

Pensieve header: The PBW multiplication tensor for upper nilpotent matrices using the λ -tangent formalism. Some material from pensieve://Projects/UEA.

```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

Prolog

```
In[ ]:= BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**", B, m, SetAlgebra, U, UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ },
"."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
UEA` implements / extends {**, B, m, SetAlgebra, U, UB, UProducts, USimp, UU, δ , \$Basis, \$PBWRule}.

Utilities

```
In[ ]:=  $\delta_{i,j}$  := If[i == j, 1, 0];
```

Implementing general universal enveloping algebras

```
In[ ]:= B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
In[*]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_p-] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs_] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
In[*]:= m_s_[0] = 0;
m_s_[x_Plus] := m_s_/@x;
m_s_[sd_SeriesData] := MapAt[m_s_, sd, {3, All}];
m_i->j_[ε_] := ε /. U_i -> U_j;
```

```
In[*]:= m_i,j->k_[c_. U_i[x_] U_j[]] := c U_k[x];
m_i,j->k_[c_. U_i[] U_j[y_]] := c U_k[y];
m_i,j->k_[c_. U_i[xx_, x_] U_j[y_, yy_]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i) U_j[yy] // Expand // m_i,j->k)
  c // USimp
];
```

```
In[*]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is_}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[is, d - j]}];
```

```
In[*]:= Supp[ε_] := Union@Cases[ε, U_i[___] => i, ∞];
```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = m_i->σ@i[z], {i, is}];
  z = Expand[y z]; Do[z = m_σ@i,i->i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

Epilog

```
In[*]:= End[]; EndPackage[];
```

Predefined Algebras

sl(2)

```
In[*]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
In[*]:= SetAlgebra["sl2"] := (
  Print["In sl2: <e,h,f>/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Basis = {e, h, f};
  $PBWRule = {e -> 1, h -> 2, f -> 3};
);
```

un_n

```
In[*]:= Print["UEA`SetAlgebra knows the nilpotent algebra un_n."];
```

UEA`SetAlgebra knows the nilpotent algebra un_n.

```
In[*]:= SetAlgebra[un_n_] := (
  $Basis = Flatten@Table[xα,β, {β, 2, n}, {α, 1, β - 1}];
  $PBWRule = Thread[$Basis -> Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
);
```

```
In[*]:= SetAlgebra[un5];
$PBWRule
MatrixForm@Table[{b1, b2} -> B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

```
Out[*]:= {x1,2 -> 1, x1,3 -> 2, x2,3 -> 3, x1,4 -> 4, x2,4 -> 5, x3,4 -> 6, x1,5 -> 7, x2,5 -> 8, x3,5 -> 9, x4,5 -> 10}
```

```
Out[*]//MatrixForm=
```

{x _{1,2} , x _{1,2} } -> 0	{x _{1,2} , x _{1,3} } -> 0	{x _{1,2} , x _{2,3} } -> x _{1,3}	{x _{1,2} , x _{1,4} } -> 0	{x _{1,2} , x _{2,4} } -> x _{1,4}
{x _{1,3} , x _{1,2} } -> 0	{x _{1,3} , x _{1,3} } -> 0	{x _{1,3} , x _{2,3} } -> 0	{x _{1,3} , x _{1,4} } -> 0	{x _{1,3} , x _{2,4} } -> 0
{x _{2,3} , x _{1,2} } -> -x _{1,3}	{x _{2,3} , x _{1,3} } -> 0	{x _{2,3} , x _{2,3} } -> 0	{x _{2,3} , x _{1,4} } -> 0	{x _{2,3} , x _{2,4} } -> 0
{x _{1,4} , x _{1,2} } -> 0	{x _{1,4} , x _{1,3} } -> 0	{x _{1,4} , x _{2,3} } -> 0	{x _{1,4} , x _{1,4} } -> 0	{x _{1,4} , x _{2,4} } -> 0
{x _{2,4} , x _{1,2} } -> -x _{1,4}	{x _{2,4} , x _{1,3} } -> 0	{x _{2,4} , x _{2,3} } -> 0	{x _{2,4} , x _{1,4} } -> 0	{x _{2,4} , x _{2,4} } -> 0
{x _{3,4} , x _{1,2} } -> 0	{x _{3,4} , x _{1,3} } -> -x _{1,4}	{x _{3,4} , x _{2,3} } -> -x _{2,4}	{x _{3,4} , x _{1,4} } -> 0	{x _{3,4} , x _{2,4} } -> 0
{x _{1,5} , x _{1,2} } -> 0	{x _{1,5} , x _{1,3} } -> 0	{x _{1,5} , x _{2,3} } -> 0	{x _{1,5} , x _{1,4} } -> 0	{x _{1,5} , x _{2,4} } -> 0
{x _{2,5} , x _{1,2} } -> -x _{1,5}	{x _{2,5} , x _{1,3} } -> 0	{x _{2,5} , x _{2,3} } -> 0	{x _{2,5} , x _{1,4} } -> 0	{x _{2,5} , x _{2,4} } -> 0
{x _{3,5} , x _{1,2} } -> 0	{x _{3,5} , x _{1,3} } -> -x _{1,5}	{x _{3,5} , x _{2,3} } -> -x _{2,5}	{x _{3,5} , x _{1,4} } -> 0	{x _{3,5} , x _{2,4} } -> 0
{x _{4,5} , x _{1,2} } -> 0	{x _{4,5} , x _{1,3} } -> 0	{x _{4,5} , x _{2,3} } -> 0	{x _{4,5} , x _{1,4} } -> -x _{1,5}	{x _{4,5} , x _{2,4} } -> -x _{2,5}

```
In[*]:= x0 = ($Basis /. x -> xi) . $Basis;
y0 = ($Basis /. x -> eta) . $Basis
```

Out[*]=

$$x_{1,2} \eta_{1,2} + x_{1,3} \eta_{1,3} + x_{1,4} \eta_{1,4} + x_{1,5} \eta_{1,5} + x_{2,3} \eta_{2,3} + x_{2,4} \eta_{2,4} + x_{2,5} \eta_{2,5} + x_{3,4} \eta_{3,4} + x_{3,5} \eta_{3,5} + x_{4,5} \eta_{4,5}$$

```
In[*]:= adExp[x_][y_] := Expand@Module[{A, b, c},
  A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
  $Basis.MatrixExp[A].Table[Coefficient[y, b], {b, $Basis}]
]
```

```
In[*]:= adExp[x0][y0]
```

Out[*]=

$$\begin{aligned} & x_{1,2} \eta_{1,2} + x_{1,3} \eta_{1,3} + x_{1,4} \eta_{1,4} + x_{1,5} \eta_{1,5} + x_{2,3} \eta_{2,3} + x_{2,4} \eta_{2,4} + x_{2,5} \eta_{2,5} + x_{3,4} \eta_{3,4} + x_{3,5} \eta_{3,5} + x_{4,5} \eta_{4,5} - \\ & x_{1,3} \eta_{2,3} \xi_{1,2} - x_{1,4} \eta_{2,4} \xi_{1,2} - x_{1,5} \eta_{2,5} \xi_{1,2} - x_{1,4} \eta_{3,4} \xi_{1,3} - x_{1,5} \eta_{3,5} \xi_{1,3} - x_{1,5} \eta_{4,5} \xi_{1,4} + \\ & x_{1,3} \eta_{1,2} \xi_{2,3} - x_{2,4} \eta_{3,4} \xi_{2,3} - x_{2,5} \eta_{3,5} \xi_{2,3} + \frac{1}{2} x_{1,4} \eta_{3,4} \xi_{1,2} \xi_{2,3} + \frac{1}{2} x_{1,5} \eta_{3,5} \xi_{1,2} \xi_{2,3} + \\ & x_{1,4} \eta_{1,2} \xi_{2,4} - x_{2,5} \eta_{4,5} \xi_{2,4} + \frac{1}{2} x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,4} + x_{1,5} \eta_{1,2} \xi_{2,5} + x_{1,4} \eta_{1,3} \xi_{3,4} + \\ & x_{2,4} \eta_{2,3} \xi_{3,4} - x_{3,5} \eta_{4,5} \xi_{3,4} - x_{1,4} \eta_{2,3} \xi_{1,2} \xi_{3,4} + \frac{1}{2} x_{1,5} \eta_{4,5} \xi_{1,3} \xi_{3,4} + \frac{1}{2} x_{1,4} \eta_{1,2} \xi_{2,3} \xi_{3,4} + \\ & \frac{1}{2} x_{2,5} \eta_{4,5} \xi_{2,3} \xi_{3,4} - \frac{1}{6} x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + x_{1,5} \eta_{1,3} \xi_{3,5} + x_{2,5} \eta_{2,3} \xi_{3,5} - x_{1,5} \eta_{2,3} \xi_{1,2} \xi_{3,5} + \\ & \frac{1}{2} x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,5} + x_{1,5} \eta_{1,4} \xi_{4,5} + x_{2,5} \eta_{2,4} \xi_{4,5} + x_{3,5} \eta_{3,4} \xi_{4,5} - x_{1,5} \eta_{2,4} \xi_{1,2} \xi_{4,5} - \\ & x_{1,5} \eta_{3,4} \xi_{1,3} \xi_{4,5} - x_{2,5} \eta_{3,4} \xi_{2,3} \xi_{4,5} + \frac{1}{2} x_{1,5} \eta_{3,4} \xi_{1,2} \xi_{2,3} \xi_{4,5} + \frac{1}{2} x_{1,5} \eta_{1,2} \xi_{2,4} \xi_{4,5} + \\ & \frac{1}{2} x_{1,5} \eta_{1,3} \xi_{3,4} \xi_{4,5} + \frac{1}{2} x_{2,5} \eta_{2,3} \xi_{3,4} \xi_{4,5} - \frac{1}{2} x_{1,5} \eta_{2,3} \xi_{1,2} \xi_{3,4} \xi_{4,5} + \frac{1}{6} x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

```
In[*]:= Coefficient[adExp[x0][y0], x1,5]
```

Out[*]=

$$\begin{aligned} & \eta_{1,5} - \eta_{2,5} \xi_{1,2} - \eta_{3,5} \xi_{1,3} - \eta_{4,5} \xi_{1,4} + \frac{1}{2} \eta_{3,5} \xi_{1,2} \xi_{2,3} + \frac{1}{2} \eta_{4,5} \xi_{1,2} \xi_{2,4} + \eta_{1,2} \xi_{2,5} + \frac{1}{2} \eta_{4,5} \xi_{1,3} \xi_{3,4} - \\ & \frac{1}{6} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \eta_{1,3} \xi_{3,5} - \eta_{2,3} \xi_{1,2} \xi_{3,5} + \frac{1}{2} \eta_{1,2} \xi_{2,3} \xi_{3,5} + \eta_{1,4} \xi_{4,5} - \eta_{2,4} \xi_{1,2} \xi_{4,5} - \eta_{3,4} \xi_{1,3} \xi_{4,5} + \\ & \frac{1}{2} \eta_{3,4} \xi_{1,2} \xi_{2,3} \xi_{4,5} + \frac{1}{2} \eta_{1,2} \xi_{2,4} \xi_{4,5} + \frac{1}{2} \eta_{1,3} \xi_{3,4} \xi_{4,5} - \frac{1}{2} \eta_{2,3} \xi_{1,2} \xi_{3,4} \xi_{4,5} + \frac{1}{6} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

```
In[*]:= lambdaTangent[] = 0;
```

```
lambdaTangent[xs___, x_] := adExp[x][lambdaTangent[xs]] + partial_lambda x;
```

```
lambdaTangent[xs_List] := lambdaTangent@@xs
```

```
In[*]:= $Basis ($Basis /. x -> xi)
```

Out[*]=

$$\{x_{1,2} \xi_{1,2}, x_{1,3} \xi_{1,3}, x_{2,3} \xi_{2,3}, x_{1,4} \xi_{1,4}, x_{2,4} \xi_{2,4}, x_{3,4} \xi_{3,4}, x_{1,5} \xi_{1,5}, x_{2,5} \xi_{2,5}, x_{3,5} \xi_{3,5}, x_{4,5} \xi_{4,5}\}$$

In[*]:= $\lambda \text{Tangent} @ (\lambda \$\text{Basis} (\$Basis /. x \rightarrow \xi))$

Out[*]=

$$\begin{aligned} & x_{1,2} \xi_{1,2} + x_{1,3} \xi_{1,3} + x_{1,4} \xi_{1,4} + x_{1,5} \xi_{1,5} + x_{2,3} \xi_{2,3} + \lambda x_{1,3} \xi_{1,2} \xi_{2,3} + \\ & x_{2,4} \xi_{2,4} + \lambda x_{1,4} \xi_{1,2} \xi_{2,4} + x_{2,5} \xi_{2,5} + \lambda x_{1,5} \xi_{1,2} \xi_{2,5} + x_{3,4} \xi_{3,4} + \lambda x_{1,4} \xi_{1,3} \xi_{3,4} + \\ & \lambda x_{2,4} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + x_{3,5} \xi_{3,5} + \lambda x_{1,5} \xi_{1,3} \xi_{3,5} + \lambda x_{2,5} \xi_{2,3} \xi_{3,5} + \\ & \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + x_{4,5} \xi_{4,5} + \lambda x_{1,5} \xi_{1,4} \xi_{4,5} + \lambda x_{2,5} \xi_{2,4} \xi_{4,5} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \\ & \lambda x_{3,5} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= $\text{lhs} = \text{Join}[\lambda \$\text{Basis} (\$Basis /. x \rightarrow \xi), \lambda \$\text{Basis} (\$Basis /. x \rightarrow \eta)] // \lambda \text{Tangent}$

Out[*]=

$$\begin{aligned} & x_{1,2} \eta_{1,2} + x_{1,3} \eta_{1,3} + x_{1,4} \eta_{1,4} + x_{1,5} \eta_{1,5} + x_{2,3} \eta_{2,3} + \lambda x_{1,3} \eta_{1,2} \eta_{2,3} + x_{2,4} \eta_{2,4} + \lambda x_{1,4} \eta_{1,2} \eta_{2,4} + \\ & x_{2,5} \eta_{2,5} + \lambda x_{1,5} \eta_{1,2} \eta_{2,5} + x_{3,4} \eta_{3,4} + \lambda x_{1,4} \eta_{1,3} \eta_{3,4} + \lambda x_{2,4} \eta_{2,3} \eta_{3,4} + \lambda^2 x_{1,4} \eta_{1,2} \eta_{2,3} \eta_{3,4} + \\ & x_{3,5} \eta_{3,5} + \lambda x_{1,5} \eta_{1,3} \eta_{3,5} + \lambda x_{2,5} \eta_{2,3} \eta_{3,5} + \lambda^2 x_{1,5} \eta_{1,2} \eta_{2,3} \eta_{3,5} + x_{4,5} \eta_{4,5} + \lambda x_{1,5} \eta_{1,4} \eta_{4,5} + \\ & \lambda x_{2,5} \eta_{2,4} \eta_{4,5} + \lambda^2 x_{1,5} \eta_{1,2} \eta_{2,4} \eta_{4,5} + \lambda x_{3,5} \eta_{3,4} \eta_{4,5} + \lambda^2 x_{1,5} \eta_{1,3} \eta_{3,4} \eta_{4,5} + \lambda^2 x_{2,5} \eta_{2,3} \eta_{3,4} \eta_{4,5} + \\ & \lambda^3 x_{1,5} \eta_{1,2} \eta_{2,3} \eta_{3,4} \eta_{4,5} + x_{1,2} \xi_{1,2} + \lambda x_{1,3} \eta_{2,3} \xi_{1,2} + \lambda x_{1,4} \eta_{2,4} \xi_{1,2} + \lambda x_{1,5} \eta_{2,5} \xi_{1,2} + \\ & \lambda^2 x_{1,4} \eta_{2,3} \eta_{3,4} \xi_{1,2} + \lambda^2 x_{1,5} \eta_{2,3} \eta_{3,5} \xi_{1,2} + \lambda^2 x_{1,5} \eta_{2,4} \eta_{4,5} \xi_{1,2} + \lambda^3 x_{1,5} \eta_{2,3} \eta_{3,4} \eta_{4,5} \xi_{1,2} + \\ & x_{1,3} \xi_{1,3} + \lambda x_{1,4} \eta_{3,4} \xi_{1,3} + \lambda x_{1,5} \eta_{3,5} \xi_{1,3} + \lambda^2 x_{1,5} \eta_{3,4} \eta_{4,5} \xi_{1,3} + x_{1,4} \xi_{1,4} + \lambda x_{1,5} \eta_{4,5} \xi_{1,4} + \\ & x_{1,5} \xi_{1,5} + x_{2,3} \xi_{2,3} - \lambda x_{1,3} \eta_{1,2} \xi_{2,3} + \lambda x_{2,4} \eta_{3,4} \xi_{2,3} - \lambda^2 x_{1,4} \eta_{1,2} \eta_{3,4} \xi_{2,3} + \lambda x_{2,5} \eta_{3,5} \xi_{2,3} - \\ & \lambda^2 x_{1,5} \eta_{1,2} \eta_{3,5} \xi_{2,3} + \lambda^2 x_{2,5} \eta_{3,4} \eta_{4,5} \xi_{2,3} - \lambda^3 x_{1,5} \eta_{1,2} \eta_{3,4} \eta_{4,5} \xi_{2,3} + \lambda x_{1,3} \xi_{1,2} \xi_{2,3} + \\ & \lambda^2 x_{1,4} \eta_{3,4} \xi_{1,2} \xi_{2,3} + \lambda^2 x_{1,5} \eta_{3,5} \xi_{1,2} \xi_{2,3} + \lambda^3 x_{1,5} \eta_{3,4} \eta_{4,5} \xi_{1,2} \xi_{2,3} + x_{2,4} \xi_{2,4} - \lambda x_{1,4} \eta_{1,2} \xi_{2,4} + \\ & \lambda x_{2,5} \eta_{4,5} \xi_{2,4} - \lambda^2 x_{1,5} \eta_{1,2} \eta_{4,5} \xi_{2,4} + \lambda x_{1,4} \xi_{1,2} \xi_{2,4} + \lambda^2 x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,4} + x_{2,5} \xi_{2,5} - \\ & \lambda x_{1,5} \eta_{1,2} \xi_{2,5} + \lambda x_{1,5} \xi_{1,2} \xi_{2,5} + x_{3,4} \xi_{3,4} - \lambda x_{1,4} \eta_{1,3} \xi_{3,4} - \lambda x_{2,4} \eta_{2,3} \xi_{3,4} + \lambda x_{3,5} \eta_{4,5} \xi_{3,4} - \\ & \lambda^2 x_{1,5} \eta_{1,3} \eta_{4,5} \xi_{3,4} - \lambda^2 x_{2,5} \eta_{2,3} \eta_{4,5} \xi_{3,4} + \lambda x_{1,4} \xi_{1,3} \xi_{3,4} + \lambda^2 x_{1,5} \eta_{4,5} \xi_{1,3} \xi_{3,4} + \lambda x_{2,4} \xi_{2,3} \xi_{3,4} - \\ & \lambda^2 x_{1,4} \eta_{1,2} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{2,5} \eta_{4,5} \xi_{2,3} \xi_{3,4} - \lambda^3 x_{1,5} \eta_{1,2} \eta_{4,5} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \\ & \lambda^3 x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + x_{3,5} \xi_{3,5} - \lambda x_{1,5} \eta_{1,3} \xi_{3,5} - \lambda x_{2,5} \eta_{2,3} \xi_{3,5} + \lambda x_{1,5} \xi_{1,3} \xi_{3,5} + \\ & \lambda x_{2,5} \xi_{2,3} \xi_{3,5} - \lambda^2 x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + x_{4,5} \xi_{4,5} - \lambda x_{1,5} \eta_{1,4} \xi_{4,5} - \\ & \lambda x_{2,5} \eta_{2,4} \xi_{4,5} - \lambda x_{3,5} \eta_{3,4} \xi_{4,5} + \lambda x_{1,5} \xi_{1,4} \xi_{4,5} + \lambda x_{2,5} \xi_{2,4} \xi_{4,5} - \lambda^2 x_{1,5} \eta_{1,2} \xi_{2,4} \xi_{4,5} + \\ & \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \lambda x_{3,5} \xi_{3,4} \xi_{4,5} - \lambda^2 x_{1,5} \eta_{1,3} \xi_{3,4} \xi_{4,5} - \lambda^2 x_{2,5} \eta_{2,3} \xi_{3,4} \xi_{4,5} + \\ & \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} - \lambda^3 x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= $\text{rhs} = \lambda \text{Tangent} @ (\$Basis (\$Basis /. x_{\alpha\beta} \rightarrow f_{\alpha\beta}[\lambda]))$

Out[*]=

$$\begin{aligned} & x_{1,2} f_{1,2}'[\lambda] + x_{1,3} f_{2,3}[\lambda] f_{1,2}'[\lambda] + x_{1,4} f_{2,4}[\lambda] f_{1,2}'[\lambda] + x_{1,5} f_{2,5}[\lambda] f_{1,2}'[\lambda] + \\ & x_{1,4} f_{2,3}[\lambda] f_{3,4}[\lambda] f_{1,2}'[\lambda] + x_{1,5} f_{2,3}[\lambda] f_{3,5}[\lambda] f_{1,2}'[\lambda] + x_{1,5} f_{2,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] + \\ & x_{1,5} f_{2,3}[\lambda] f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] + x_{1,3} f_{1,3}'[\lambda] + x_{1,4} f_{3,4}[\lambda] f_{1,3}'[\lambda] + x_{1,5} f_{3,5}[\lambda] f_{1,3}'[\lambda] + \\ & x_{1,5} f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,3}'[\lambda] + x_{1,4} f_{1,4}'[\lambda] + x_{1,5} f_{4,5}[\lambda] f_{1,4}'[\lambda] + x_{1,5} f_{1,5}'[\lambda] + x_{2,3} f_{2,3}'[\lambda] + \\ & x_{2,4} f_{3,4}[\lambda] f_{2,3}'[\lambda] + x_{2,5} f_{3,5}[\lambda] f_{2,3}'[\lambda] + x_{2,5} f_{3,4}[\lambda] f_{4,5}[\lambda] f_{2,3}'[\lambda] + x_{2,4} f_{2,4}'[\lambda] + \\ & x_{2,5} f_{4,5}[\lambda] f_{2,4}'[\lambda] + x_{2,5} f_{2,5}'[\lambda] + x_{3,4} f_{3,4}'[\lambda] + x_{3,5} f_{4,5}[\lambda] f_{3,4}'[\lambda] + x_{3,5} f_{3,5}'[\lambda] + x_{4,5} f_{4,5}'[\lambda] \end{aligned}$$

In[*]:= (Coefficient[lhs - rhs, #] == 0) & /@ \$Basis

Out[*]=

$$\left\{ \begin{aligned} &\eta_{1,2} + \xi_{1,2} - f_{1,2}'[\lambda] = 0, \\ &\eta_{1,3} + \lambda \eta_{1,2} \eta_{2,3} + \lambda \eta_{2,3} \xi_{1,2} + \xi_{1,3} - \lambda \eta_{1,2} \xi_{2,3} + \lambda \xi_{1,2} \xi_{2,3} - f_{2,3}[\lambda] f_{1,2}'[\lambda] - f_{1,3}'[\lambda] = 0, \\ &\eta_{2,3} + \xi_{2,3} - f_{2,3}'[\lambda] = 0, \\ &\eta_{1,4} + \lambda \eta_{1,2} \eta_{2,4} + \lambda \eta_{1,3} \eta_{3,4} + \lambda^2 \eta_{1,2} \eta_{2,3} \eta_{3,4} + \lambda \eta_{2,4} \xi_{1,2} + \lambda^2 \eta_{2,3} \eta_{3,4} \xi_{1,2} + \lambda \eta_{3,4} \xi_{1,3} + \xi_{1,4} - \\ &\quad \lambda^2 \eta_{1,2} \eta_{3,4} \xi_{2,3} + \lambda^2 \eta_{3,4} \xi_{1,2} \xi_{2,3} - \lambda \eta_{1,2} \xi_{2,4} + \lambda \xi_{1,2} \xi_{2,4} - \lambda \eta_{1,3} \xi_{3,4} + \lambda \xi_{1,3} \xi_{3,4} - \lambda^2 \eta_{1,2} \xi_{2,3} \xi_{3,4} + \\ &\quad \lambda^2 \xi_{1,2} \xi_{2,3} \xi_{3,4} - f_{2,4}[\lambda] f_{1,2}'[\lambda] - f_{2,3}[\lambda] f_{3,4}[\lambda] f_{1,2}'[\lambda] - f_{3,4}[\lambda] f_{1,3}'[\lambda] - f_{1,4}'[\lambda] = 0, \\ &\eta_{2,4} + \lambda \eta_{2,3} \eta_{3,4} + \lambda \eta_{3,4} \xi_{2,3} + \xi_{2,4} - \lambda \eta_{2,3} \xi_{3,4} + \lambda \xi_{2,3} \xi_{3,4} - f_{3,4}[\lambda] f_{2,3}'[\lambda] - f_{2,4}'[\lambda] = 0, \\ &\eta_{3,4} + \xi_{3,4} - f_{3,4}'[\lambda] = 0, \\ &\eta_{1,5} + \lambda \eta_{1,2} \eta_{2,5} + \lambda \eta_{1,3} \eta_{3,5} + \lambda^2 \eta_{1,2} \eta_{2,3} \eta_{3,5} + \lambda \eta_{1,4} \eta_{4,5} + \lambda^2 \eta_{1,2} \eta_{2,4} \eta_{4,5} + \lambda^2 \eta_{1,3} \eta_{3,4} \eta_{4,5} + \\ &\quad \lambda^3 \eta_{1,2} \eta_{2,3} \eta_{3,4} \eta_{4,5} + \lambda \eta_{2,5} \xi_{1,2} + \lambda^2 \eta_{2,3} \eta_{3,5} \xi_{1,2} + \lambda^2 \eta_{2,4} \eta_{4,5} \xi_{1,2} + \lambda^3 \eta_{2,3} \eta_{3,4} \eta_{4,5} \xi_{1,2} + \\ &\quad \lambda \eta_{3,5} \xi_{1,3} + \lambda^2 \eta_{3,4} \eta_{4,5} \xi_{1,3} + \lambda \eta_{4,5} \xi_{1,4} + \xi_{1,5} - \lambda^2 \eta_{1,2} \eta_{3,5} \xi_{2,3} - \lambda^3 \eta_{1,2} \eta_{3,4} \eta_{4,5} \xi_{2,3} + \\ &\quad \lambda^2 \eta_{3,5} \xi_{1,2} \xi_{2,3} + \lambda^3 \eta_{3,4} \eta_{4,5} \xi_{1,2} \xi_{2,3} - \lambda^2 \eta_{1,2} \eta_{4,5} \xi_{2,4} + \lambda^2 \eta_{4,5} \xi_{1,2} \xi_{2,4} - \\ &\quad \lambda \eta_{1,2} \xi_{2,5} + \lambda \xi_{1,2} \xi_{2,5} - \lambda^2 \eta_{1,3} \eta_{4,5} \xi_{3,4} + \lambda^2 \eta_{4,5} \xi_{1,3} \xi_{3,4} - \lambda^3 \eta_{1,2} \eta_{4,5} \xi_{2,3} \xi_{3,4} + \\ &\quad \lambda^3 \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} - \lambda \eta_{1,3} \xi_{3,5} + \lambda \xi_{1,3} \xi_{3,5} - \lambda^2 \eta_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 \xi_{1,2} \xi_{2,3} \xi_{3,5} - \\ &\quad \lambda \eta_{1,4} \xi_{4,5} + \lambda \xi_{1,4} \xi_{4,5} - \lambda^2 \eta_{1,2} \xi_{2,4} \xi_{4,5} + \lambda^2 \xi_{1,2} \xi_{2,4} \xi_{4,5} - \lambda^2 \eta_{1,3} \xi_{3,4} \xi_{4,5} + \\ &\quad \lambda^2 \xi_{1,3} \xi_{3,4} \xi_{4,5} - \lambda^3 \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} - f_{2,5}[\lambda] f_{1,2}'[\lambda] - \\ &\quad f_{2,3}[\lambda] f_{3,5}[\lambda] f_{1,2}'[\lambda] - f_{2,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] - f_{2,3}[\lambda] f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] - \\ &\quad f_{3,5}[\lambda] f_{1,3}'[\lambda] - f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,3}'[\lambda] - f_{4,5}[\lambda] f_{1,4}'[\lambda] - f_{1,5}'[\lambda] = 0, \\ &\eta_{2,5} + \lambda \eta_{2,3} \eta_{3,5} + \lambda \eta_{2,4} \eta_{4,5} + \lambda^2 \eta_{2,3} \eta_{3,4} \eta_{4,5} + \lambda \eta_{3,5} \xi_{2,3} + \lambda^2 \eta_{3,4} \eta_{4,5} \xi_{2,3} + \lambda \eta_{4,5} \xi_{2,4} + \xi_{2,5} - \\ &\quad \lambda^2 \eta_{2,3} \eta_{4,5} \xi_{3,4} + \lambda^2 \eta_{4,5} \xi_{2,3} \xi_{3,4} - \lambda \eta_{2,3} \xi_{3,5} + \lambda \xi_{2,3} \xi_{3,5} - \lambda \eta_{2,4} \xi_{4,5} + \lambda \xi_{2,4} \xi_{4,5} - \lambda^2 \eta_{2,3} \xi_{3,4} \xi_{4,5} + \\ &\quad \lambda^2 \xi_{2,3} \xi_{3,4} \xi_{4,5} - f_{3,5}[\lambda] f_{2,3}'[\lambda] - f_{3,4}[\lambda] f_{4,5}[\lambda] f_{2,3}'[\lambda] - f_{4,5}[\lambda] f_{2,4}'[\lambda] - f_{2,5}'[\lambda] = 0, \\ &\eta_{3,5} + \lambda \eta_{3,4} \eta_{4,5} + \lambda \eta_{4,5} \xi_{3,4} + \xi_{3,5} - \lambda \eta_{3,4} \xi_{4,5} + \lambda \xi_{3,4} \xi_{4,5} - f_{4,5}[\lambda] f_{3,4}'[\lambda] - f_{3,5}'[\lambda] = 0, \\ &\eta_{4,5} + \xi_{4,5} - f_{4,5}'[\lambda] = 0 \end{aligned} \right\}$$

In[*]:= {sol} = DSolve[

```
Join[
  (Coefficient[lhs - rhs, #] == 0) & /@ $Basis,
  $Basis /. xαβ -> fαβ[0] == 0
],
  $Basis /. xαβ -> fαβ[λ],
  λ
]
```

Out[*]=

$$\left\{ \left\{ \begin{aligned} &f_{1,2}[\lambda] \rightarrow \lambda \eta_{1,2} + \lambda \xi_{1,2}, \quad f_{1,3}[\lambda] \rightarrow \lambda \eta_{1,3} + \lambda \xi_{1,3} - \lambda^2 \eta_{1,2} \xi_{2,3}, \\ &f_{2,3}[\lambda] \rightarrow \lambda \eta_{2,3} + \lambda \xi_{2,3}, \quad f_{1,4}[\lambda] \rightarrow -\lambda (-\eta_{1,4} - \xi_{1,4} + \lambda \eta_{1,2} \xi_{2,4} + \lambda \eta_{1,3} \xi_{3,4}), \\ &f_{2,4}[\lambda] \rightarrow -\lambda (-\eta_{2,4} - \xi_{2,4} + \lambda \eta_{2,3} \xi_{3,4}), \quad f_{3,4}[\lambda] \rightarrow \lambda (\eta_{3,4} + \xi_{3,4}), \quad f_{4,5}[\lambda] \rightarrow \lambda \eta_{4,5} + \lambda \xi_{4,5}, \\ &f_{3,5}[\lambda] \rightarrow \lambda \eta_{3,5} + \lambda \xi_{3,5} - \lambda^2 \eta_{3,4} \xi_{4,5}, \quad f_{2,5}[\lambda] \rightarrow \lambda \eta_{2,5} + \lambda \xi_{2,5} - \lambda^2 \eta_{2,3} \xi_{3,5} - \lambda^2 \eta_{2,4} \xi_{4,5}, \\ &f_{1,5}[\lambda] \rightarrow \lambda \eta_{1,5} + \lambda \xi_{1,5} - \lambda^2 \eta_{1,2} \xi_{2,5} - \lambda^2 \eta_{1,3} \xi_{3,5} - \lambda^2 \eta_{1,4} \xi_{4,5} \end{aligned} \right\} \right\}$$

In[*]:= **sol** /. $\lambda \rightarrow 1$

Out[*]=

$$\begin{aligned} &\{f_{1,2}[1] \rightarrow \eta_{1,2} + \xi_{1,2}, f_{1,3}[1] \rightarrow \eta_{1,3} + \xi_{1,3} - \eta_{1,2} \xi_{2,3}, f_{2,3}[1] \rightarrow \eta_{2,3} + \xi_{2,3}, \\ & f_{1,4}[1] \rightarrow \eta_{1,4} + \xi_{1,4} - \eta_{1,2} \xi_{2,4} - \eta_{1,3} \xi_{3,4}, f_{2,4}[1] \rightarrow \eta_{2,4} + \xi_{2,4} - \eta_{2,3} \xi_{3,4}, \\ & f_{3,4}[1] \rightarrow \eta_{3,4} + \xi_{3,4}, f_{4,5}[1] \rightarrow \eta_{4,5} + \xi_{4,5}, f_{3,5}[1] \rightarrow \eta_{3,5} + \xi_{3,5} - \eta_{3,4} \xi_{4,5}, \\ & f_{2,5}[1] \rightarrow \eta_{2,5} + \xi_{2,5} - \eta_{2,3} \xi_{3,5} - \eta_{2,4} \xi_{4,5}, f_{1,5}[1] \rightarrow \eta_{1,5} + \xi_{1,5} - \eta_{1,2} \xi_{2,5} - \eta_{1,3} \xi_{3,5} - \eta_{1,4} \xi_{4,5}\} \end{aligned}$$

In[*]:= **osol** = $\mathbb{E}_{\{1,2\} \rightarrow \{2\}}$ [$x_{1,1}[2] \xi_{1,1}[1] + x_{1,1}[2] \xi_{1,1}[2] + e^{-\xi_{1,1}[2]} x_{1,2}[2] \xi_{1,2}[1] + e^{-\xi_{2,2}[1]} x_{1,2}[2] \xi_{1,2}[2] +$
 $e^{-\xi_{1,1}[2]} x_{1,3}[2] \xi_{1,3}[1] + e^{-\xi_{3,3}[1]} x_{1,3}[2] \xi_{1,3}[2] + e^{-\xi_{1,1}[2]} x_{1,4}[2] \xi_{1,4}[1] +$
 $e^{-\xi_{4,4}[1]} x_{1,4}[2] \xi_{1,4}[2] + e^{-\xi_{1,1}[2]} x_{1,5}[2] \xi_{1,5}[1] + e^{-\xi_{5,5}[1]} x_{1,5}[2] \xi_{1,5}[2] +$
 $x_{2,2}[2] \xi_{2,2}[1] + x_{2,2}[2] \xi_{2,2}[2] + e^{-\xi_{2,2}[2]} x_{2,3}[2] \xi_{2,3}[1] - x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] +$
 $e^{-\xi_{3,3}[1]} x_{2,3}[2] \xi_{2,3}[2] + e^{-\xi_{2,2}[2]} x_{2,4}[2] \xi_{2,4}[1] - x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] +$
 $e^{-\xi_{4,4}[1]} x_{2,4}[2] \xi_{2,4}[2] + e^{-\xi_{2,2}[2]} x_{2,5}[2] \xi_{2,5}[1] - x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] +$
 $e^{-\xi_{5,5}[1]} x_{2,5}[2] \xi_{2,5}[2] + x_{3,3}[2] \xi_{3,3}[1] + x_{3,3}[2] \xi_{3,3}[2] + e^{-\xi_{3,3}[2]} x_{3,4}[2] \xi_{3,4}[1] -$
 $x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] - x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] + e^{-\xi_{4,4}[1]} x_{3,4}[2] \xi_{3,4}[2] +$
 $e^{-\xi_{3,3}[2]} x_{3,5}[2] \xi_{3,5}[1] - x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] - x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] +$
 $e^{-\xi_{5,5}[1]} x_{3,5}[2] \xi_{3,5}[2] + x_{4,4}[2] \xi_{4,4}[1] + x_{4,4}[2] \xi_{4,4}[2] + e^{-\xi_{4,4}[2]} x_{4,5}[2] \xi_{4,5}[1] -$
 $x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] - x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] - x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] +$
 $e^{-\xi_{5,5}[1]} x_{4,5}[2] \xi_{4,5}[2] + x_{5,5}[2] \xi_{5,5}[1] + x_{5,5}[2] \xi_{5,5}[2], \theta, \theta$] [**1**] /. $\xi_{i_,i}_[] \rightarrow \theta$

Out[*]=

$$\begin{aligned} &x_{1,2}[2] \xi_{1,2}[1] + x_{1,2}[2] \xi_{1,2}[2] + x_{1,3}[2] \xi_{1,3}[1] + x_{1,3}[2] \xi_{1,3}[2] + \\ & x_{1,4}[2] \xi_{1,4}[1] + x_{1,4}[2] \xi_{1,4}[2] + x_{1,5}[2] \xi_{1,5}[1] + x_{1,5}[2] \xi_{1,5}[2] + x_{2,3}[2] \xi_{2,3}[1] - \\ & x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] + x_{2,3}[2] \xi_{2,3}[2] + x_{2,4}[2] \xi_{2,4}[1] - x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] + \\ & x_{2,4}[2] \xi_{2,4}[2] + x_{2,5}[2] \xi_{2,5}[1] - x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] + x_{2,5}[2] \xi_{2,5}[2] + x_{3,4}[2] \xi_{3,4}[1] - \\ & x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] - x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] + x_{3,4}[2] \xi_{3,4}[2] + x_{3,5}[2] \xi_{3,5}[1] - \\ & x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] - x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] + x_{3,5}[2] \xi_{3,5}[2] + x_{4,5}[2] \xi_{4,5}[1] - \\ & x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] - x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] - x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] + x_{4,5}[2] \xi_{4,5}[2] \end{aligned}$$

In[*]:= **Expand@Total**[**sol** /. { $\xi_{\alpha\beta}_[] \Rightarrow \xi_{\alpha\beta}[1]$, $\eta_{\alpha\beta}_[] \Rightarrow \xi_{\alpha\beta}[2]$, **Rule** \rightarrow **Times**, $\lambda \rightarrow 1$, $f_{\alpha\beta}_[] \Rightarrow x_{\alpha\beta}[2]$ }] - **osol**

Out[*]=

0