# AN UNEXPECTED CYCLIC SYMMETRY OF $sl^\epsilon_{n+}$

DROR BAR-NATAN AND ROLAND VAN DER VEEN

Abstract. We introduce $sl^\epsilon_{n+}$, a one-parameter family of Lie algebras that encodes an approximation of the semi-simple Lie algebra $sl_n$ by solvable algebras (this is useful elsewhere; see [?]). We find that $sl^\epsilon_{n+}$ has an unanticipated order $n$ automorphism $\Psi$. Why is it there?

Contents



## 1. Statement

We start with some conventions, then define our main stars the Lie algebras $gl^\epsilon_{n+}$ and $sl^\epsilon_{n+}$ in completely explicit terms, then exhibit the completely obvious automorphism $\Psi$ of $gl^\epsilon_{n+}$ and of $sl^\epsilon_{n+}$, and then go back to the abstract origins of $gl^\epsilon_{n+}$ and $sl^\epsilon_{n+}$, where the presence of $\Psi$ becomes surprising and unexplained.

**Convention 1.1.** Let $n$ be a fixed positive integer and let $\epsilon$ be a formal parameter. For the formal symbol $x_{ij}$ where $1 \leqslant i \neq j \leqslant n$ define its "length" $\lambda(x_{ij}) := \begin{cases} j-i & i<j \\ n-(i-j) & i>j \end{cases}$. Let $\chi_\epsilon$ be the function that assigns $\epsilon$ to `True` and 1 to `False`. For example, $\chi(5<7)=\epsilon$ while $\chi(7<5)=1$. Let $\delta_{ij}$ be the Kronecker $\delta$-function.

**Definition 1.2.** Let $gl^\epsilon_{n+}$ be the Lie algebra with generators $\{x_{ij}\}_{1\leqslant i\neq j\leqslant n} \cup \{a_i, b_i\}_{1\leqslant i\leqslant n}$ and with commutation relations

$$\begin{aligned}
[x_{ij}, x_{kl}] &= \chi_\epsilon(\lambda(x_{ij})+\lambda(x_{kl})>n)(\delta_{jk}x_{il}-\delta_{li}x_{kj}) \quad \text{unless both } j=k \text{ and } l=i, \\
[x_{ij}, x_{ji}] &= \frac{1}{2}(b_i-b_j+\epsilon(a_i-a_j)), \\
[a_i, x_{jk}] &= (\delta_{ij}-\delta_{ik})x_{jk}, \\
[b_i, x_{jk}] &= \epsilon(\delta_{ij}-\delta_{ik})x_{jk}.
\end{aligned} \tag{1}$$

Let $sl^\epsilon_{n+}$ be the subalgebra of $gl^\epsilon_{n+}$ generated by the $x_{ij}$'s and by the differences $\{a_i-a_j\}$ and $\{b_i-b_j\}$. 1.2

*Date*: First edition Not Yet, 2020, this edition January 23, 2020.
2010 *Mathematics Subject Classification.* 57M25.
*Key words and phrases.* Lie algebras, Lie bi-algberas, solvable approximation .
This work was partially supported by NSERC grant RGPIN-2018-04350.

It is easy to verify that the commutation relations in (1) respect the Jacobi identity and hence $gl^\epsilon_{n+}$ and $sl^\epsilon_{n+}$ are indeed Lie algebras.

It is easy to verify that if $\epsilon = 1$ then $t_i := b_i - a_i$ is central in $gl^\epsilon_{n+}$, that $gl^1_{n+} \cong \langle x_{ij}, h_i = (b_i + a_i)/2\rangle \oplus \langle t_i\rangle$, and that the first summand, $\langle x_{ij}, h_i\rangle$, is isomorphic to the general linear Lie algebra $gl_n$ by mapping $x_{ij}$ to the matrix that has 1 in position $(ij)$ and 0 everywhere else and $h_i$ to the diagonal matrix that has 1 in position $(ii)$ and 0 everywhere else. Hence $gl^\epsilon_{n+}$ is an $\epsilon$-dependent variant of $gl_n$ plus an Abelian summand, explaining its name $gl^\epsilon_{n+}$. Nearly identical observations hold for $sl^\epsilon_{n+}$: at $\epsilon = 1$ it is a sum of $_n$ and an Abelian summand.

It is also easy to verify that the map $\Phi_\epsilon\colon gl^\epsilon_{n+} \to gl^1_{n+}$ defined by $a_i \mapsto a_i$, $b_i \mapsto \epsilon b_i$, and $x_{ij} \mapsto \chi_\epsilon(i > j)x_{ij}$ is a morphism of Lie algebras, and it is clearly invertible if $\epsilon$ is invertible. Hence for invertible $\epsilon$ our $gl^\epsilon_{n+}$ is always a sum of $gl_n$ with an Abelian factor, and so $gl^\epsilon_{n+}$ is most interesting at $\epsilon = 0$ or in a formal neighborhood of $\epsilon = 0$ (namely, over a ring like $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$). Similarly for $sl^\epsilon_{n+}$.

**Theorem 1.3.** *The map $\Psi\colon gl^\epsilon_{n+} \to gl^\epsilon_{n+}$ which increments all indices modulo $n$ is a Lie-algebra automorphism of $gl^\epsilon_{n+}$ and/or $sl^\epsilon_{n+}$. (Precisely, if $\psi$ is the single-cycle permutation $\psi = (123\dots n)$ then $\Psi$ is defined by $\Psi(x_{ij}) = x_{\psi(i)\psi(j)}$, $\Psi(a_i) = a_{\psi(i)}$, and $\Psi(b_i) = b_{\psi(i)}$).*

*Proof.* By case checking the length $\lambda(x_{ij})$ is $\Psi$-invariant, and hence everything in (1) is $\Psi$-equivariant. □

Thus our main theorem is a complete triviality. More precisely, Definition 1.2 was set up so that Theorem 1.3 would be a complete triviality. But $gl^\epsilon_{n+}$ also has an abstract origin which we describe in the next section. From the abstract perspective the existence of $\Psi$ remains mysterious to us.

Note that at $\epsilon = 1$ the automorphism $\Psi$ induces an inner automorphism of $gl_n$ — namely, it is conjugation $C(P_\psi)$ by a permutation matrix $P_\psi$. But at other values of $\epsilon$ the automorphism $\Psi$ does not restrict to conjugation by $P_\psi$, and at/near our point of interest $\epsilon = 0$ the phrase "conjugation by $P_\psi$" stops making sense — namely, $\lim_{\epsilon\to 0} \Phi_{1/\epsilon} \circ C(P_\psi) \circ \Phi_\epsilon$ does not exist.

Finally, for any permutation $\sigma$ one may define a map $A_\sigma\colon gl^\epsilon_{n+} \to gl^\epsilon_{n+}$ by permuting the indices: $A_\sigma(x_{ij}) = x_{\sigma(i)\sigma(j)}$, $A_\sigma(a_i) = a_{\sigma(i)}$, and $A_\sigma(b_i) = b_{\sigma(i)}$). At $\epsilon = 1$ the map $A_\sigma$ always respects the Lie bracket. Yet it is easy to verify that at $\epsilon \neq 1$ the only permutations $\sigma$ for which $A_\sigma$ is a morphism of Lie algebras are the powers of $\psi$, and obviously, $A_{\psi^p} = \Psi^p$.

## 2. Wherefore $gl^\epsilon_{n+}$?

Let $\mathbb{F}$ be some ground field. A semi-simple Lie algebra over $\mathbb{F}$ (say, $gl_n$ or $sl_n$) can be reconstructed from its half: if $\mathfrak{g}$ is a semi-simple Lie algebra (say, $gl_n$ or $sl_n$) and $\mathfrak{b}^+$ is an upper Borel subalgebra (the upper triangular matrices if $\mathfrak{g}$ is $gl_n$ or $sl_n$), then $\mathfrak{g}$ can be recovered from $\mathfrak{b}^+$, which has roughly half the dimension of $\mathfrak{g}$.

Let us go through the process in some detail. The "half" $\mathfrak{b}^+$ has its own Lie bracket $B^+\colon \mathfrak{b}^+ \otimes \mathfrak{b}^+ \to \mathfrak{b}^+$. In addition, $\mathfrak{b}^+$ is dual to a lower Borel subalgebra $\mathfrak{b}^-$ (lower triangular matrices for $gl_n$ or $sl_n$, with the duality pairing $P\colon \mathfrak{b}^- \otimes \mathfrak{b}^+ \to \mathbb{F}$ given by $P(L,U) = \mathrm{tr}(LU)$). Now $\mathfrak{b}^-$ also has a bracket $B^-\colon \mathfrak{b}^- \otimes \mathfrak{b}^- \to \mathfrak{b}^-$ and its adjoint relative to the duality $P$ is a "cobracket" map $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ which satisfies three conditions:

- $\delta$ is anti-symmetric: $\delta + \sigma \circ \delta = 0$, where $\sigma\colon \mathfrak{b}^+ \otimes \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ swaps the two tensor factors.
- $\delta$ satisfies a co-Jacobi identity: $(1 + \tau + \tau^2) \circ (1 \otimes \delta) \circ \delta = 0$, where $\tau\colon \mathfrak{b}^+ \otimes \mathfrak{b}^+ \otimes \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+ \otimes \mathfrak{b}^+$ is the cyclic permutation of the tensor factors.

- Along with the bracket $[\cdot,\cdot] = B^+$, $\delta$ satisfies a cocycle identity:

$$\forall x_1, x_2 \in \mathfrak{b}^+, \quad \delta([x_1, x_2]) = (\mathrm{ad}_{x_1} \otimes 1 + 1 \otimes \mathrm{ad}_{x_1})(\delta(x_2)) - (\mathrm{ad}_{x_2} \otimes 1 + 1 \otimes \mathrm{ad}_{x_2})(\delta(x_1)).$$

One may show that given any finite dimensional Lie algebra $\mathfrak{b}$ and a co-bracket $\delta\colon \mathfrak{b} \to \mathfrak{b}\otimes\mathfrak{b}$ satisfying the above conditions, then the adjoint $\delta^*\colon \mathfrak{b}^* \otimes \mathfrak{b}^* \to \mathfrak{b}^*$ of the cobracket defines a bracket on $\mathfrak{b}$, and then the "double" $\mathcal{D}(\mathfrak{b}, \delta) = \mathfrak{b} \oplus \mathfrak{b}^*$ is also a Lie algebra, with bracket

$$[x_1 \oplus y_1, x_2 \oplus y_2] := ([x_1, x_2] - \mathrm{ad}_{y_1}(x_2) + \mathrm{ad}_{y_2}(x_1)) \oplus ([y_1, y_2] + \mathrm{ad}_{x_1}(y_2) - \mathrm{ad}_{x_2}(y_1)),$$

where we have used $\mathrm{ad}_x(y)$ to denote the coadjoint action of $\mathfrak{b}$ on $\mathfrak{b}^*$ (whose definition depends only on the bracket of $\mathfrak{b}$) and $\mathrm{ad}_y(x)$ to denote the coadjoint action of $\mathfrak{b}^*$ on $\mathfrak{b}^{**} = \mathfrak{b}$ (whose definition depends only on the bracket of $\mathfrak{b}^*$ or the cobracket of $\mathfrak{b}$).

With all this in mind, $\mathfrak{g}_+ := \mathcal{D}(\mathfrak{b}^+) \cong \mathfrak{g} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is an "extra" copy of the Cartan subalgebra of $\mathfrak{g}$. In the case of $gl_n$ (and similarly for $sl_n$), this becomes the statement that the upper triangular matrices direct sum the lower triangular matrices make all matrices, but with the diagonal matrices repeated twice.

Clearly, if $\delta$ satisfies the three conditions above, then so does $\epsilon\delta$, where $\epsilon$ is any scalar. Hence we can define $\mathfrak{g}^{\epsilon}_+ := \mathcal{D}(\mathfrak{b}^+, \epsilon\delta)$. At $\epsilon = 1$ this is $\mathfrak{g} \oplus \mathfrak{h}$, and by scaling the $(\mathfrak{b}^+)^* = \mathfrak{b}^-$ component of $\mathcal{D}(\mathfrak{b}^+, \epsilon\delta)$ by a factor of $\epsilon$, the same is true whenever $\epsilon$ is invertible. Yet the one-parameter family $\mathfrak{g}^{\epsilon}_+$ is not constant: at $\epsilon = 0$ the double construction degenerates to the semi-direct product $\mathfrak{b}^+ \ltimes (\mathfrak{b}^+)^*$, where $(\mathfrak{b}^+)^*$ is taken as an Abelian Lie algebra and $\mathfrak{b}^+$ acts on $(\mathfrak{b}^+)^*$ using the coadjoint action. A Borel subalgebra is solvable, and its semi-direct product with an Abelian factor remains solvable. Hence $\mathfrak{g}^0_+$ cannot be isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$.

It is an elementary exercise to verify that if $\mathfrak{g} = gl_n$ or $\mathfrak{g} = sl_n$ then the resulting $\mathfrak{g}^{\epsilon}_+$ is indeed $gl^{\epsilon}_{n+}$ or $sl^{\epsilon}_{n+}$ of the previous section.

## References

Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada
*Email address*: drorbn@math.toronto.edu
*URL*: http://www.math.toronto.edu/drorbn

Mathematisch Instituut, Universiteit Leiden, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands
*Email address*: roland.mathematics@gmail.com
*URL*: http://www.rolandvdv.nl/