AN UNEXPECTED CYCLIC SYMMETRY OF $I\mathfrak{u}_n$

DROR BAR-NATAN AND ROLAND VAN DER VEEN

Abstract. We find and discuss an unexpected (to us) order n cyclic group of automorphisms of the Lie algebra $I\mathfrak{u}_n := \mathfrak{u}_n \ltimes \mathfrak{u}_n^*$, where $\mathfrak{u}_n$ is the Lie algebra of upper triangular $n\times n$ matrices. Our results also extend to $\mathfrak{gl}_{n+}^\epsilon$, a "solvable approximation" of $\mathfrak{gl}_n$, as defined within.

Given any Lie algebra $\mathfrak{a}$ one may form its "inhomogeneous version" $I\mathfrak{a} := \mathfrak{a} \ltimes \mathfrak{a}^*$, its semidirect product with its dual $\mathfrak{a}^*$ where $\mathfrak{a}^*$ is considered as an Abelian Lie[1] algebra and $\mathfrak{a}$ acts on $\mathfrak{a}^*$ via the coadjoint action. (Over $\mathbb{R}$ if $\mathfrak{a} = \mathfrak{so}_3$ then $\mathfrak{a}^* = \mathbb{R}^3$ and so $I\mathfrak{a} = \mathfrak{so}_3 \ltimes \mathbb{R}^3$ is the Lie algebra of the Euclidean group of rotations and translations, explaining the name).

In general, we care about $I\mathfrak{a}$. It is a special case of the Drinfel'd double / Manin triple construction [Dr, ES] when the cobracket is 0. These Lie algebras occur in the study of the Kashiwara-Vergne problem [BN1, BND2] and they provide the simplest quantum algebra context for the Alexander polynomial [BN2, BND1]. We care especially for the case where $\mathfrak{a}$ is a Borel subalgebra of a semi-simple Lie algebra (e.g., upper triangular matrices) as then the algebras $I\mathfrak{a}$ are the $\epsilon = 0$ "base case" for "solvable approximation" [BV1, BV2, BV3, BN3, BN4, BN5], and their automorphisms are expected to become symmetries of the resulting knot invariants.

Let $\mathfrak{u}_n$ be the Lie algebra of upper triangular $n \times n$ matrices over a field in which 2 is invertible. Beyond inner automorphisms, $\mathfrak{u}_n$ and hence $I\mathfrak{u}_n$ has one obvious and expected anti-automorphism Φ corresponding to flipping matrices along their anti-main-diagonal, as shown in the first image of Figure 1. With x_{ij} denoting the $n \times n$ matrix with 1 in position (ij) and zero everywhere else ($i \leqslant j$ in $\mathfrak{u}_n$), Φ is given by $x_{ij} \mapsto x_{n+1-j,n+1-i}$.

There clearly isn't an automorphism of $\mathfrak{u}_n$ that acts by "sliding down and right parallel to the main diagonal", as in the second image in Figure 1. Where would the last column go? Yet the sliding map, when restricted to where it is clearly defined ($\mathfrak{u}_n$ with the last column excluded), does extend to an automorphism of $I\mathfrak{u}_n$ as in the theorem below.

Theorem 1. *With the basis $\{x_{ij}\}_{1\leqslant i<j\leqslant n} \cup \{a_i = x_{ii}\}_{1\leqslant i\leqslant n}$ for $\mathfrak{u}_n$ and dual basis $\{x_{ji}\}_{1\leqslant i<j\leqslant n} \cup \{b_i\}_{1\leqslant i\leqslant n}$ for $\mathfrak{u}_n^*$ (and duality $\langle x_{kl}, x_{ij}\rangle = \delta_{li}\delta_{jk}$, $\langle b_i, a_j\rangle = 2\delta_{ij}$,[2] and $\langle x_{ji}, a_k\rangle = \langle b_k, x_{ij}\rangle = 0$), the map $\Psi\colon I\mathfrak{u}_n \to I\mathfrak{u}_n$ defined by "incrementing all indices by 1 mod n" (precisely, if ψ is the single-cycle permutation $\psi = (123\dots n)$ then Ψ is defined by $\Psi(x_{ij}) = x_{\psi(i)\psi(j)}$, $\Psi(a_i) = a_{\psi(i)}$, and $\Psi(b_i) = b_{\psi(i)}$) is a Lie algebra automorphism of $I\mathfrak{u}_n$.*

Date: First edition February 3, 2020, this edition February 25, 2021.

2010 *Mathematics Subject Classification.* 57M25.

Key words and phrases. Lie algebras, Lie bialgebras, Lie algebra automorphism, solvable approximation, triangular matrices.

This work was partially supported by NSERC grant RGPIN-2018-04350. It is available in electronic form, along with source files and a verification *Mathematica* notebook at http://drorbn.net/UnexpectedCyclic and at arXiv:2002.00697.

[1]Two Norwegians!

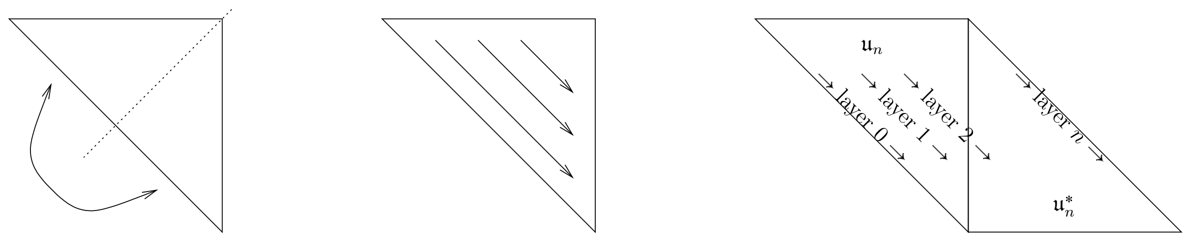


FIGURE 1. An expected anti-automorphism (left), an unexpected automorphism (middle), and an alternative presentation of the "layers" table (right).

Note that our choice of bases, using similar symbols x_{ij} / x_{ji} for the non-diagonal matrices and their duals, hides the intricacy of Ψ; e.g., $\Psi\colon x_{n-1,n} \mapsto x_{n1}$ maps an element of $\mathfrak{u}_n$ to an element of $\mathfrak{u}_n^*$ (also see Figure 1, right).

It may be tempting to think that Ψ has a simple explanation in $\mathfrak{gl}_n$ language: $\mathfrak{u}_n$ is a subset of $\mathfrak{gl}_n$, $\mathfrak{gl}_n$ has a metric (the Killing form) such that the dual of x_{ij} is x_{ji} as is the case for us, and every permutation of the indices induces an automorphism of $\mathfrak{gl}_n$. But this explains nothing and too much: nothing because the bracket of $I\mathfrak{u}_n$ simply isn't the bracket of $\mathfrak{gl}_n$ (even away from the diagonal matrices), and too much because *every* permutation of indices induces an automorphism of $\mathfrak{gl}_n$, whereas only ψ and its powers induce automorphisms of $I\mathfrak{u}_n$.

Proof of Theorem 1. Recall that as a vector space $I\mathfrak{u}_n = \mathfrak{u}_n \oplus \mathfrak{u}_n^*$, yet with bracket $[(x,f),(y,g)] = ([x,y], x\cdot g - y\cdot f)$ where $\cdot$ denotes the coadjoint action, $(x\cdot f)(v) = f([v,x])$. With that and some case checking and explicit computations, the commutation relations of $I\mathfrak{u}_n$ are given by

$$\begin{aligned} &[x_{ij},x_{kl}] = \chi_{\lambda(x_{ij})+\lambda(x_{kl})<n}(\delta_{jk}x_{il} - \delta_{li}x_{kj}) \quad \text{unless both } j=k \text{ and } l=i,\\ &[x_{ij},x_{ji}] = \frac{1}{2}(b_i - b_j), \quad [a_i,x_{jk}] = (\delta_{ij}-\delta_{ik})x_{jk}, \quad [b_i,x_{jk}] = 0,\\ &\quad [a_i,a_j] = [b_i,b_j] = [a_i,b_j] = 0. \end{aligned} \tag{1}$$

Here χ is the indicator function of truth (so $\chi_{5<7} = 1$ while $\chi_{7<5} = 0$), and $\lambda(x_{ij})$ is the "length" of x_{ij}, defined by $\lambda(x_{ij}) := \begin{cases} j-i & i<j \\ n-(i-j) & i>j \end{cases}$.

It is easy to verify that the length $\lambda(x_{ij})$ is Ψ-invariant, and hence everything in (1) is Ψ-equivariant. □

$I\mathfrak{u}_n$ is a solvable Lie algebra (as a semi-direct product of solvable with Abelian, and as will be obvious from the table below). It is therefore interesting to look at the structure of its commutator subalgebras. This structure is summarized in the following table (an alternative view is in Figure 1):

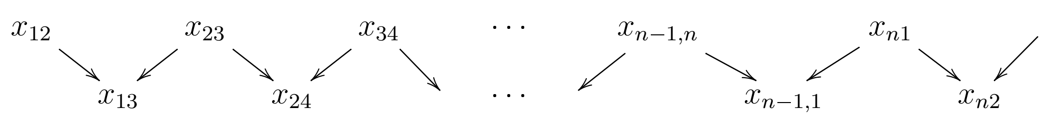
[2]The awkward factor of 2 in $\langle b_i, a_j\rangle$ is irrelevant for Theorem 1 yet crucial for Theorem 2. Removing this factor removes the factor $\frac{1}{2}$ in (1).

layer 0	$\mathfrak{g} = I\mathfrak{u}_n$	a_1	$\to$	a_2	$\to \cdots \to$	a_{n-2}	$\to$	a_{n-1}	$\to$	a_n	$\to$
layer 1	$\mathfrak{g}'_1 = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$	x_{12}	$\to$	x_{23}	$\to \cdots \to$	$x_{n-2,n-1}$	$\to$	$x_{n-1,n}$	$\to$	x_{n1}	$\to$
layer 2	$\mathfrak{g}'_2 = [\mathfrak{g}', \mathfrak{g}'_1]$	x_{13}	$\to$	x_{24}	$\to \cdots \to$	$x_{n-2,n}$	$\to$	$x_{n-1,1}$	$\to$	x_{n2}	$\to$
layer 3	$\mathfrak{g}'_3 = [\mathfrak{g}', \mathfrak{g}'_2]$	x_{14}	$\to$	x_{25}	$\to \cdots \to$	$x_{n-2,1}$	$\to$	$x_{n-1,2}$	$\to$	x_{n3}	$\to$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$	
layer $(n-1)$	$\mathfrak{g}'_{n-1} = [\mathfrak{g}', \mathfrak{g}'_{n-2}]$	x_{1n}	$\to$	x_{21}	$\to \cdots \to$	$x_{n-2,n-1}$	$\to$	$x_{n-1,n-2}$	$\to$	$x_{n,n-1}$	$\to$
layer n	$\mathfrak{g}'_n = [\mathfrak{g}', \mathfrak{g}'_{n-1}]$	b_1	$\to$	b_2	$\to \cdots \to$	b_{n-2}	$\to$	b_{n-1}	$\to$	b_n	$\to$

In this table (all assertions are easy to verify):

- Apart for the treatment of the a_i's and the b_i's, layer=length= $\lambda(x_{ij})$.
- The layers indicate a filtration; each layer should be considered to contain all the ones below it. The generators marked at each layer generate it modulo the layers below.
- The bracket of an element at layer p with an element of layer q is in layer $p+q$ (and it must vanish if $p+q>n$).
- If $p \geqslant 2$, every generator in layer p is the bracket of a generator in layer 1 with a generator in layer $p-1$.
- In layer p, the first $n-p$ generators indicated belong to $\mathfrak{u}_n$ and the last p belong to $\mathfrak{u}_n^*$. So as we go down, $\mathfrak{u}_n^*$ slowly "overtakes" the table.
- The automorphism Ψ acts by following the arrows and shifting every generator one step to the right (and pushing the rightmost generator in each layer back to the left).
- The anti-automorphism Φ acts by mirroring the $\mathfrak{u}_n$ part of each layer left to right and by doing the same to the $\mathfrak{u}_n^*$ part, without mixing the two parts.
- Note that $I\mathfrak{u}_n$ can be metrized by pairing the $\mathfrak{u}_n$ summand with the $\mathfrak{u}_n^*$ one. The metric only pairs generators indicated in layer p with generators indicated in layer $(n-p)$.

Note also that the brackets of the generators indicated in layer 1 yield the generators indicated in layer 2 as follows:



(with the diagram continued cyclically). The symmetry group of the above cycle is the dihedral group D_n and this strongly suggests that the group of outer automorphisms and anti-automorphisms of $I\mathfrak{u}_n$ (all automorphisms and anti-automorphisms modulo inner automorphisms) is D_n, generated by Φ and Ψ. We did not endeavor to prove this formally.

Extension. The Drinfel'd double / Manin triple construction [Dr, ES], when applied to $\mathfrak{u}_n$, is a way to reconstruct $\mathfrak{gl}_n$ from its subalgebras of upper triangular matrices $\mathfrak{u}_n$ and lower triangular matrices $\mathfrak{l}_n$. Precisely, one endows the vector space $\mathfrak{g} = \mathfrak{u}_n \oplus \mathfrak{l}_n$ with a non-degenerate symmetric bilinear form by declaring that the subspaces $\mathfrak{u}_n$ and $\mathfrak{l}_n$ are isotropic ($\langle \mathfrak{u}_n, \mathfrak{u}_n \rangle = \langle \mathfrak{l}_n, \mathfrak{l}_n \rangle = 0$) and by setting $\langle x_{kl}, x_{ij} \rangle = \delta_{li}\delta_{jk}$, $\langle b_i, a_j \rangle = 2\delta_{ij}$, and $\langle x_{ji}, a_k \rangle = \langle b_k, x_{ij} \rangle = 0$ as in Theorem 1 and where a_i stands for the diagonal matrix x_{ii} considered as an element of $\mathfrak{u}_n$ and b_i stands for the same matrix as an element of $\mathfrak{l}_n$. There is then a unique bracket on $\mathfrak{g}$ that extends the brackets on the summands $\mathfrak{u}_n$ and $\mathfrak{l}_n$ and relative to which the inner product of $\mathfrak{g}$ is invariant[3]. With our judicious choice of bilinear form, this bracket on $\mathfrak{g}$ satisfies the Jacobi identity and turns $\mathfrak{g}$ into a Lie algebra isomorphic to $\mathfrak{gl}_{n+} = \mathfrak{gl}_n \oplus \mathfrak{h}'_n$, where $\mathfrak{h}'_n$ denotes a second copy of the diagonal matrices in $\mathfrak{gl}_n$.

We let $\mathfrak{gl}^\epsilon_{n+}$ be the Inonu-Wigner [IW] contraction of $\mathfrak{g}$ along its $\mathfrak{l}_n$ summand, with parameter ϵ.[4] All that this means is that the bracket of $\mathfrak{l}_n$ gets multiplied by ϵ to give $\mathfrak{l}^\epsilon_n$, and then the Drinfel'd double / Manin triple construction is repeated starting with $\mathfrak{u}_n \oplus \mathfrak{l}^\epsilon_n$, without changing the bilinear form. The result is a Lie algebra $\mathfrak{gl}^\epsilon_{n+}$ over the ring of polynomials in ϵ which specializes to $I\mathfrak{u}_n$ at $\epsilon = 0$ and which is isomorphic to $\mathfrak{gl}_n \oplus \mathfrak{h}'_n$ when ϵ is invertible.[5] We care about $\mathfrak{gl}^\epsilon_{n+}$ a lot [BV1, BV2, BV3, BN3, BN4, BN5]; when reduced modulo $\epsilon^{k+1} = 0$ for some natural number k it becomes solvable, and hence a "solvable approximation" of $\mathfrak{gl}_n$ with applications to computability of knot invariants.

Theorem 2. *With the same conventions as in Theorem 1 the map Ψ is also a Lie algebra automorphism of $\mathfrak{gl}^\epsilon_{n+}$.*

Proof. By some case checking and explicit computations, the commutation relations of $\mathfrak{gl}^\epsilon_{n+}$ are given by

$$[x_{ij}, x_{kl}] = \chi^\epsilon_{\lambda(x_{ij})+\lambda(x_{kl})<n}(\delta_{jk}x_{il} - \delta_{li}x_{kj}) \quad \text{unless both } j = k \text{ and } l = i,$$
$$[x_{ij}, x_{ji}] = \frac{1}{2}(b_i - b_j) + \frac{\epsilon}{2}(a_i - a_j), \quad [a_i, x_{jk}] = (\delta_{ij} - \delta_{ik})x_{jk}, \quad [b_i, x_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})x_{jk},$$
$$[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 0,$$

where $\chi^\epsilon_{\texttt{True}} = 1$ and $\chi^\epsilon_{\texttt{False}} = \epsilon$. These relations are clearly Ψ-equivariant. □

Note 3. There is of course an "$\mathfrak{sl}$" version of everything, in which linear combinations $\sum \alpha_i a_i$ and $\sum \beta_i b_i$ are allowed only if $\sum \alpha_i = \sum \beta_i = 0$, with obvious modifications throughout.

Note 4. At $n = 2$ and $\epsilon = 0$, the algebra $\mathfrak{sl}^0_{2+}$ is the "diamond Lie algebra" of [Ki, Chapter 4.3], which is sometimes called "the Nappi-Witten algebra" [NW]: With $a = (a_1 - a_2)/2$, $x = x_{12}$, $y = x_{21}$, and $b = (b_1 - b_2)/2$, it is

$$\langle a, x, y, b\rangle / ([a, x] = x,\ [a, y] = -y,\ [x, y] = b,\ [b, -] = 0)\,.$$

Here $\Phi\colon (a, x, y, b) \mapsto (-a, x, y, -b)$ and $\Psi\colon (a, x, y, b) \mapsto (-a, y, x, -b)$.

Note 5. Upon circulating this paper as an eprint we received a note from A. Knutson informing us of [KZ, esp. sec. 2.3], where the algebra $I\mathfrak{u}_n$ (except reduced modulo $\langle b_i\rangle$ and considered globally rather than infinitesimally) is considered from a different perspective. It is shown to be a subquotient of the affine algebra $\widehat{\mathfrak{gl}_n}$ in a manner preserved by its automorphisms corresponding to its Dynkin diagram, which is a cycle. Similar comments apply to the other algebras considered here.

Note 6. A day later we received a note [BR] from M. Bulois and N. Ressayre reporting on an explanation of Theorem 1 in terms of affine Kac-Moody Lie algebras, similarly to Note 5.

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[3] Indeed we only need to determine $[u, l]$ for $u \in \mathfrak{u}_n$ and $l \in \mathfrak{l}_n$. Writing $[u, l] = u' + l'$ with $u' \in \mathfrak{u}_n$ and $l' \in \mathfrak{l}_n$, we determine u' using the non-degeneracy of the inner product from the relation $\langle u', l''\rangle = \langle [u, l], l''\rangle = \langle u, [l, l'']\rangle$ which holds for every $l'' \in \mathfrak{l}_n$ due to the invariance of $\langle\,,\,\rangle$. Similarly l' is determined from $\langle l', u''\rangle = \langle [u, l], u''\rangle = \langle l, [u'', u]\rangle$.

[4] Alternatively, make $\mathfrak{u}_n$ into a Lie bialgebra with cobracket δ using its given duality with $\mathfrak{l}_n$, and double it as in [Dr, ES] but using the cobracket $\epsilon\delta$.

[5] Hence $\mathfrak{gl}^\epsilon_{n+} \to I\mathfrak{u}_n$ is a counter-example to the feel-true statement "a contraction of a direct sum is a direct sum". Indeed with notation as in Theorem 2, as $\epsilon \to 0$ the decomposition $\mathfrak{gl}^\epsilon_{n+} = \mathfrak{gl}_n \oplus \mathfrak{h}'_n = \langle x_{ij}, b_i + \epsilon a_i\rangle \oplus \langle b_i - \epsilon a_i\rangle$ collapses.

Acknowledgement. We wish to thank M. Bulois, A. Knutson, A. Referees, N. Ressayre, and N. Williams for comments and suggestions.

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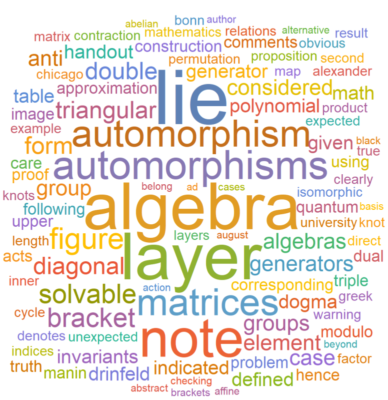

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An Unexpected Cyclic Symmetry of $\mathfrak{Iu}_n$ - Verification Notebook

Pensieve header: This is a feel-good verification notebook for the paper “An Unexpected Cyclic Symmetry of Iu_n” by Dror Bar-Natan and Roland van der Veen, written only to ascertain that the paper contains no calculation errors. It is available web-only at http://drorbn.net/UnexpectedCyclic, and is not a part of the paper itself. Continues pensieve://2020-01/ .

Only Theorem 2 is tested; Theorem 1 is simply the case where $\epsilon = 0$, so it does not require independent testing.

Definitions.

General definitions - brackets *B* and pairings *P* are bilinear, brackets are anti-symmetric:

```
In[∘]:= B[0, _] = 0; B[_, 0] = 0;
B[c_ * x : (x | a | b)__, y_] := Expand[c B[x, y]];
B[y_, c_ * x : (x | a | b)__] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
```

```
In[∘]:= P[0, _] = 0; P[_, 0] = 0;
P[c_ * x : (x | a | b)__, y_] := Expand[c P[x, y]];
P[y_, c_ * x : (x | a | b)__] := Expand[c P[y, x]];
P[x_Plus, y_] := P[#, y] & /@ x;
P[x_, y_Plus] := P[x, #] & /@ y;
```

```
In[∘]:= B[y_, x_] := Expand[-B[x, y]];
```

The default value of *n* (can be changed):

```
In[∘]:= n = 5;
```

The “length” λ and the “truth indicator” χ_ϵ, and the Kronecker δ-function δ:

$$\text{In[∘]:= } \lambda[x_{i\_,j\_}] := \begin{cases} j - i & i < j \\ n - (i - j) & i > j \end{cases}$$

```
χ_ϵ_[cond_] := If[TrueQ@cond, 1, ϵ];
δ_{i_,j_} := χ_0[i == j];
```

The bracket:

$$\text{In[∘]:= } B[x_{i\_,j\_}, x_{k\_,l\_}] := \begin{cases} \chi_\epsilon[\lambda[x_{i,j}] + \lambda[x_{k,l}] < n]\ (\delta_{j,k}\, x_{i,l} - \delta_{l,i}\, x_{k,j}) & j \neq k \lor l \neq i \\ \frac{1}{2} b_i - \frac{1}{2} b_j + \frac{\epsilon}{2} a_i - \frac{\epsilon}{2} a_j & j == k \land l == i \end{cases};$$

$$B[a_{i\_}, x_{j\_,k\_}] := (\delta_{i,j} - \delta_{i,k})\, x_{j,k};$$
$$B[b_{i\_}, x_{j\_,k\_}] := \epsilon\, (\delta_{i,j} - \delta_{i,k})\, x_{j,k};$$

```
B[(a | b)_, (a | b)_] = 0;
```

The duality pairing:

```
In[∘]:= P[x_{i_,j_}, x_{k_,l_}] := δ_{j,k} δ_{l,i};
        P[x__, (a | b)_] = 0; P[(a | b)_, x__] = 0;
        P[a_{i_}, b_{j_}] := 2 δ_{i,j}; P[b_{j_}, a_{i_}] := 2 δ_{i,j};
        P[a_, a_] = 0; P[b_, b_] = 0;
```

The permutation ψ and the automorphism Ψ:

```
In[∘]:= ψ[k_Integer] := { k+1   k < n
                        { 1     k == n ;
        Ψ[ℰ_] := ℰ /. {x_{i_,j_} :→ x_{ψ@i,ψ@j}, a_{i_} :→ a_{ψ@i}, b_{i_} :→ b_{ψ@i}}
```

The basis of $\mathfrak{lu}_n$ / $\mathrm{gl}_{n+}^{\epsilon}$:

```
In[∘]:= Basis[n_] := Flatten@{
          Table[{x_{i,j}, x_{j,i}}, {i, n-1}, {j, i+1, n}],
          Table[{a_i, b_i}, {i, n}] }
```

Testing.

```
In[∘]:= Basis[4]
```

Out[∘]= $\{x_{1,2}, x_{2,1}, x_{1,3}, x_{3,1}, x_{1,4}, x_{4,1}, x_{2,3}, x_{3,2}, x_{2,4}, x_{4,2}, x_{3,4}, x_{4,3}, a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4\}$

A full bracket-table for $n = 2$:

```
In[∘]:= n = 2; MatrixForm[
         Table[B[u, v], {u, Basis[n]}, {v, Basis[n]}],
         TableHeadings → {Basis[n], Basis[n]}]
```

Out[∘]//MatrixForm=

	$x_{1,2}$	$x_{2,1}$	a_1	b_1	a_2	b_2
$x_{1,2}$	0	$\frac{\in a_1}{2} - \frac{\in a_2}{2} + \frac{b_1}{2} - \frac{b_2}{2}$	$-x_{1,2}$	$-\in x_{1,2}$	$x_{1,2}$	$\in x_{1,2}$
$x_{2,1}$	$-\frac{\in a_1}{2} + \frac{\in a_2}{2} - \frac{b_1}{2} + \frac{b_2}{2}$	0	$x_{2,1}$	$\in x_{2,1}$	$-x_{2,1}$	$-\in x_{2,1}$
a_1	$x_{1,2}$	$-x_{2,1}$	0	0	0	0
b_1	$\in x_{1,2}$	$-\in x_{2,1}$	0	0	0	0
a_2	$-x_{1,2}$	$x_{2,1}$	0	0	0	0
b_2	$-\in x_{1,2}$	$\in x_{2,1}$	0	0	0	0

The bracket is anti-symmetric at $n = 4$:

```
In[∘]:= n = 4; Short@Table[
          {u, v} = t; B[u, v] + B[v, u],
          {t, Tuples[Basis[n], 2]}]
```

```
Out[∘]//Short= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, <<364>>, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

The bracket satisfies the Jacobi identity (strictly, we very Jacobi only for $n = 6$, but three basis elements may involve at most 6 distinct indices, so this is general):

In[•]:=
```
n = 6; DeleteCases[0]@Table[
  {u, v, w} = t; B[u, B[v, w]] + B[v, B[w, u]] + B[w, B[u, v]],
  {t, Tuples[Basis[n], 3]}]
```
Out[•]= { }

The pairing is invariant:

In[•]:=
```
n = 6; DeleteCases[0]@Table[
  {u, v, w} = t; P[B[u, v], w] + P[v, B[u, w]],
  {t, Tuples[Basis[n], 3]}]
```
Out[•]= { }

The action of Ψ:

In[•]:=
```
(# → Ψ[#]) & /@ Basis[4]
```
Out[•]= $\{x_{1,2} \to x_{2,3},\ x_{2,1} \to x_{3,2},\ x_{1,3} \to x_{2,4},\ x_{3,1} \to x_{4,2},\ x_{1,4} \to x_{2,1},\ x_{4,1} \to x_{1,2},\ x_{2,3} \to x_{3,4},\ x_{3,2} \to x_{4,3},\ x_{2,4} \to x_{3,1},\ x_{4,2} \to x_{1,3},\ x_{3,4} \to x_{4,1},\ x_{4,3} \to x_{1,4},\ a_1 \to a_2,\ b_1 \to b_2,\ a_2 \to a_3,\ b_2 \to b_3,\ a_3 \to a_4,\ b_3 \to b_4,\ a_4 \to a_1,\ b_4 \to b_1\}$

Ψ is an automorphism:

In[•]:=
```
n = 4; DeleteCases[0]@Table[
  {u, v} = t; Ψ[B[u, v]] - B[Ψ[u], Ψ[v]],
  {t, Tuples[Basis[n], 2]}]
```
Out[•]= { }

Ψ respects the pairing:

In[•]:=
```
n = 4; DeleteCases[0]@Table[
  {u, v} = t; Ψ[P[u, v]] - P[Ψ[u], Ψ[v]],
  {t, Tuples[Basis[n], 2]}]
```
Out[•]= { }

Bonus Tests

Acting by arbitrary index-permutations:

In[•]:=
```
Act_{σ_List}[ℰ_] := ℰ /. {x_{i_,j_} :> x_{σ[[i]],σ[[j]]}, a_{i_} :> a_{σ[[i]]}, b_{i_} :> b_{σ[[i]]}}
```

At $n = 5$, only cyclic permutations induce automorphisms:

In[•]:=
```
n = 5;
Select[Permutations[Range[n]],
 σ ↦ And @@ Flatten[Table[
     Act_σ[B[u, v]] === B[Act_σ[u], Act_σ[v]],
     {u, Basis[n]}, {v, Basis[n]}
    ]]
]
```
Out[•]= {{1, 2, 3, 4, 5}, {2, 3, 4, 5, 1}, {3, 4, 5, 1, 2}, {4, 5, 1, 2, 3}, {5, 1, 2, 3, 4}}

Yet in the case of gl_n, meaning when $\epsilon = 1$, all permutations induce automorphisms:

In[∘]:=
```
n = 4;
Block[{ϵ = 1}, Select[Permutations[Range[n]],
   σ ↦ And @@ Flatten[Table[
       Act_σ[B[u, v]] === B[Act_σ[u], Act_σ[v]],
       {u, Basis[n]}, {v, Basis[n]}
      ]]
 ]]
```

Out[∘]= {{1, 2, 3, 4}, {1, 2, 4, 3}, {1, 3, 2, 4}, {1, 3, 4, 2}, {1, 4, 2, 3}, {1, 4, 3, 2}, {2, 1, 3, 4}, {2, 1, 4, 3}, {2, 3, 1, 4}, {2, 3, 4, 1}, {2, 4, 1, 3}, {2, 4, 3, 1}, {3, 1, 2, 4}, {3, 1, 4, 2}, {3, 2, 1, 4}, {3, 2, 4, 1}, {3, 4, 1, 2}, {3, 4, 2, 1}, {4, 1, 2, 3}, {4, 1, 3, 2}, {4, 2, 1, 3}, {4, 2, 3, 1}, {4, 3, 1, 2}, {4, 3, 2, 1}}

If ϵ is invertible, the isomorphism class of gl^ϵ_{n+} is independent of ϵ, using Inonu-Wigner contractions:

In[∘]:=
```
IW_λ_[ℰ_] := ℰ /. {x_{i_,j_} /; i > j ⧴ λ x_{i,j}, b_i_ ⧴ λ b_i};
```

In[∘]:=
```
n = 4;
Union@Flatten@Table[
   (B[u, v] /. ϵ → 1) == IW_ϵ@B[IW_{1/ϵ}@u, IW_{1/ϵ}@v],
   {u, Basis[n]}, {v, Basis[n]}]
```

Out[∘]= {True}

Even cyclic index permutations become singular at $\epsilon = 0$ when conjugated by Inonu-Wigner contractions (so Ψ simply isn't that):

In[∘]:=
```
n = 4; Table[u → IW_{1/ϵ}@Act_{2,3,4,1}@IW_ϵ@u, {u, Basis[n]}]
```

Out[∘]=

$$\left\{x_{1,2}\to x_{2,3},\ x_{2,1}\to x_{3,2},\ x_{1,3}\to x_{2,4},\ x_{3,1}\to x_{4,2},\ x_{1,4}\to \frac{x_{2,1}}{\epsilon},\ x_{4,1}\to \epsilon\, x_{1,2},\ x_{2,3}\to x_{3,4},\ x_{3,2}\to x_{4,3},\ x_{2,4}\to \frac{x_{3,1}}{\epsilon},\ x_{4,2}\to \epsilon\, x_{1,3},\ x_{3,4}\to \frac{x_{4,1}}{\epsilon},\ x_{4,3}\to \epsilon\, x_{1,4},\ a_1\to a_2,\ b_1\to b_2,\ a_2\to a_3,\ b_2\to b_3,\ a_3\to a_4,\ b_3\to b_4,\ a_4\to a_1,\ b_4\to b_1\right\}$$

The same is true for all other permutations (except the identity):

In[∘]:=
```
n = 3; MatrixForm[
 Table[IW_{1/ϵ}@Act_σ@IW_ϵ@u, {σ, rows = Permutations@Range@n}, {u, cols = Basis[n]}],
 TableHeadings → {rows, cols}]
```

Out[∘]//MatrixForm=

	$x_{1,2}$	$x_{2,1}$	$x_{1,3}$	$x_{3,1}$	$x_{2,3}$	$x_{3,2}$	a_1	b_1	a_2	b_2	a_3	b_3
{1, 2, 3}	$x_{1,2}$	$x_{2,1}$	$x_{1,3}$	$x_{3,1}$	$x_{2,3}$	$x_{3,2}$	a_1	b_1	a_2	b_2	a_3	b_3
{1, 3, 2}	$x_{1,3}$	$x_{3,1}$	$x_{1,2}$	$x_{2,1}$	$\frac{x_{3,2}}{\epsilon}$	$\epsilon\, x_{2,3}$	a_1	b_1	a_3	b_3	a_2	b_2
{2, 1, 3}	$\frac{x_{2,1}}{\epsilon}$	$\epsilon\, x_{1,2}$	$x_{2,3}$	$x_{3,2}$	$x_{1,3}$	$x_{3,1}$	a_2	b_2	a_1	b_1	a_3	b_3
{2, 3, 1}	$x_{2,3}$	$x_{3,2}$	$\frac{x_{2,1}}{\epsilon}$	$\epsilon\, x_{1,2}$	$\frac{x_{3,1}}{\epsilon}$	$\epsilon\, x_{1,3}$	a_2	b_2	a_3	b_3	a_1	b_1
{3, 1, 2}	$\frac{x_{3,1}}{\epsilon}$	$\epsilon\, x_{1,3}$	$\frac{x_{3,2}}{\epsilon}$	$\epsilon\, x_{2,3}$	$x_{1,2}$	$x_{2,1}$	a_3	b_3	a_1	b_1	a_2	b_2
{3, 2, 1}	$\frac{x_{3,2}}{\epsilon}$	$\epsilon\, x_{2,3}$	$\frac{x_{3,1}}{\epsilon}$	$\epsilon\, x_{1,3}$	$\frac{x_{2,1}}{\epsilon}$	$\epsilon\, x_{1,2}$	a_3	b_3	a_2	b_2	a_1	b_1