

Pensieve header: A package for computations in general universal enveloping algebras.

Prolog

```
BeginPackage["UEA`"];
Print[
  "UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the
  public domain, available at http://drorbn.net/AcademicPensieve/Projects/UEA/. Dror Bar-Natan
  is committed to support it within reason until March 18, 2022. This is version 170423."];
Print["UEA` implements / extends ",
  Sort@{"**", B, m, SetAlgebra, U, UB, UProducts, USimp, UU, $Basis, $PBWRule},
  "."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/UEA/>. Dror Bar-Natan is committed to support it within reason until March 18, 2022. This is version 170423.

UEA` implements / extends {**, B, m, SetAlgebra, U, UB, UProducts, USimp, UU, \$Basis, \$PBWRule}.

Implementing general universal enveloping algebras

```
B[0, _] = 0; B[_, 0] = 0;
B[c_*x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_*x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i[_] := ε /. {
  U[xx_] => U_i[xx],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
USimp[_] := Collect[ε, Times[U[___] ..], Expand];
USimp[ε] := Expand[ε];
```

```
m_s_[0] = 0; m_s_[x_Plus] := m_s_ /@ x;
m_i->j[_] := ε /. U_i → U_j;
```

```
m_i,j->k[c_*U_i[x___] U_j[]] := c U_k[x];
m_i,j->k[c_*U_i[] U_j[y___]] := c U_k[y];
m_i,j->k[c_*U_i[xx___, x_] U_j[y_, yy___]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i U_j[yy] // Expand // m_i,j->k) c // USimp
];
```

```
UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] :=
  Sort@Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
S[_] := Union@Cases[ε, U_i[___] => i, ∞];
```

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = S[x] ∩ S[y], σ, z},
  z = x; Do[z = mi→σ[i][z], {i, is}];
  z = Expand[y z]; Do[z = mσ[i], i→i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];

```

Epilog

```
End[]; EndPackage[];
```

Predefined Algebras

```
Print["UEA`SetAlgebra knows \"s12\"."];
```

UEA`SetAlgebra knows "s12".

```

SetAlgebra["s12"] := (
  Print["In s12: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
);

```

Implementing $tg^\epsilon = \langle b, e, g, f \rangle / ([g, e] = 2e, [g, f] = -2f, [e, f] = b + \epsilon g, [b, *] = 0)$

```

B[g, e] = 2 e; B[g, f] = -2 f; B[e, f] = b + ε g; B[b, _] = 0;
$Basis = {b, e, g, f};
$PBWRule = {b → 1, e → 2, g → 3, f → 4};

```

```
Table[{x, y} → B[x, y], {x, $Basis}, {y, $Basis}] // MatrixForm
```

$$\begin{pmatrix} \{b, b\} \rightarrow 0 & \{b, e\} \rightarrow 0 & \{b, g\} \rightarrow 0 & \{b, f\} \rightarrow 0 \\ \{e, b\} \rightarrow 0 & \{e, e\} \rightarrow 0 & \{e, g\} \rightarrow -2e & \{e, f\} \rightarrow b + g\epsilon \\ \{g, b\} \rightarrow 0 & \{g, e\} \rightarrow 2e & \{g, g\} \rightarrow 0 & \{g, f\} \rightarrow -2f \\ \{f, b\} \rightarrow 0 & \{f, e\} \rightarrow -b - g\epsilon & \{f, g\} \rightarrow 2f & \{f, f\} \rightarrow 0 \end{pmatrix}$$

```
Module[{x, y}, Union@Table[{x, y} = t; B[x, y] + B[y, x], {t, Tuples[$Basis, 2]}]]
{0}
```

```

Module[{x, y, z}, DeleteCases[Table[
  ({x, y, z} = t) → B[x, B[y, z]] + B[y, B[z, x]] + B[z, B[x, y]],
  {t, Tuples[$Basis, 3]}
], _ → 0]]
{0}

```

```

Union[{u → m1,3→1[m1,2→1[u]] - m1,2→1[m2,3→2[u]]} /@ UProducts[{1, 2, 3, 4}, 4]]
{0}

```

$$\mathbf{r}_{i,j} := \text{USimp}[U_i[f] U_j[e] + \frac{1}{4} (-\epsilon^{-1} \delta U_i[b] U_j[b] + 2 U_i[b] U_j[g] + \epsilon U_i[g] U_j[g]) + \alpha (U_i[b] U_j[g] - U_i[g] U_j[b])];$$

 $\mathbf{r}_{1,2}$

$$- \frac{\delta U_1[b] U_2[b]}{4 \epsilon} - \alpha U_1[g] U_2[b] + U_1[f] U_2[e] + \frac{1}{2} U_1[b] U_2[g] + \alpha U_1[b] U_2[g] + \frac{1}{4} \epsilon U_1[g] U_2[g]$$

$$\mathbf{r}_{1,2} /. \{\delta \rightarrow 0, \alpha \rightarrow \frac{-1}{4}\}$$

$$\frac{1}{4} U_1[g] U_2[b] + U_1[f] U_2[e] + \frac{1}{4} U_1[b] U_2[g] + \frac{1}{4} \epsilon U_1[g] U_2[g]$$

$$\mathbf{r}_{1,2} /. \{\delta \rightarrow 0, \alpha \rightarrow \frac{-1}{2}\}$$

$$\frac{1}{2} U_1[g] U_2[b] + U_1[f] U_2[e] + \frac{1}{4} \epsilon U_1[g] U_2[g]$$

 $\text{UB}[\mathbf{r}_{1,2}, \mathbf{r}_{1,3}] + \text{UB}[\mathbf{r}_{1,2}, \mathbf{r}_{2,3}] + \text{UB}[\mathbf{r}_{1,3}, \mathbf{r}_{2,3}] // \text{USimp}$
 0
 $\text{UB}[\mathbf{r}_{1,2}, \mathbf{r}_{1,3}]$

$$-2 \alpha U_1[f] U_2[e] U_3[b] + 2 \alpha U_1[f] U_2[b] U_3[e] - \frac{1}{2} \epsilon U_1[f] U_2[g] U_3[e] + \frac{1}{2} \epsilon U_1[f] U_2[e] U_3[g]$$

 $\text{UB}[\mathbf{r}_{1,2}, \mathbf{r}_{2,3}]$

$$2 \alpha U_1[f] U_2[e] U_3[b] + U_1[f] U_2[b] U_3[e] - U_1[b] U_2[f] U_3[e] - 2 \alpha U_1[b] U_2[f] U_3[e] - \frac{1}{2} \epsilon U_1[g] U_2[f] U_3[e] + \epsilon U_1[f] U_2[g] U_3[e] - \frac{1}{2} \epsilon U_1[f] U_2[e] U_3[g]$$

 $\text{UB}[\mathbf{r}_{1,3}, \mathbf{r}_{2,3}]$

$$-U_1[f] U_2[b] U_3[e] - 2 \alpha U_1[f] U_2[b] U_3[e] + U_1[b] U_2[f] U_3[e] + 2 \alpha U_1[b] U_2[f] U_3[e] + \frac{1}{2} \epsilon U_1[g] U_2[f] U_3[e] - \frac{1}{2} \epsilon U_1[f] U_2[g] U_3[e]$$