Pensieve header: Triality attempt 2, continuing TrialityComputations.nb at pensieve://2013-11/DoubleTree/ and MisguidedTriality.nb, here.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
<< WKO4.m
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence,
  BracketForm, BS, CC, Crop, CW, CWS, CWSeries, D, Deg, DegreeScale,
  DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA,
  LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism,
  New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t,
  tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ħ, ⟞, ⟝}.

FreeLie` is in the public domain. Dror Bar-Natan
  is committed to support it within reason until July 15, 2022.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm,
  dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hη, hσ, tA, tha, tm, tS, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan
  is committed to support it within reason until July 15, 2022.
```

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

$$\text{Es}\left[\left\langle x \to \text{LS}[0,\,0,\,0,\,0,\,\ldots]\right\rangle,\ \text{CWS}\left[0,\ \frac{\overline{xx}}{24},\ 0,\ -\frac{\overline{xxxx}}{2880},\ \ldots\right]\right]$$

```
ρ3[ξ_Es] := ξ // dS[y] // dΔ[y, y, z] // dm[x, z, x] // dσ[{x, y} → {y, x}];
V0 // ρ3
```

$$\text{Es}\left[\left\langle x \to \text{LS}\left[-\frac{\overline{y}}{2},\ \frac{\overline{xy}}{12},\ 0,\ -\frac{1}{720}\overline{x\,\overline{x\,\overline{xy}}}+\frac{1}{720}\overline{x\,\overline{\overline{xy}\,y}}-\frac{\overline{\overline{\overline{xy}\,y}\,y}}{5760},\ \ldots\right],\right.$$
$$\left. y \to \text{LS}\left[-\frac{\overline{y}}{2},\ \frac{\overline{xy}}{24},\ -\frac{1}{96}\overline{\overline{xy}\,y},\ -\frac{\overline{x\,\overline{x\,\overline{xy}}}}{1440}+\frac{7\overline{x\,\overline{\overline{xy}\,y}}}{5760}-\frac{\overline{\overline{\overline{xy}\,y}\,y}}{2880},\ \ldots\right]\right\rangle,$$
$$\text{CWS}\left[-\frac{\widehat{y}}{2},\ \frac{\widehat{xy}}{48}+\frac{\widehat{yy}}{48},\ 0,\ -\frac{\widehat{xxxy}}{2880}-\frac{\widehat{xxyy}}{2880}-\frac{\widehat{xyxy}}{5760}-\frac{\widehat{xyyy}}{2880}-\frac{\widehat{yyyy}}{5760},\ \ldots\right]\right]$$

```
V0@{5}
```

$$\text{Es}\left[\left\langle x \to \text{LS}\left[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\overline{x\,\overline{x\,\overline{xy}}}}{5760}-\frac{7\overline{x\,\overline{\overline{xy}\,y}}}{5760}+\frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440},\ 0,\ \ldots\right],\ y \to \text{LS}\left[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\right.\right.$$
$$\left.\left.\frac{\overline{x\,\overline{x\,\overline{xy}}}}{5760}-\frac{1}{720}\overline{x\,\overline{\overline{xy}\,y}}+\frac{1}{720}\overline{\overline{\overline{xy}\,y}\,y},\ -\frac{\overline{x\,\overline{x\,\overline{x\,\overline{xy}}}}}{7680}+\frac{\overline{x\,\overline{x\,\overline{\overline{xy}\,y}}}}{3840}-\frac{\overline{\overline{x\,\overline{xy}}\,\overline{xy}}}{6912},\ \ldots\right]\right\rangle,$$
$$\text{CWS}\left[0,\ -\frac{\widehat{xy}}{48},\ 0,\ \frac{\widehat{xxxy}}{2880}+\frac{\widehat{xxyy}}{2880}+\frac{\widehat{xyxy}}{5760}+\frac{\widehat{xyyy}}{2880},\ 0,\ \ldots\right]\right]$$

```
V0 // ρ3 // ρ3 // ρ3
```

$$\text{Es}\Big[\Big\langle x \to \text{LS}\Big[0, -\frac{\overline{xy}}{24}, 0, \frac{7\,\overline{x\,\overline{x\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}y}}}{5760} + \frac{\overline{\overline{\overline{xy}y}\,y}}{1440}, \ldots\Big],$$
$$y \to \text{LS}\Big[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \frac{\overline{x\,\overline{x\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{xy}y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}y}\,y}, \ldots\Big]\Big\rangle,$$
$$\text{CWS}\Big[0, -\frac{\widehat{xy}}{48}, 0, \frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880}, \ldots\Big]\Big]$$

```
θ[x_, s_] := Module[{y}, θs[x, y, -s] // dS[y] // dm[x, y, x]];
θ[1, 1]@{6}
```

$$\text{Es}\Big[\big\langle 1 \to \text{LS}\big[2\,\overline{1}, 0, 0, 0, 0, 0, \ldots\big]\big\rangle, \text{CWS}\big[\widehat{1}, 0, 0, 0, 0, 0, \ldots\big]\Big]$$

```
Vpp = (V0 ** θs[x, y, -1/2]) θ[z, -1/4] // dm[y, z, y] // ρ3 // ρ3
```

$$\text{Es}\Big[\Big\langle x \to \text{LS}\Big[0, -\frac{\overline{xy}}{24}, 0, \frac{7\,\overline{x\,\overline{x\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}y}}}{5760} + \frac{\overline{\overline{\overline{xy}y}\,y}}{1440}, \ldots\Big],$$
$$y \to \text{LS}\Big[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \frac{\overline{x\,\overline{x\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{xy}y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}y}\,y}, \ldots\Big]\Big\rangle,$$
$$\text{CWS}\Big[\frac{\widehat{x}}{4}, \frac{\widehat{xx}}{48} + \frac{\widehat{xy}}{48}, 0, -\frac{\widehat{xxxx}}{5760} - \frac{\widehat{xxxy}}{2880} - \frac{\widehat{xxyy}}{2880} - \frac{\widehat{xyxy}}{5760} - \frac{\widehat{xyyy}}{2880}, \ldots\Big]\Big]$$

```
(V0 // dA[x, y]) ** (
  (((V0 ** θs[x, y, -1/2]) // ρ3 // ρ3)
      (Es[⟨z → LS[-LW@z/2]⟩, CWS[-CW@x/2]]) // dm[x, z, x]) **
   (Cap ** Cap // dΔ[x, x, y]) ** Es[⟨x → LS@0, y → LS@0⟩, (-2 Cap[[2]]) // tσ[x, y]]
 )
```

$$\text{Es}[\langle x \to \text{LS}[0, 0, 0, 0, \ldots], y \to \text{LS}[0, 0, 0, 0, \ldots]\rangle, \text{CWS}[0, 0, 0, 0, \ldots]]$$

```
(V0 ≡ (
    (((V0 ** θs[x, y, -1/2]) // ρ3 // ρ3)
        (Es[⟨z → LS[-LW@z/2]⟩, CWS[-CW@x/2]]) // dm[x, z, x]) ** (Cap ** Cap //
      dΔ[x, x, y]) ** Es[⟨x → LS@0, y → LS@0⟩, (-2 Cap[[2]]) // tσ[x, y]]
  ))@
 {8}
```

```
$Aborted
```

```
V1 = V0 ** Θs[x, y, -1/4];
{V1, ρ3[V1], V1 ≡ ρ3[V1]}
```

$$\left\{\text{Es}\left[\left\langle x \to \text{LS}\left[-\frac{\overline{y}}{4}, -\frac{\overline{xy}}{96}, \frac{1}{128}\overline{x\overline{xy}} - \frac{1}{256}\overline{\overline{xy}y}, \frac{7\overline{x\overline{x\overline{xy}}}}{92\,160} - \frac{7\overline{x\overline{\overline{xy}y}}}{92\,160} + \frac{11\overline{\overline{\overline{xy}y}y}}{23\,040}, \ldots\right],\right.\right.$$
$$\left.\left. y \to \text{LS}\left[\frac{\overline{x}}{4}, \frac{\overline{xy}}{96}, \frac{1}{256}\overline{x\overline{xy}} - \frac{1}{128}\overline{\overline{xy}y}, -\frac{11\overline{x\overline{x\overline{xy}}}}{23\,040} + \frac{7\overline{x\overline{\overline{xy}y}}}{92\,160} - \frac{7\overline{\overline{\overline{xy}y}y}}{92\,160}, \ldots\right]\right\rangle,$$
$$\text{CWS}\left[0, -\frac{\widehat{xy}}{48}, 0, \frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880}, \ldots\right]\Big],$$
$$\text{Es}\left[\left\langle x \to \text{LS}\left[-\frac{\overline{y}}{4}, -\frac{\overline{xy}}{96}, \frac{1}{128}\overline{x\overline{xy}} - \frac{1}{256}\overline{\overline{xy}y}, \frac{7\overline{x\overline{x\overline{xy}}}}{92\,160} - \frac{7\overline{x\overline{\overline{xy}y}}}{92\,160} + \frac{11\overline{\overline{\overline{xy}y}y}}{23\,040}, \ldots\right],\right.\right.$$
$$\left.\left. y \to \text{LS}\left[\frac{\overline{x}}{4}, \frac{\overline{xy}}{96}, \frac{1}{256}\overline{x\overline{xy}} - \frac{1}{128}\overline{\overline{xy}y}, -\frac{11\overline{x\overline{x\overline{xy}}}}{23\,040} + \frac{7\overline{x\overline{\overline{xy}y}}}{92\,160} - \frac{7\overline{\overline{\overline{xy}y}y}}{92\,160}, \ldots\right]\right\rangle,$$
$$\text{CWS}\left[-\frac{\widehat{y}}{4}, \frac{\widehat{xy}}{48} + \frac{\widehat{yy}}{48}, 0, -\frac{\widehat{xxxy}}{2880} - \frac{\widehat{xxyy}}{2880} - \frac{\widehat{xyxy}}{5760} - \frac{\widehat{xyyy}}{2880} - \frac{\widehat{yyyy}}{5760}, \ldots\right]\Big],$$
$$\text{BS}\Big[\text{True}, 0 == -\frac{\widehat{y}}{4}, 0 == -\frac{\widehat{y}}{4} \;\&\&\; -\frac{\widehat{xy}}{48} == \frac{\widehat{xy}}{48} + \frac{\widehat{yy}}{48},$$
$$0 == -\frac{\widehat{y}}{4} \;\&\&\; -\frac{\widehat{xy}}{48} == \frac{\widehat{xy}}{48} + \frac{\widehat{yy}}{48}, 0 == -\frac{\widehat{y}}{4} \;\&\&\; -\frac{\widehat{xy}}{48} == \frac{\widehat{xy}}{48} + \frac{\widehat{yy}}{48} \;\&\&$$
$$\frac{\widehat{xxxy}}{2880} + \frac{\widehat{xxyy}}{2880} + \frac{\widehat{xyxy}}{5760} + \frac{\widehat{xyyy}}{2880} == -\frac{\widehat{xxxy}}{2880} - \frac{\widehat{xxyy}}{2880} - \frac{\widehat{xyxy}}{5760} - \frac{\widehat{xyyy}}{2880} - \frac{\widehat{yyyy}}{5760}, \ldots\Big]\Big\}$$

```
(2 Cap[[2]]) @ {6}
```

$$\text{CWS}\left[0, -\frac{\widehat{xx}}{48}, 0, \frac{\widehat{xxxx}}{5760}, 0, -\frac{\widehat{xxxxxx}}{362\,880}, \ldots\right]$$

```
W0s[y] = s;
W0 = CWS[{y}, W0s];
W1 = CWS[{z}, W1s];
W2 = CWS[{x}, W2s];
V2 = V1 ** Es[⟨x → LS@0, y → LS@0⟩, W0 - (W1 // tΔ[z, x, y]) + W2];
msgs = SeriesSolve[{W0, W1, W2}, V2 ≡ ρ3[V2]];
```

```
(W1 // tΔ[z, x, y])
```

$$\text{CWS}\left[0, -\frac{\widehat{xx}}{48} - \frac{\widehat{xy}}{24} - \frac{\widehat{yy}}{48}, 0, \frac{\widehat{xxxx}}{5760} + \frac{\widehat{xxxy}}{1440} + \frac{\widehat{xxyy}}{1440} + \frac{\widehat{xyxy}}{2880} + \frac{\widehat{xyyy}}{1440} + \frac{\widehat{yyyy}}{5760}, \ldots\right]$$

```
{W0, W1@{6}, W2}
```

$$\left\{\text{CWS}[s\,\widehat{y}, 0, 0, 0, \ldots], \text{CWS}\left[\frac{1}{12}(1 + 12\,s)\,\widehat{z}, -\frac{\widehat{zz}}{48}, 0, \frac{\widehat{zzzz}}{5760}, 0, -\frac{\widehat{zzzzzz}}{362\,880}, \ldots\right],\right.$$
$$\left.\text{CWS}\left[\frac{1}{6}(1 + 6\,s)\,\widehat{x}, 0, 0, 0, \ldots\right]\right\}$$

```
V3 = V0 ** Θs[x, y, -1/4] **
   Es[⟨x → LS@0, y → LS@0⟩, CWS[CW[x]/12 - CW[y]/12] - (2 Cap[[2]] // tΔ[x, x, y])];
(V3 ≡ ρ3[V3]) @
 {8}
```

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, y, x, y, y] → 0}.

```
$Aborted
```

```
{s, 1/12 (1 + 12 s), 1/6 (1 + 6 s)} /. s → -1/12
```

$$\left\{-\frac{1}{12},\ 0,\ \frac{1}{12}\right\}$$

```
Solve[s == 1/6 (1 + 6 s), s]
```

$$\{\}$$

```
Θs[x, y, -1/4] **
 Es[⟨x → LS@0, y → LS@0⟩, CWS[CW[x]/12 - CW[y]/12] - (2 Cap[[2]] // tΔ[x, x, y])]
```

$$\mathrm{Es}\Big[\Big\langle x \to \mathrm{LS}\Big[-\frac{\overline{y}}{4},\ \frac{\overline{xy}}{32},\ -\frac{1}{384}\,\overline{x\overline{xy}}+\frac{1}{768}\,\overline{\overline{xy}y},\ \frac{\overline{x\,\overline{x\overline{xy}}}}{6144}-\frac{\overline{x\,\overline{\overline{xy}y}}}{6144},\ \ldots\Big],$$
$$y \to \mathrm{LS}\Big[-\frac{\overline{x}}{4},\ -\frac{\overline{xy}}{32},\ \frac{1}{768}\,\overline{x\overline{xy}}-\frac{1}{384}\,\overline{\overline{xy}y},\ \frac{\overline{x\,\overline{\overline{xy}y}}}{6144}-\frac{\overline{\overline{\overline{xy}y}\,y}}{6144},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[\frac{\widehat{x}}{12}-\frac{\widehat{y}}{12},\ \frac{\widehat{xx}}{48}+\frac{\widehat{xy}}{24}+\frac{\widehat{yy}}{48},\ 0,\ -\frac{\widehat{xxxx}}{5760}-\frac{\widehat{xxxy}}{1440}-\frac{\widehat{xxyy}}{1440}-\frac{\widehat{xyxy}}{2880}-\frac{\widehat{xyyy}}{1440}-\frac{\widehat{yyyy}}{5760},\ \ldots\Big]\Big]$$

```
Θs[x, y, -1/4] // ρ3
```

$$\mathrm{Es}\Big[\Big\langle x \to \mathrm{LS}\Big[\frac{\overline{y}}{4},\ \frac{\overline{xy}}{32},\ \frac{1}{384}\,\overline{x\overline{xy}}-\frac{1}{768}\,\overline{\overline{xy}y},\ \frac{\overline{x\,\overline{x\overline{xy}}}}{6144}-\frac{\overline{x\,\overline{\overline{xy}y}}}{6144},\ \ldots\Big],$$
$$y \to \mathrm{LS}\Big[\frac{\overline{x}}{4}+\frac{\overline{y}}{2},\ \frac{\overline{xy}}{32},\ \frac{1}{768}\,\overline{x\overline{xy}},\ \frac{\overline{x\,\overline{\overline{xy}y}}}{6144}-\frac{\overline{\overline{\overline{xy}y}\,y}}{6144},\ \ldots\Big]\Big\rangle,\ \mathrm{CWS}\Big[\frac{\widehat{y}}{4},\ 0,\ 0,\ 0,\ \ldots\Big]\Big]$$