

Pensieve header: Aw calculations for the WKO4 paper.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section I - Introduction

Initialization

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

```
x1 = LW[1]; x2 = LW[2];
{alpha, beta, gamma} = LS /@ {x1 + b[x1, x2], x2 - b[x1, b[x1, x2]], x1 + x2 - 2 b[x1, x2]}
```

alphabetagamma

```
{LS[1, 12, 0, ___], LS[2, 0, -1 112, ___], LS[1 + 2, -2 12, 0, ___]}
```

BracketExample

```
{b[alpha, beta], b[alpha, b[beta, gamma]] + b[beta, b[gamma, alpha]] + b[gamma, b[alpha, beta]}}
```

BracketExample

```
{LS[0, 12, 122, -1 112, ___], LS[0, 0, 0, 0, ___]}
```

bch

```
bch = BCH[LW@x, LW@y]
```

bch

```
LS[x + y, x y / 2, 1 / 12 x x y + 1 / 12 x y y, 1 / 24 x x y y, ___]
```

bch16

```
Timing@{Length@(bch@16), (bch@16)[[1090 ;; 1092]] // TopBracketForm}
```

bch16

```
{39.140651, {2181,  $\frac{53 \times x \times x \times y \times y \times x \times y \times x \times y \times x \times y \times y}{1089728640}$  -  $\frac{17 \times x \times x \times y \times y \times x \times y \times y}{179625600}$  +  $\frac{389 \times x \times x \times y \times y \times x \times y \times x \times y \times x \times y}{1320883200}$ }}
```

omegas

$$\{\omega_1, \omega_2\} = \text{CWS} / @ \{ \text{CW}["1"] - 3 \text{CW}["121"], \text{CW}["2"] + \text{CW}["22"] \}$$

omegas

$$\{ \text{CWS}[\widehat{1}, 0, -3 \widehat{121}, 0, ___], \text{CWS}[\widehat{2}, \widehat{22}, 0, 0, ___] \}$$

DegreeScale

$$\text{DegreeScale}[\mathbf{h}] / @ \{ \omega_1, \omega_2 \}$$

DegreeScale

$$\{ \text{CWS}[h \widehat{1}, 0, -3 h^3 \widehat{121}, 0, ___], \text{CWS}[h \widehat{2}, h^2 \widehat{22}, 0, 0, ___] \}$$

TangentialDerivative

$$\{\lambda = \langle 1 \rightarrow \alpha, 2 \rightarrow \beta \rangle, \gamma // D_\lambda\}$$

TangentialDerivative

$$\left\{ \left\langle 1 \rightarrow \text{LS}[\widehat{1}, \widehat{12}, 0, 0, ___], 2 \rightarrow \text{LS}[\widehat{2}, 0, -\widehat{112}, 0, ___] \right\rangle, \text{LS}[0, 0, \widehat{112}, -\widehat{1122}, ___] \right\}$$

tb

$$\lambda_1 = \lambda; \lambda_2 = \langle 1 \rightarrow \beta, 2 \rightarrow \gamma \rangle; \text{tb}[\lambda_1, \lambda_2]$$

tb

$$\left\langle 1 \rightarrow \text{LS}[0, 0, \widehat{112}, -\widehat{1122}, ___], 2 \rightarrow \text{LS}[0, 0, \widehat{112}, -\widehat{1122}, ___] \right\rangle$$

$$\{ \mathbf{D}_{\text{tb}[\lambda_1, \lambda_2]}[\omega_1] @ \{8\}, \mathbf{b}[\mathbf{D}_{\lambda_1}, \mathbf{D}_{\lambda_2}][\omega_1] @ \{8\} \}$$

$$\{ \text{CWS}[0, 0, 0, 0, -12 \widehat{11122} + 12 \widehat{11212}, -9 \widehat{112122} + 9 \widehat{112212}, \\ 12 \widehat{1111122} - 36 \widehat{1111212} - 6 \widehat{1111222} + 24 \widehat{1112112} + 6 \widehat{1112122} + 6 \widehat{1112212} + 6 \widehat{1121122} - \\ 12 \widehat{1121212}, 6 \widehat{11112122} - 6 \widehat{11112212} - 12 \widehat{11121122} + 12 \widehat{11122112}, ___], \\ \text{CWS}[0, 0, 0, 0, 0, 0, 0, 0, 18 \widehat{11112122} - 18 \widehat{11112212} - 36 \widehat{11121122} + 36 \widehat{11122112}, ___] \}$$

tb2

$$\{ \mathbf{D}_{\text{tb}[\lambda_1, \lambda_2]}[\omega_1] @ \{8\}, \mathbf{b}[\mathbf{D}_{\lambda_1}, \mathbf{D}_{\lambda_2}][\omega_1] @ \{8\} \}$$

tb2

$$\{ \text{CWS}[0, 0, 0, 0, 0, 0, 0, 0, 18 \widehat{11112122} - 18 \widehat{11112212} - 36 \widehat{11121122} + 36 \widehat{11122112}, ___], \\ \text{CWS}[0, 0, 0, 0, 0, 0, 0, 0, 18 \widehat{11112122} - 18 \widehat{11112212} - 36 \widehat{11121122} + 36 \widehat{11122112}, ___] \}$$

TestingGammaODE

$$\{ \Gamma_0[\lambda], \partial_t \Gamma_t[\lambda], \lambda // e^{-t D_\lambda} // \text{adSeries} \left[\frac{\text{ad}}{e^{\text{ad}} - 1}, \Gamma_t[\lambda] \right] \}$$

TestingGammaODE

$$\left\{ \left\langle 1 \rightarrow \text{LS}[0, 0, 0, 0, ___], 2 \rightarrow \text{LS}[0, 0, 0, 0, ___] \right\rangle, \right. \\ \left. \left\langle 1 \rightarrow \text{LS}[\widehat{1}, \widehat{12}, -t \widehat{112}, \frac{1}{4} t^2 \widehat{1112} - t \widehat{1122}, ___], \right. \right. \\ \left. \left. 2 \rightarrow \text{LS}[\widehat{2}, 0, -\widehat{112}, -t \widehat{1122}, ___] \right\rangle, \right. \\ \left. \left\langle 1 \rightarrow \text{LS}[\widehat{1}, \widehat{12}, -t \widehat{112}, \frac{1}{4} t^2 \widehat{1112} - t \widehat{1122}, ___], \right. \right. \\ \left. \left. 2 \rightarrow \text{LS}[\widehat{2}, 0, -\widehat{112}, -t \widehat{1122}, ___] \right\rangle \right\}$$

TestingGamma

$$\{\gamma // e^{-tD\lambda}, \gamma // CC[\Gamma_t[\lambda]]\}$$

TestingGamma

$$\{LS[\overline{1+2}, -2\overline{12}, -t\overline{112}, t\overline{1122}, \dots], LS[\overline{1+2}, -2\overline{12}, -t\overline{112}, t\overline{1122}, \dots]\}$$

TestingLambdaODE

$$\{\Lambda_0[\lambda], \partial_t \Lambda_t[\lambda], \lambda // e^{D\Lambda_t[\lambda]} // adSeries\left[\frac{ad}{e^{ad}-1}, \Lambda_t[\lambda], tb\right]\}$$

TestingLambdaODE

$$\begin{aligned} &\langle 1 \rightarrow LS[0, 0, 0, 0, \dots], 2 \rightarrow LS[0, 0, 0, 0, \dots] \rangle, \\ &\left\langle 1 \rightarrow LS[\overline{1}, \overline{12}, t\overline{112}, \frac{1}{2}t^2\overline{1112} + t\overline{1122}, \dots], 2 \rightarrow LS[\overline{2}, 0, -\overline{112}, t\overline{1122}, \dots] \right\rangle, \\ &\left\langle 1 \rightarrow LS[\overline{1}, \overline{12}, t\overline{112}, \frac{1}{2}t^2\overline{1112} + t\overline{1122}, \dots], 2 \rightarrow LS[\overline{2}, 0, -\overline{112}, t\overline{1122}, \dots] \right\rangle \end{aligned}$$

TestingLambda

$$\{\gamma // CC[t\lambda], \gamma // e^{-D\Lambda_t[\lambda]}\}$$

TestingLambda

$$\begin{aligned} &LS[\overline{1+2}, -2\overline{12}, -t\overline{112}, -\frac{1}{2}t^2\overline{1112} + t\overline{1122}, \dots], \\ &LS[\overline{1+2}, -2\overline{12}, -t\overline{112}, -\frac{1}{2}t^2\overline{1112} + t\overline{1122}, \dots] \end{aligned}$$

CCAndRC

$$\{\alpha // CC_1[-\gamma], \alpha // CC_1[-\gamma] // RC_1[\gamma], \alpha // CC_1[-\gamma] // CC_1[\gamma]\}$$

CCAndRC

$$\begin{aligned} &LS[\overline{1}, 2\overline{12}, -\frac{5}{2}\overline{112} + \frac{3}{2}\overline{122}, \frac{7}{6}\overline{1112} - \frac{23}{6}\overline{1122} + \frac{2}{3}\overline{1222}, \dots], \\ &LS[\overline{1}, \overline{12}, 0, 0, \dots], LS[\overline{1}, \overline{12}, -\overline{112}, 2\overline{1112} + \overline{1122}, \dots] \end{aligned}$$

divu

$$\text{With}[\{\gamma = b[b[LW@v, LW@u], LW@u]\}, \text{div}_u[\gamma]] // \text{TopBracketForm}$$

divu

$$-\widehat{uv}$$

Ju

$$\text{With}[\{u = 1\}, \{J_u[\gamma], \int_0^1 (\gamma // RC_u[s\gamma] // \text{div}_u // CC_u[-s\gamma]) ds\}]$$

Ju

$$\begin{aligned} &CWS[\widehat{1}, \frac{5\widehat{12}}{2}, -\frac{7\widehat{112}}{6} + \frac{7\widehat{122}}{6}, \frac{3\widehat{1112}}{8} - \frac{11\widehat{1122}}{4} - \frac{3\widehat{1212}}{4} + \frac{3\widehat{1222}}{8}, \dots], \\ &CWS[\widehat{1}, \frac{5\widehat{12}}{2}, -\frac{7\widehat{112}}{6} + \frac{7\widehat{122}}{6}, \frac{3\widehat{1112}}{8} - \frac{11\widehat{1122}}{4} - \frac{3\widehat{1212}}{4} + \frac{3\widehat{1222}}{8}, \dots] \end{aligned}$$

```
j
{div[λ]@{5}, j[λ]@{5}}
j
{CWS[1̄ + 2̄, -1̄2̄, -1̄12̄, 0, 0, ___],
 CWS[1̄ + 2̄, -1̄2̄, -1̄12̄, -1̄122̄ + 1̄212̄, -1̄1122̄ + 1̄1212̄, ___]}
```

Section 2.3 - The [AT]-inspired presentation E_I of A^W_{exp}

EISetup

```
x1 = LW[1]; x2 = LW[2];
{ξa = EI[⟨1 → LS[x1 + b[x1, x2]], 2 → LS[x2 - b[x1, b[x1, x2]]],
 CWS[CW["1"] - 3 CW["121"]]],
 ξb = EI[⟨1 → LS[x2 - b[x1, x2]], 2 → LS[x1 + x2 + b[x2, b[x1, x2]]],
 CWS[CW["2"] - 2 CW["12"]]],
 ξc = EI[⟨1 → LS[x1 - b[b[x1, x2], b[x1, x2]], 2 → LS[x2 + 3 b[x1, b[x1, x2]]],
 CWS[CW["1"] - 2 CW["12"] + CW["121"]]]}
```

EISetup

```
{EI[⟨1 → LS[1̄, 1̄2̄, 0, ___], 2 → LS[2̄, 0, -1̄1̄2̄, ___]⟩, CWS[1̄, 0, -3 1̄21̄, ___]],
 EI[⟨1 → LS[2̄, -1̄2̄, 0, ___], 2 → LS[1̄ + 2̄, 0, -1̄2̄2̄, ___]⟩, CWS[2̄, -2 1̄2̄, 0, ___]],
 EI[⟨1 → LS[1̄, 0, 0, ___], 2 → LS[2̄, 0, 3 1̄1̄2̄, ___]⟩, CWS[1̄, -2 1̄2̄, 1̄21̄, ___]]}
```

```
ξa = EI[
 ⟨1 → RandomLieSeries[{1, 2}], 2 → RandomLieSeries[{1, 2}]⟩,
 RandomCWSeries[{1, 2}]
]
```

```
EI[⟨1 → LS[-2 1̄, -2 1̄2̄, 11 1̄1̄2̄ - 2 1̄2̄2̄, 43 1̄1̄1̄2̄ - 1 1̄1̄2̄2̄ + 11 1̄2̄2̄2̄, ___],
 2 → LS[2 1̄, -1̄2̄/2, 4 1̄1̄2̄/3 - 5 1̄2̄2̄/3, 1 1̄1̄2̄ - 25 1̄1̄2̄2̄/24 - 13 1̄2̄2̄2̄/24, ___]⟩,
 CWS[-2 2̄, 11 - 1̄2̄/2, -11 1̄1̄1̄/6 + 7 1̄1̄2̄/6 - 1̄2̄2̄ - 5 2̄2̄2̄/3,
 -5 1̄1̄1̄1̄/12 + 2 1̄1̄1̄2̄/3 + 41 1̄1̄2̄2̄/24 - 43 1̄2̄1̄2̄/24 + 37 1̄2̄2̄2̄/24 - 7 2̄2̄2̄2̄/4, ___]]]
```

```

ξb = E1 [
  <3 → RandomLieSeries [{3, 4}], 4 → RandomLieSeries [{3, 4}],
  RandomCWSeries [{3, 4}]
]

```

$$\begin{aligned}
 & \text{E1} \left[\left(3 \rightarrow \text{LS} \left[\overline{3+4}, -2 \overline{34}, \overline{334} - \overline{344}, \frac{23}{24} \overline{3334} - \frac{11}{6} \overline{3344} + \frac{19}{24} \overline{3444}, \dots \right], \right. \\
 & \quad \left. 4 \rightarrow \text{LS} \left[-\overline{3}, \overline{34}, -\frac{1}{2} \overline{334} - \overline{344}, \frac{11}{8} \overline{3334} + \frac{4}{3} \overline{3344} - \frac{9}{8} \overline{3444}, \dots \right] \right), \\
 & \text{CWS} \left[\widehat{4}, \frac{3 \overline{33}}{2} - \frac{3 \overline{34}}{2} - 2 \widehat{44}, -\frac{11 \overline{333}}{6} + \frac{\overline{334}}{6} + \widehat{344} + \widehat{444}, \right. \\
 & \quad \left. -\frac{13 \overline{3333}}{12} - \frac{5 \overline{3334}}{3} + \frac{\overline{3344}}{8} + \frac{5 \overline{3434}}{24} - \frac{5 \overline{3444}}{6} - \frac{15 \overline{4444}}{8}, \dots \right]
 \end{aligned}$$

```

ξc = E1 [
  <1 → RandomLieSeries [{1, 2}], 2 → RandomLieSeries [{1, 2}],
  RandomCWSeries [{1, 2}]
]

```

$$\begin{aligned}
 & \text{E1} \left[\left(1 \rightarrow \text{LS} \left[\overline{1+2}, -\frac{3 \overline{12}}{2}, \frac{2}{3} \overline{112} - \frac{3}{2} \overline{122}, \frac{37}{24} \overline{1112} - \frac{13}{8} \overline{1122} - \frac{13}{24} \overline{1222}, \dots \right], \right. \\
 & \quad \left. 2 \rightarrow \text{LS} \left[2 \overline{1} + 2 \overline{2}, -\frac{3 \overline{12}}{2}, -\frac{1}{3} \overline{112} + \frac{5}{3} \overline{122}, -\frac{13}{8} \overline{1112} - \frac{5}{12} \overline{1122} - \frac{5}{6} \overline{1222}, \dots \right] \right), \\
 & \text{CWS} \left[2 \widehat{1}, 2 \widehat{11} + \widehat{12} - \frac{3 \overline{22}}{2}, \frac{\overline{111}}{6} + \frac{5 \overline{112}}{3} + \frac{\overline{122}}{3} - \frac{5 \overline{222}}{6}, \right. \\
 & \quad \left. -\frac{47 \overline{1111}}{24} + 2 \overline{1112} + \frac{\overline{1122}}{3} - \frac{3 \overline{1212}}{8} - \frac{25 \overline{1222}}{24} - \frac{7 \overline{2222}}{4}, \dots \right]
 \end{aligned}$$

Testing the Union Operation

$\xi_a \cup \xi_b$

$$\begin{aligned}
 \text{E1} & \left[\left(1 \rightarrow \text{LS} \left[-2 \overline{1}, -2 \overline{12}, \frac{11}{6} \overline{112} - 2 \overline{122}, \frac{43}{24} \overline{1112} - \overline{1122} + \frac{11}{8} \overline{1222}, \dots \right], \right. \\
 & 2 \rightarrow \text{LS} \left[2 \overline{1}, -\frac{\overline{12}}{2}, \frac{4}{3} \overline{112} - \frac{5}{3} \overline{122}, \overline{1112} - \frac{25}{24} \overline{1122} - \frac{13}{24} \overline{1222}, \dots \right], \\
 & 3 \rightarrow \text{LS} \left[\overline{3} + \overline{4}, -2 \overline{34}, \overline{334} - \overline{344}, \frac{23}{24} \overline{3334} - \frac{11}{6} \overline{3344} + \frac{19}{24} \overline{3444}, \dots \right], \\
 & 4 \rightarrow \text{LS} \left[-\overline{3}, \overline{34}, -\frac{1}{2} \overline{334} - \overline{344}, \frac{11}{8} \overline{3334} + \frac{4}{3} \overline{3344} - \frac{9}{8} \overline{3444}, \dots \right] \left. \right), \\
 \text{CWS} & \left[-2 \widehat{2} + \widehat{4}, \widehat{11} - \frac{\widehat{12}}{2} + \frac{3 \widehat{33}}{2} - \frac{3 \widehat{34}}{2} - 2 \widehat{44}, -\frac{11 \widehat{111}}{6} + \frac{7 \widehat{112}}{6} - \widehat{122} - \frac{5 \widehat{222}}{3} - \right. \\
 & \frac{11 \widehat{333}}{6} + \frac{\widehat{334}}{6} + \widehat{344} + \widehat{444}, -\frac{5 \widehat{1111}}{12} + \frac{2 \widehat{1112}}{3} + \frac{41 \widehat{1122}}{24} - \frac{43 \widehat{1212}}{24} + \frac{37 \widehat{1222}}{24} - \\
 & \left. \frac{7 \widehat{2222}}{4} - \frac{13 \widehat{3333}}{12} - \frac{5 \widehat{3334}}{3} + \frac{\widehat{3344}}{8} + \frac{5 \widehat{3434}}{24} - \frac{5 \widehat{3444}}{6} - \frac{15 \widehat{4444}}{8}, \dots \right]
 \end{aligned}$$

$\text{Es}[\xi_a] \cup \text{Es}[\xi_b]$

$$\begin{aligned}
 \text{Es} & \left[\left(1 \rightarrow \text{LS} \left[-2 \overline{1}, -2 \overline{12}, -\frac{13}{6} \overline{112} - 2 \overline{122}, \frac{35}{24} \overline{1112} - \frac{13}{2} \overline{1122} + \frac{11}{8} \overline{1222}, \dots \right], \right. \\
 & 2 \rightarrow \text{LS} \left[2 \overline{1}, -\frac{\overline{12}}{2}, \frac{17}{6} \overline{112} - \frac{5}{3} \overline{122}, \overline{1112} - \frac{13}{12} \overline{1122} - \frac{13}{24} \overline{1222}, \dots \right], \\
 & 3 \rightarrow \text{LS} \left[\overline{3} + \overline{4}, -3 \overline{34}, \frac{7}{2} \overline{334} + \frac{2}{3} \overline{344}, -\frac{29}{24} \overline{3334} - \frac{41}{6} \overline{3344} + \frac{13}{24} \overline{3444}, \dots \right], \\
 & 4 \rightarrow \text{LS} \left[-\overline{3}, \frac{3 \overline{34}}{2}, -\frac{25}{12} \overline{334} - \frac{5}{3} \overline{344}, \frac{19}{8} \overline{3334} + \frac{101}{24} \overline{3344} - \frac{5}{12} \overline{3444}, \dots \right] \left. \right), \\
 \text{CWS} & \left[-2 \widehat{2} + \widehat{4}, \widehat{11} - \frac{\widehat{12}}{2} + \frac{3 \widehat{33}}{2} - \frac{3 \widehat{34}}{2} - 2 \widehat{44}, \right. \\
 & -\frac{11 \widehat{111}}{6} + \frac{7 \widehat{112}}{6} - \widehat{122} - \frac{5 \widehat{222}}{3} - \frac{11 \widehat{333}}{6} + \frac{\widehat{334}}{6} + \widehat{344} + \widehat{444}, \\
 & -\frac{5 \widehat{1111}}{12} + \frac{2 \widehat{1112}}{3} + \frac{41 \widehat{1122}}{24} - \frac{43 \widehat{1212}}{24} + \frac{37 \widehat{1222}}{24} - \frac{7 \widehat{2222}}{4} - \\
 & \left. \frac{13 \widehat{3333}}{12} - \frac{5 \widehat{3334}}{3} + \frac{\widehat{3344}}{8} + \frac{5 \widehat{3434}}{24} - \frac{5 \widehat{3444}}{6} - \frac{15 \widehat{4444}}{8}, \dots \right]
 \end{aligned}$$

Es [$\xi_a \cup \xi_b$]

$$\begin{aligned}
 & \text{E1} \left[\left(1 \rightarrow \text{LS} \left[-2 \overline{1}, -2 \overline{12}, -\frac{13}{6} \overline{112} - 2 \overline{122}, \frac{35}{24} \overline{1112} - \frac{13}{2} \overline{1122} + \frac{11}{8} \overline{1222}, \dots \right], \right. \\
 & \quad 2 \rightarrow \text{LS} \left[2 \overline{1}, -\frac{\overline{12}}{2}, \frac{17}{6} \overline{112} - \frac{5}{3} \overline{122}, \overline{1112} - \frac{13}{12} \overline{1122} - \frac{13}{24} \overline{1222}, \dots \right], \\
 & \quad 3 \rightarrow \text{LS} \left[\overline{3} + \overline{4}, -3 \overline{34}, \frac{7}{2} \overline{334} + \frac{2}{3} \overline{344}, -\frac{29}{24} \overline{3334} - \frac{41}{6} \overline{3344} + \frac{13}{24} \overline{3444}, \dots \right], \\
 & \quad \left. 4 \rightarrow \text{LS} \left[-\overline{3}, \frac{3 \overline{34}}{2}, -\frac{25}{12} \overline{334} - \frac{5}{3} \overline{344}, \frac{19}{8} \overline{3334} + \frac{101}{24} \overline{3344} - \frac{5}{12} \overline{3444}, \dots \right] \right), \\
 & \text{CWS} \left[-2 \widehat{2} + \widehat{4}, \widehat{11} - \frac{\widehat{12}}{2} + \frac{3 \widehat{33}}{2} - \frac{3 \widehat{34}}{2} - 2 \widehat{44}, \right. \\
 & \quad -\frac{11 \widehat{111}}{6} + \frac{7 \widehat{112}}{6} - \frac{\widehat{122}}{3} - \frac{5 \widehat{222}}{3} - \frac{11 \widehat{333}}{6} + \frac{\widehat{334}}{6} + \widehat{344} + \widehat{444}, \\
 & \quad -\frac{5 \widehat{1111}}{12} + \frac{2 \widehat{1112}}{3} + \frac{41 \widehat{1122}}{24} - \frac{43 \widehat{1212}}{24} + \frac{37 \widehat{1222}}{24} - \frac{7 \widehat{2222}}{4} - \\
 & \quad \left. \frac{13 \widehat{3333}}{12} - \frac{5 \widehat{3334}}{3} + \frac{\widehat{3344}}{8} + \frac{5 \widehat{3434}}{24} - \frac{5 \widehat{3444}}{6} - \frac{15 \widehat{4444}}{8}, \dots \right]
 \end{aligned}$$

Testing the Product

$\xi_a ** \xi_c$

$$\begin{aligned}
 & \text{E1} \left[\left(1 \rightarrow \text{LS} \left[-\overline{1} + \overline{2}, -\frac{9 \overline{12}}{2}, 2 \overline{112} - \frac{37}{12} \overline{122}, \frac{29}{6} \overline{1112} - \frac{5}{2} \overline{1122} + \frac{13}{8} \overline{1222}, \dots \right], \right. \\
 & \quad \left. 2 \rightarrow \text{LS} \left[4 \overline{1} + 2 \overline{2}, -3 \overline{12}, \frac{1}{2} \overline{112} + \frac{5}{12} \overline{122}, \frac{7}{8} \overline{1112} - \frac{4}{3} \overline{1122} - \frac{7}{12} \overline{1222}, \dots \right] \right), \\
 & \text{CWS} \left[2 \widehat{1} - 2 \widehat{2}, 3 \widehat{11} + \frac{\widehat{12}}{2} - \frac{3 \widehat{22}}{2}, -\frac{5 \widehat{111}}{3} + \frac{17 \widehat{112}}{6} - \frac{2 \widehat{122}}{3} - \frac{5 \widehat{222}}{2}, \right. \\
 & \quad \left. -\frac{19 \widehat{1111}}{8} + \frac{8 \widehat{1112}}{3} - \frac{23 \widehat{1122}}{24} + \frac{5 \widehat{1212}}{6} + \frac{\widehat{1222}}{2} - \frac{7 \widehat{2222}}{2}, \dots \right]
 \end{aligned}$$

\$SeriesShowDegree = 5;

Es [$\xi_a ** \xi_c$]

$$\begin{aligned}
 & \text{Es} \left[\left(1 \rightarrow \text{LS} \left[-\widehat{1} + \widehat{2}, -2\widehat{12}, -\frac{11}{2}\widehat{112} - \frac{11}{4}\widehat{122}, \right. \right. \right. \\
 & \quad -\frac{23}{8}\widehat{1112} - \frac{89}{8}\widehat{1122} + \frac{13}{4}\widehat{1222}, \frac{913}{240}\widehat{11112} - \frac{2935}{144}\widehat{11122} + \\
 & \quad \left. \left. \frac{2203}{720}\widehat{11222} + \frac{1451}{720}\widehat{12122} - \frac{1001}{180}\widehat{11212} - \frac{1081}{720}\widehat{12222}, \dots \right], \right. \\
 & \quad 2 \rightarrow \text{LS} \left[4\widehat{1} + 2\widehat{2}, -\widehat{12}, \frac{29}{6}\widehat{112} - \frac{13}{12}\widehat{122}, -\frac{1}{8}\widehat{1112} + \frac{13}{3}\widehat{1122} - \frac{13}{24}\widehat{1222}, \right. \\
 & \quad -\frac{604}{45}\widehat{11112} + \frac{1023}{80}\widehat{11122} - \frac{1549}{180}\widehat{11222} - \\
 & \quad \left. \left. \frac{277}{180}\widehat{12122} + \frac{4391}{720}\widehat{11212} + \frac{159}{80}\widehat{12222}, \dots \right] \right), \\
 & \quad \text{CWS} \left[2\widehat{1} - 2\widehat{2}, 3\widehat{11} + \frac{\widehat{12}}{2} - \frac{3\widehat{22}}{2}, -\frac{5\widehat{111}}{3} + \frac{17\widehat{112}}{6} - \frac{2\widehat{122}}{3} - \frac{5\widehat{222}}{2}, \right. \\
 & \quad -\frac{19\widehat{1111}}{8} + \frac{8\widehat{1112}}{3} - \frac{23\widehat{1122}}{24} + \frac{5\widehat{1212}}{6} + \frac{\widehat{1222}}{2} - \frac{7\widehat{2222}}{2}, \\
 & \quad \frac{13\widehat{11111}}{40} - \frac{83\widehat{11112}}{120} - \frac{427\widehat{11122}}{60} + \frac{509\widehat{11212}}{120} - \\
 & \quad \left. \left. \frac{23\widehat{11222}}{15} + \frac{341\widehat{12122}}{120} + \frac{41\widehat{12222}}{120} - \frac{143\widehat{22222}}{120}, \dots \right] \right]
 \end{aligned}$$

Es [ζ_a] ** Es [ζ_c]

$$\begin{aligned}
 & \text{Es} \left[\left(1 \rightarrow \text{LS} \left[-\overline{1} + \overline{2}, -2\overline{12}, -\frac{11}{2}\overline{112} - \frac{11}{4}\overline{122}, \right. \right. \right. \\
 & \quad -\frac{23}{8}\overline{1112} - \frac{89}{8}\overline{1122} + \frac{13}{4}\overline{1222}, \frac{913}{240}\overline{11112} - \frac{2935}{144}\overline{11122} + \\
 & \quad \left. \left. \frac{2203}{720}\overline{11222} + \frac{1451}{720}\overline{12122} - \frac{1001}{180}\overline{11212} - \frac{1081}{720}\overline{12222}, \dots \right] \right), \\
 & 2 \rightarrow \text{LS} \left[4\overline{1} + 2\overline{2}, -\overline{12}, \frac{29}{6}\overline{112} - \frac{13}{12}\overline{122}, -\frac{1}{8}\overline{1112} + \frac{13}{3}\overline{1122} - \frac{13}{24}\overline{1222}, \right. \\
 & \quad -\frac{604}{45}\overline{11112} + \frac{1023}{80}\overline{11122} - \frac{1549}{180}\overline{11222} - \\
 & \quad \left. \left. \frac{277}{180}\overline{12122} + \frac{4391}{720}\overline{11212} + \frac{159}{80}\overline{12222}, \dots \right] \right), \\
 & \text{CWS} \left[2\widehat{1} - 2\widehat{2}, 3\widehat{11} + \frac{\widehat{12}}{2} - \frac{3\widehat{22}}{2}, -\frac{5\widehat{111}}{3} + \frac{17\widehat{112}}{6} - \frac{2\widehat{122}}{3} - \frac{5\widehat{222}}{2}, \right. \\
 & \quad -\frac{19\widehat{1111}}{8} + \frac{8\widehat{1112}}{3} + \frac{49\widehat{1122}}{24} - \frac{13\widehat{1212}}{6} + \frac{\widehat{1222}}{2} - \frac{7\widehat{2222}}{2}, \\
 & \quad \frac{13\widehat{11111}}{40} - \frac{83\widehat{11112}}{120} - \frac{127\widehat{11122}}{60} - \frac{91\widehat{11212}}{120} - \\
 & \quad \left. \left. \frac{8\widehat{11222}}{15} + \frac{221\widehat{12122}}{120} + \frac{41\widehat{12222}}{120} - \frac{143\widehat{22222}}{120}, \dots \right] \right]
 \end{aligned}$$

\$SeriesShowDegree = 3;

ζ_a

$$\begin{aligned}
 & \text{E1} \left[\left(1 \rightarrow \text{LS} \left[-2\overline{1}, -2\overline{12}, \frac{11}{6}\overline{112} - 2\overline{122}, \dots \right], 2 \rightarrow \text{LS} \left[2\overline{1}, -\frac{\overline{12}}{2}, \frac{4}{3}\overline{112} - \frac{5}{3}\overline{122}, \dots \right] \right), \\
 & \text{CWS} \left[-2\widehat{2}, \widehat{11} - \frac{\widehat{12}}{2}, -\frac{11\widehat{111}}{6} + \frac{7\widehat{112}}{6} - \widehat{122} - \frac{5\widehat{222}}{3}, \dots \right]
 \end{aligned}$$

ζ_a // Es

$$\begin{aligned}
 & \text{Es} \left[\left(1 \rightarrow \text{LS} \left[-2\overline{1}, -2\overline{12}, -\frac{13}{6}\overline{112} - 2\overline{122}, \dots \right], \right. \right. \\
 & \quad \left. \left. 2 \rightarrow \text{LS} \left[2\overline{1}, -\frac{\overline{12}}{2}, \frac{17}{6}\overline{112} - \frac{5}{3}\overline{122}, \dots \right] \right) \right), \\
 & \text{CWS} \left[-2\widehat{2}, \widehat{11} - \frac{\widehat{12}}{2}, -\frac{11\widehat{111}}{6} + \frac{7\widehat{112}}{6} - \widehat{122} - \frac{5\widehat{222}}{3}, \dots \right]
 \end{aligned}$$

ζ_a // Es // E1

$$\begin{aligned}
& \text{E1} \left[\right. \\
& \left. \left\langle 1 \rightarrow \text{LS} \left[-2 \overline{1}, -2 \overline{12}, \frac{11}{6} \overline{112} - 2 \overline{122}, \dots \right], 2 \rightarrow \text{LS} \left[2 \overline{1}, -\frac{\overline{12}}{2}, \frac{4}{3} \overline{112} - \frac{5}{3} \overline{122}, \dots \right] \right\rangle, \right. \\
& \left. \text{CWS} \left[-2 \widehat{2}, \widehat{11} - \frac{\widehat{12}}{2}, -\frac{11 \widehat{111}}{6} + \frac{7 \widehat{112}}{6} - \widehat{122} - \frac{5 \widehat{222}}{3}, \dots \right] \right]
\end{aligned}$$