat{12},\ -\widehat{112},\ 0,\ 0,\ \ldots\right],\right.$$
$$\left.\text{CWS}\left[\widehat{1}+\widehat{2},\ -\widehat{12},\ -\widehat{112},\ -\widehat{1122}+\widehat{1212},\ -\widehat{11122}+\widehat{11212},\ \ldots\right]\right\}$$

cocycle4j

```
lhs = j[BCH_tb[λ1, λ2]]; rhs = j[λ1] + e^Dλ1[j[λ2]];
{lhs, (lhs ≡ rhs)@{8}}
```

cocycle4j

$$\left\{\text{CWS}\left[\widehat{1}+2\,\widehat{2},\ -3\,\widehat{12},\ 0,\ -9\,\widehat{1122}+9\,\widehat{1212},\ \ldots\right],\ \text{BS}[9\ \text{True},\ \ldots]\right\}$$

```
lhs = j[BCH_b[λ1, λ2]]; rhs = j[λ1] + e^Dλ1[j[λ2]];
{lhs, (lhs ≡ rhs)}
```

$$\left\{\text{CWS}\left[\widehat{1}+2\,\widehat{2},\ -4\,\widehat{12},\ -\frac{5\,\widehat{122}}{12},\ \widehat{1112}-\frac{101\,\widehat{1122}}{6}+\frac{53\,\widehat{1212}}{3}-\frac{\widehat{1222}}{24},\ \ldots\right],\right.$$
$$\text{BS}\left[2\ \text{True},\ -4\,\text{CW}[12]==-3\,\text{CW}[12],\ -4\,\text{CW}[12]==-3\,\text{CW}[12]\ \&\&\ -\frac{5\,\text{CW}[122]}{12}==0,\right.$$
$$-4\,\text{CW}[12]==-3\,\text{CW}[12]\ \&\&\ -\frac{5\,\text{CW}[122]}{12}==0\ \&\&$$
$$\left.\left.\text{CW}[1112]-\frac{101\,\text{CW}[1122]}{6}+\frac{53\,\text{CW}[1212]}{3}-\frac{\text{CW}[1222]}{24}==-9\,\text{CW}[1122]+9\,\text{CW}[1212],\ \ldots\right]\right\}$$

dj

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

dj

$$\left\{\text{CWS}\left[\epsilon\,\widehat{1}+\epsilon\,\widehat{2},\ -\epsilon\,\widehat{12},\ -\epsilon\,\widehat{112},\ 0,\ \ldots\right],\ \text{BS}[5\ \text{True},\ \ldots]\right\}$$

Section 2.3 - The [AT]-inspired presentation E_l of A^w_{exp}

ElSetup

```
x1 = LW[1]; x2 = LW[2];
{ξa = El[⟨1 → LS[x1 + b[x1, x2]], 2 → LS[x2 - b[x1, b[x1, x2]]]⟩,
   CWS[CW["1"] - 3 CW["121"]]],
 ξb = El[⟨1 → LS[x2 - b[x1, x2]], 2 → LS[x1 + x2 + b[x2, b[x1, x2]]]⟩,
   CWS[CW["2"] - 2 CW["12"]]],
 ξc = El[⟨1 → LS[x1 - b[b[x1, x2], b[x1, x2]]], 2 → LS[x2 + 3 b[x1, b[x1, x2]]]⟩,
   CWS[CW["1"] - 2 CW["12"] + CW["121"]]]}
```

ElSetup

```
{El[⟨1 → LS[1, 12, 0, 0, ...], 2 → LS[2, 0, -1112, 0, ...]⟩,
  CWS[1, 0, -3 112, 0, ...]],
 El[⟨1 → LS[2, -12, 0, 0, ...], 2 → LS[1 + 2, 0, -1222, 0, ...]⟩,
  CWS[2, -2 12, 0, 0, ...]], El[
  ⟨1 → LS[1, 0, 0, 0, ...], 2 → LS[2, 0, 3 1112, 0, ...]⟩, CWS[1, -2 12, 112, 0, ...]]}
```

ElAssociativity

```
lhs = ξa ** (ξb ** ξc); rhs = (ξa ** ξb) ** ξc;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

ElAssociativity

```
{El[⟨1 → LS[2 1 + 2, 0, 1/2 1112, ...], 2 → LS[1 + 3 2, 0, 5/2 1112 - 1222, ...]⟩,
  CWS[2 1 + 2, -4 12, -2 112, ...]], BS[9 True, ...]}
```

detaExample

```
{ξa // dη[1], ξa // dη[2]}
```

detaExample

```
{El[⟨2 → LS[2, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
 El[⟨1 → LS[1, 0, 0, 0, ...]⟩, CWS[1, 0, 0, 0, ...]]}
```

dA1

```
{ξd = El[λ, CWS[0]], ξd // dA}
```

dA1

```
{El[⟨1 → LS[1, 12, 0, 0, ...], 2 → LS[2, 0, -1112, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
 El[⟨1 → LS[-1, -12, 0, 0, ...], 2 → LS[-2, 0, 1112, 0, ...]⟩,
  CWS[-1 - 2, 12, 112, 1122 - 1212, ...]]}
```

dA2

```
(ξd ≡ (ξd // dA // dA))@{8}
```

dA2

```
BS[9 True, ...]
```

dA3

```
lhs = (ξa ** ξb) // dA; rhs = (ξb // dA) ** (ξa // dA);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dA3

$$\left\{\mathrm{El}\left[\left\langle 1\to \mathrm{LS}\left[-\overline{1}-\overline{2},\ 0,\ -\frac{1}{2}\,\overline{1\,\overline{1\,2}},\ \ldots\right],\ 2\to \mathrm{LS}\left[-\overline{1}-2\,\overline{2},\ 0,\ \frac{1}{2}\,\overline{1\,\overline{1\,2}}+\overline{\overline{1\,2}\,2},\ \ldots\right]\right\rangle,\right.\right.$$
$$\left.\left.\mathrm{CWS}\left[-\widehat{2},\ -2\,\widehat{12},\ -2\,\widehat{112}-\widehat{122},\ \ldots\right]\right],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\right\}$$

dS

```
ξd // dS
```

dS

$$\mathrm{El}\left[\left\langle 1\to \mathrm{LS}\left[\overline{1},\ -\overline{1\,2},\ 0,\ 0,\ \ldots\right],\ 2\to \mathrm{LS}\left[\overline{2},\ 0,\ -\overline{1\,\overline{1\,2}},\ 0,\ \ldots\right]\right\rangle,\right.$$
$$\left.\mathrm{CWS}\left[\widehat{1}+\widehat{2},\ \widehat{12},\ -\widehat{112},\ \widehat{1122}-\widehat{1212},\ \ldots\right]\right]$$

dD1

```
{ξa, ξa // dΔ[2, 2, 3]}
```

dD1

$$\left\{\mathrm{El}\left[\left\langle 1\to \mathrm{LS}\left[\overline{1},\ \overline{1\,2},\ 0,\ 0,\ \ldots\right],\ 2\to \mathrm{LS}\left[\overline{2},\ 0,\ -\overline{1\,\overline{1\,2}},\ 0,\ \ldots\right]\right\rangle,\right.\right.$$
$$\left.\mathrm{CWS}\left[\widehat{1},\ 0,\ -3\,\widehat{112},\ 0,\ \ldots\right]\right],$$
$$\mathrm{El}\left[\left\langle 1\to \mathrm{LS}\left[\overline{1},\ \overline{1\,2}+\overline{1\,3},\ 0,\ 0,\ \ldots\right],\ 2\to \mathrm{LS}\left[\overline{2}+\overline{3},\ 0,\ -\overline{1\,\overline{1\,2}}-\overline{1\,\overline{1\,3}},\ 0,\ \ldots\right],\right.\right.$$
$$\left.\left.3\to \mathrm{LS}\left[\overline{2}+\overline{3},\ 0,\ -\overline{1\,\overline{1\,2}}-\overline{1\,\overline{1\,3}},\ 0,\ \ldots\right]\right\rangle,\ \mathrm{CWS}\left[\widehat{1},\ 0,\ -3\,\widehat{112}-3\,\widehat{113},\ 0,\ \ldots\right]\right]\right\}$$

dD2

```
lhs = (ξa ** ξb) // dΔ[2, 2, 3]; rhs = (ξa // dΔ[2, 2, 3]) ** (ξb // dΔ[2, 2, 3]);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dD2

$$\left\{\mathrm{El}\left[\left\langle 1\to \mathrm{LS}\left[\overline{1}+\overline{2}+\overline{3},\ 0,\ \frac{1}{2}\,\overline{1\,\overline{1\,2}}+\frac{1}{2}\,\overline{1\,\overline{1\,3}},\ \ldots\right],\right.\right.$$
$$2\to \mathrm{LS}\left[\overline{1}+2\,\overline{2}+2\,\overline{3},\ 0,\ -\frac{1}{2}\,\overline{1\,\overline{1\,2}}-\frac{1}{2}\,\overline{1\,\overline{1\,3}}-\overline{1\,\overline{2\,3}}-\overline{\overline{1\,2}\,2}-2\,\overline{\overline{1\,3}\,2}-\overline{\overline{1\,3}\,3},\ \ldots\right],$$
$$3\to \mathrm{LS}\left[\overline{1}+2\,\overline{2}+2\,\overline{3},\ 0,\ -\frac{1}{2}\,\overline{1\,\overline{1\,2}}-\frac{1}{2}\,\overline{1\,\overline{1\,3}}-\overline{1\,\overline{2\,3}}-\overline{\overline{1\,2}\,2}-2\,\overline{\overline{1\,3}\,2}-\overline{\overline{1\,3}\,3},\ \ldots\right]\Big\rangle,$$
$$\left.\mathrm{CWS}\left[\widehat{1}+\widehat{2}+\widehat{3},\ -2\,\widehat{12}-2\,\widehat{13},\ -3\,\widehat{112}-3\,\widehat{113},\ \ldots\right]\right],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\Big\}$$

Section 2.4 - The factored presentation E_f of A^w_{exp} and its stronger precursor E_s

EsSetup1

```
u = LW@u; v = LW@v;
ξa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
  CWS[CW["u"] - 3 CW["uvu"]]]
```

EsSetup1

$$\text{Es}\left[\left\langle 1 \to \text{LS}[\overline{u}, \overline{uv}, 0, 0, \ldots], 2 \to \text{LS}\left[\overline{v}, 0, -\overline{u\overline{uv}}, 0, \ldots\right], 3 \to \text{LS}[\overline{u}, 0, 0, 0, \ldots]\right\rangle, \text{CWS}[\widehat{u}, 0, -3\,\widehat{uuv}, 0, \ldots]\right]$$

EsSetup2

```
SeedRandom[0]; ξb = Es[
  ⟨Table[i → RandomLieSeries[{1, 2, 3, 4}], {i, 4}]⟩, RandomCWSeries[{1, 2, 3, 4}]];
ξb@
 {2}
```

EsSetup2

$$\text{Es}\left[\left\langle 1 \to \text{LS}\left[-\overline{1} - 2\,\overline{2} + 2\,\overline{3} - 2\,\overline{4}, 2\,\overline{12} + \frac{\overline{13}}{2} + \overline{14} - \frac{\overline{23}}{2} - \frac{\overline{24}}{2} + 2\,\overline{34}, \ldots\right],\right.$$

$$2 \to \text{LS}\left[2\,\overline{1} - \overline{2} - 2\,\overline{3} + \overline{4}, 2\,\overline{12} + \frac{3\,\overline{13}}{2} - 2\,\overline{14} - \overline{23} - \overline{24} - \frac{\overline{34}}{2}, \ldots\right],$$

$$3 \to \text{LS}\left[-\overline{1} + \overline{2} + 2\,\overline{4}, -2\,\overline{12} + 2\,\overline{13} - \overline{14} - \frac{3\,\overline{23}}{2} + 2\,\overline{24} - 2\,\overline{34}, \ldots\right],$$

$$\left. 4 \to \text{LS}\left[-2\,\overline{1} + 2\,\overline{2} + 2\,\overline{3} + \overline{4}, -\frac{\overline{12}}{2} + \frac{3\,\overline{13}}{2} - 2\,\overline{24} + \overline{34}, \ldots\right]\right\rangle,$$

$$\text{CWS}\left[\widehat{3} - \widehat{4}, \frac{3\,\widehat{11}}{2} + \frac{3\,\widehat{12}}{2} - 2\,\widehat{13} + \widehat{14} + \widehat{22} + 2\,\widehat{23} - \frac{\widehat{24}}{2} - 2\,\widehat{33} - \widehat{34} + \widehat{44}, \ldots\right]\right]$$

haction

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4];
rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

haction

$$\left\{\text{Es}\left[\left\langle 3 \to \text{LS}\left[\overline{u}, -\overline{uv}, -\overline{u\overline{uv}} + \frac{1}{2}\,\overline{\overline{uv}v}, \frac{3}{2}\,\overline{u\,\overline{u\overline{uv}}} + \overline{u\,\overline{\overline{uv}v}} - \frac{1}{6}\,\overline{\overline{\overline{uv}v}\,v}, \ldots\right],\right.\right.$$

$$\left. 4 \to \text{LS}\left[\overline{u} + \overline{v}, \frac{\overline{uv}}{2}, -\frac{23}{12}\,\overline{u\overline{uv}} - \frac{5}{12}\,\overline{\overline{uv}v}, \overline{u\,\overline{u\overline{uv}}} + \frac{13}{24}\,\overline{u\,\overline{\overline{uv}v}} + \frac{1}{12}\,\overline{\overline{\overline{uv}v}\,v}, \ldots\right]\right\rangle,$$

$$\left.\text{CWS}\left[2\,\widehat{u}, -\widehat{uv}, -\frac{3\,\widehat{uuv}}{2}, -\frac{\widehat{uuuv}}{6} + \widehat{uuvv} - \widehat{uvuv}, \ldots\right]\right], \text{BS}[9\,\text{True}, \ldots]\right\}$$

metaassoc

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

metaassoc

$$\left\{\text{Es}\left[\left\langle 1 \to \text{LS}\left[-2\,\overline{1}+\overline{4},\ -\frac{3\,\overline{1\,4}}{2},\ 20\,\overline{1\,\overline{1\,4}}-\frac{19}{3}\,\overline{\overline{1\,4}\,4},\ \ldots\right],\right.\right.$$
$$\left.\left. 4 \to \text{LS}\left[2\,\overline{1}+\overline{4},\ \overline{1\,4},\ -\frac{31}{2}\,\overline{1\,\overline{1\,4}}-\frac{13}{6}\,\overline{\overline{1\,4}\,4},\ \ldots\right]\right\rangle,\right.$$
$$\left.\text{CWS}\left[3\,\widehat{1}-\widehat{4},\ -3\,\widehat{11}+\frac{\widehat{14}}{2}+\widehat{44},\ \frac{71\,\widehat{111}}{4}+\frac{19\,\widehat{114}}{4}-\frac{7\,\widehat{144}}{6}-\frac{2\,\widehat{444}}{3},\ \ldots\right]\right],\ \text{BS}[6\,\text{True},\ \ldots]\Big\}$$

Section 3.1 - Tangle Invariants

```
Zt:(l|s)[K1_ ** K2_] := Zt[K1] ** Zt[K2]

R+[a_, b_] // Z1 := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R-[a_, b_] // Z1 := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]]

lhs = R+[1, 2] ** R+[1, 3] ** R+[2, 3] // Z1; rhs = R+[2, 3] ** R+[1, 3] ** R+[1, 2] // Z1;
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

$$\left\{\text{El}\left[\left\langle 1 \to \text{LS}[0,\ 0,\ 0,\ \ldots],\ 2 \to \text{LS}\left[\overline{1},\ 0,\ 0,\ \ldots\right],\ 3 \to \text{LS}\left[\overline{1}+\overline{2},\ 0,\ 0,\ \ldots\right]\right\rangle,\right.\right.$$
$$\left.\left.\text{CWS}[0,\ 0,\ 0,\ \ldots]\right],\ \text{BS}[6\,\text{True},\ \ldots]\right\}$$