Pensieve header: Quantum \$sl_n\$ following Rosso and Yamane.

Notes.

1. Rosso is "An Analogue of P.B.W. Theorem and the Universal \$R\$-matrix for \$U_\hbar(sl(N+1))\$'', Yamane is "A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N".
2. We call our algebra $U_q(\mathrm{sl}_{\$n})$, not $U_\hbar(\mathrm{sl}(N+1))$.
3. We dot not have a name for Rosso's X_i and Y_j. Rosso's $E_{\epsilon_i-\epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow$ xij. Likewise Rosso's $\eta_{\epsilon_i-\epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow$ yij.
4. Rosso's H_i is our a_i and Rosso's ξ_i is our b_i.

```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
x[i_, j_] := x <> i <> j;
a[i_] := a <> i;
y[i_, j_] := y <> i <> j;
b[i_] := b <> i;
```

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ⊠ y) ⊠ z;
0 ⊠ _ = _ ⊠ 0 = 0;
(x_Plus) ⊠ y_ := (# ⊠ y) & /@ x; x_ ⊠ (y_Plus) := (x ⊠ #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ⊠ y - y ⊠ x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

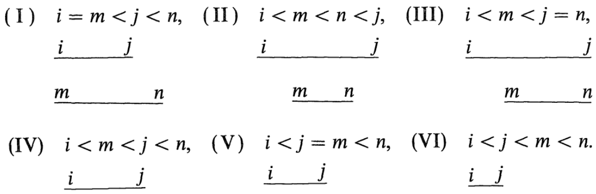
The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xayb.

```
$n = 5;
PBWGens = Flatten@{
   Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
   Table[As[i, 1, $n - 1], a[i]],
   Table[As[i, 1, $n - 1], As[j, i + 1, $n], y[i, j]],
   Table[As[i, 1, $n - 1], b[i]]
  }
(#_U = U@#) & /@ PBWGens;
SortingRule = Module[{k = 0}, Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, PBWGens}]];
```

```
{x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2,
 a3, a4, y12, y13, y14, y15, y23, y24, y25, y34, y35, y45, b1, b2, b3, b4}
```

```
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ι[s_Symbol] := Rest[Characters[SymbolName[s]]] /.
   {"0" → 0, "1" → 1, "2" → 2, "3" → 3, "4" → 4, "5" → 5, "6" → 6, "7" → 7, "8" → 8, "9" → 9};
```

We take the reduction rules RR results from Yamane page 509:



$$q^{-2} e_{ij} e_{mn} - e_{mn} e_{ij} = 0 \qquad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{III})}.$$
$$[e_{ij}, e_{mn}] = 0 \qquad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{II})} \bigcup C_{(\mathrm{VI})}.$$
$$[e_{ij}, e_{mn}] = (q^2 - q^{-2}) e_{in} e_{mj} \qquad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{IV})}.$$

```
RR[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = ι@xij; {m, n} = ι@xmn;
  Which[
   i == m ∧ j < n, q^-2 U[xij, xmn], (* I *)
   i == m ∧ j == n, U[xij, xmn],
   i < m < j ∧ n < j, U[xij, xmn], (* II *)
   i < m < j ∧ n == j, q^-2 U[xij, xmn], (* III *)
   i < m < j ∧ n > j, U[xij, xmn] + (q^-2 - q^2) U[x[i, n], x[m, j]], (* IV *)
   j == m, q^2 U[xij, xmn] - q U[x[i, n]], (* V *)
   j < m, U[xij, xmn] (* VI *)
  ]
 ]
```

```
1_U = U[];
x_ ⊠ (c_. 1_U) := CF[c x]; (c_. 1_U) ⊠ x_ := CF[c x];
(a_. U[xx___, x_]) ⊠ (b_. U[y_, yy___]) := Module[{M},
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[A_, X_] := A X;
  If[OrderedQ[{x, y} /. SortingRule],
   CF@M[a b, U[xx, x, y, yy]],
   U@xx ⊠ CF@M[a b, RR[x, y]] ⊠ U@yy ]
 ]
```