Pensieve header: A package for computations in general universal enveloping algebras.

Prolog

```
BeginPackage["UEA`"];
Print[
  "UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the
    public domain, available at http://drorbn.net/AcademicPensieve/Projects/UEA/. Dror Bar-Natan
    is committed to support it within reason until March 18, 2022. This is version 170507."];
Print["UEA` implements / extends ",
  Sort@{"**", B, m, SetAlgebra, U, UB, UProducts, USimp, UU, $Basis, $PBWRule},
  "."];
Begin["`Private`"];
```

Implementing general universal enveloping algebras

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≼ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UUi_[1] := Ui[];
UUi_[x_^p_] := UUi @@ Table[x, {p}];
UUi_[ℰ_] := ℰ /. {
    U[xs__] :⇢ Ui[xs],
    x_ /; MemberQ[$Basis, x] :⇢ Ui[x]
   };
UUi_[x_, xs__] := UUt1[x] UUt2[xs] // Expand // mt1,t2→i;
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
ms__[0] = 0;
ms__[x_Plus] := ms /@ x;
ms__[sd_SeriesData] := MapAt[ms, sd, {3, All}];
mi_→j_[ℰ_] := ℰ /. Ui → Uj;
```

```
mi_,j_→k_[c_. Ui_[x___] Uj_[]] := c Uk[x];
mi_,j_→k_[c_. Ui_[] Uj_[y___]] := c Uk[y];
mi_,j_→k_[c_. Ui_[xx___, x_] Uj_[y_, yy___]] := If[x ≼ y,
   c Uk[xx, x, y, yy],
   ((Ui[xx] (Uj[y, x] + UUj[B[x, y]]) // Expand // mi,j→i) Uj[yy] // Expand // mi,j→k) c // USimp
  ];
```

```
UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] :=
  Sort@Flatten@Table[(Ui @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
Supp[ℰ_] := Union@Cases[{ℰ}, Ui_[___] :⇢ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

Epilog

```
End[]; EndPackage[];
```

Predefined Algebras

```
Print["UEA`SetAlgebra knows \"sl2\"."];
```

```
UEA`SetAlgebra knows "sl2".
```

```
SetAlgebra["sl2"] := (
   Print["In sl2: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
   B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
   $Basis = {e, h, f};
   $PBWRule = {e → 1, h → 2, f → 3};
  );
```

Implementing $\mathfrak{tg}^\epsilon = \langle b, e, g, f\rangle / ([g, e] = 2e,\ [g, f] = -2f,\ [e, f] = b + \epsilon g,\ [b, *] = 0)$

```
B[g, e] = 2 e; B[g, f] = -2 f; B[e, f] = b + ϵ g;  B[b, _] = 0;
$Basis = {b, e, g, f};
$PBWRule = {b → 1, e → 2, g → 3, f → 3};

Table[{x, y} → B[x, y], {x, $Basis}, {y, $Basis}] // MatrixForm
```

$$\begin{pmatrix} \{b,b\}\to 0 & \{b,e\}\to 0 & \{b,g\}\to 0 & \{b,f\}\to 0 \\ \{e,b\}\to 0 & \{e,e\}\to 0 & \{e,g\}\to -2e & \{e,f\}\to b+g\epsilon \\ \{g,b\}\to 0 & \{g,e\}\to 2e & \{g,g\}\to 0 & \{g,f\}\to -2f \\ \{f,b\}\to 0 & \{f,e\}\to -b-g\epsilon & \{f,g\}\to 2f & \{f,f\}\to 0 \end{pmatrix}$$

```
Module[{x, y}, Union@Table[{x, y} = t; B[x, y] + B[y, x], {t, Tuples[$Basis, 2]}]]
```

```
{0}
```

```
Module[{x, y, z}, DeleteCases[Table[
   ({x, y, z} = t) → B[x, B[y, z]] + B[y, B[z, x]] + B[z, B[x, y]],
   {t, Tuples[$Basis, 3]}
  ], _ → 0]]
```

```
{}
```

```
Union[(u ↦ m_{1,3→1}[m_{1,2→1}[u]] - m_{1,2→1}[m_{2,3→2}[u]]) /@ UProducts[{1, 2, 3, 4}, 4]]
```

```
{0}
```

```
r_{i_,j_} := USimp[U_i[f] U_j[e] + 1/4 (-ϵ^-1 δ U_i[b] U_j[b] + 2 U_i[b] U_j[g] + ϵ U_i[g] U_j[g]) + α (U_i[b] U_j[g] - U_i[g] U_j[b])];
```

```
r_{1,2}
```

$$-\frac{\delta\, U_1[b]\, U_2[b]}{4\epsilon} - \alpha\, U_1[g]\, U_2[b] + U_1[f]\, U_2[e] + \frac{1}{2} U_1[b]\, U_2[g] + \alpha\, U_1[b]\, U_2[g] + \frac{1}{4}\epsilon\, U_1[g]\, U_2[g]$$

```
r_{1,2} /. {δ → 0, α → -1/4}
```

$$\frac{1}{4} U_1[g]\, U_2[b] + U_1[f]\, U_2[e] + \frac{1}{4} U_1[b]\, U_2[g] + \frac{1}{4}\epsilon\, U_1[g]\, U_2[g]$$

```
r_{1,2} /. {δ → 0, α → -1/2}
```

$$\frac{1}{2} U_1[g]\, U_2[b] + U_1[f]\, U_2[e] + \frac{1}{4}\epsilon\, U_1[g]\, U_2[g]$$

```
UB[r_{1,2}, r_{1,3}] + UB[r_{1,2}, r_{2,3}] + UB[r_{1,3}, r_{2,3}] // USimp
```

```
0
```

```
UB[r_{1,2}, r_{1,3}]
```

$$-2\alpha\, U_1[f]\, U_2[e]\, U_3[b] + 2\alpha\, U_1[f]\, U_2[b]\, U_3[e] - \frac{1}{2}\epsilon\, U_1[f]\, U_2[g]\, U_3[e] + \frac{1}{2}\epsilon\, U_1[f]\, U_2[e]\, U_3[g]$$

```
UB[r_{1,2}, r_{2,3}]
```

$$2\alpha\, U_1[f]\, U_2[e]\, U_3[b] + U_1[f]\, U_2[b]\, U_3[e] - U_1[b]\, U_2[f]\, U_3[e] - 2\alpha\, U_1[b]\, U_2[f]\, U_3[e] - \frac{1}{2}\epsilon\, U_1[g]\, U_2[f]\, U_3[e] + \epsilon\, U_1[f]\, U_2[g]\, U_3[e] - \frac{1}{2}\epsilon\, U_1[f]\, U_2[e]\, U_3[g]$$

```
UB[r_{1,3}, r_{2,3}]
```

$$-U_1[f]\, U_2[b]\, U_3[e] - 2\alpha\, U_1[f]\, U_2[b]\, U_3[e] + U_1[b]\, U_2[f]\, U_3[e] + 2\alpha\, U_1[b]\, U_2[f]\, U_3[e] + \frac{1}{2}\epsilon\, U_1[g]\, U_2[f]\, U_3[e] - \frac{1}{2}\epsilon\, U_1[f]\, U_2[g]\, U_3[e]$$

```
UU_1[f^3, 1, e^2]
```

$$12\epsilon^2\, U_1[f] + 18\epsilon\, U_1[b, f] + 18\epsilon^2\, U_1[g, f] + 6\, U_1[b, b, f] + 12\epsilon\, U_1[b, g, f] - 18\epsilon\, U_1[e, f, f] + 6\epsilon^2\, U_1[g, g, f] - 6\, U_1[b, e, f, f] - 6\epsilon\, U_1[e, g, f, f] + U_1[e, e, f, f, f]$$