Pensieve header: The PBW multiplication tensor for upper nilpotent matrices using the λ-tangent formalism. Some material from pensieve://Projects/UEA.

```
In[•]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
        << KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Prolog

```
In[•]:= BeginPackage["UEA`"];
        Print["UEA` does computations in general universal enveloping
            algebras and PBW algebras. It is in the public domain, available
            at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
            Dror Bar-Natan is committed to support it within
            reason until June 1, 2027. This is version 260601."];
        Print["UEA` implements / extends ",
          Sort@{"**", B, m, SetAlgebra, U, UB, UProducts, USimp, UU, $Basis, $PBWRule, δ},
          "."];
        Begin["`Private`"];
```

```
UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the
  public domain, available at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/. Dror
  Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
UEA` implements / extends {**, B, m, SetAlgebra, U, UB, UProducts, USimp, UU, δ, $Basis, $PBWRule}.
```

Utilities

```
In[•]:= δ_i_,j_ := If[i == j, 1, 0];
```

Implementing general universal enveloping algebras

```
In[•]:= B[0, _] = 0; B[_, 0] = 0;
        B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
        B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
        B[x_Plus, y_] := B[#, y] & /@ x;
        B[x_, y_Plus] := B[x, #] & /@ y;
        B[x_, x_] = 0;
        B[y_, x_] := Expand[-B[x, y]];
```

```
In[•]:= x_ ⪯ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_^p_] := UU_i @@ Table[x, {p}];
UU_i_[ℰ_] := ℰ /. {
    U[xs__] :> U_i[xs],
    x_ /; MemberQ[$Basis, x] :> U_i[x]
   };
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
In[•]:= m_s__[0] = 0;
m_s__[x_Plus] := m_s /@ x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i_→j_}[ℰ_] := ℰ /. U_i → U_j;
```

```
In[•]:= m_{i_,j_→k_}[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ⪯ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k})
     c // USimp
  ];
```

```
In[•]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] := Sort@
   Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
In[•]:= Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :> i, ∞];
```

```
In[•]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

Epilog

```
In[•]:= End[]; EndPackage[];
```

Predefined Algebras

sl(2)

```
In[•]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

```
UEA`SetAlgebra knows "sl2".
```

```
In[•]:= SetAlgebra["sl2"] := (
          Print["In sl2: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
          B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
          $Basis = {e, h, f};
          $PBWRule = {e → 1, h → 2, f → 3};
         );
```

un_n

```
In[•]:= Print["UEA`SetAlgebra knows the nilpotent algebra un_n."];
```

```
UEA`SetAlgebra knows the nilpotent algebra un_n.
```

```
In[•]:= SetAlgebra[un_n_] := (
          $Basis = Flatten@Table[x_{α,β}, {β, 2, n}, {α, 1, β - 1}];
          $PBWRule = Thread[$Basis → Range@Length@$Basis];
          B[x_{i_,j_}, x_{k_,l_}] := δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j};
         );
```

```
In[•]:= SetAlgebra[un_5];
        $PBWRule
        MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

```
Out[•]= {x_{1,2} → 1, x_{1,3} → 2, x_{2,3} → 3, x_{1,4} → 4, x_{2,4} → 5, x_{3,4} → 6, x_{1,5} → 7, x_{2,5} → 8, x_{3,5} → 9, x_{4,5} → 10}
```

```
Out[•]//MatrixForm=
⎛ {x_{1,2}, x_{1,2}} → 0     {x_{1,2}, x_{1,3}} → 0     {x_{1,2}, x_{2,3}} → x_{1,3}     {x_{1,2}, x_{1,4}} → 0     {x_{1,2}, x_{2,4}} → x_{1,4}
  {x_{1,3}, x_{1,2}} → 0     {x_{1,3}, x_{1,3}} → 0       {x_{1,3}, x_{2,3}} → 0       {x_{1,3}, x_{1,4}} → 0       {x_{1,3}, x_{2,4}} → 0
{x_{2,3}, x_{1,2}} → -x_{1,3}  {x_{2,3}, x_{1,3}} → 0     {x_{2,3}, x_{2,3}} → 0       {x_{2,3}, x_{1,4}} → 0       {x_{2,3}, x_{2,4}} → 0
  {x_{1,4}, x_{1,2}} → 0     {x_{1,4}, x_{1,3}} → 0       {x_{1,4}, x_{2,3}} → 0       {x_{1,4}, x_{1,4}} → 0       {x_{1,4}, x_{2,4}} → 0
{x_{2,4}, x_{1,2}} → -x_{1,4}  {x_{2,4}, x_{1,3}} → 0     {x_{2,4}, x_{2,3}} → 0       {x_{2,4}, x_{1,4}} → 0       {x_{2,4}, x_{2,4}} → 0
  {x_{3,4}, x_{1,2}} → 0   {x_{3,4}, x_{1,3}} → -x_{1,4}  {x_{3,4}, x_{2,3}} → -x_{2,4}  {x_{3,4}, x_{1,4}} → 0       {x_{3,4}, x_{2,4}} → 0
  {x_{1,5}, x_{1,2}} → 0     {x_{1,5}, x_{1,3}} → 0       {x_{1,5}, x_{2,3}} → 0       {x_{1,5}, x_{1,4}} → 0       {x_{1,5}, x_{2,4}} → 0
{x_{2,5}, x_{1,2}} → -x_{1,5}  {x_{2,5}, x_{1,3}} → 0     {x_{2,5}, x_{2,3}} → 0       {x_{2,5}, x_{1,4}} → 0       {x_{2,5}, x_{2,4}} → 0
  {x_{3,5}, x_{1,2}} → 0   {x_{3,5}, x_{1,3}} → -x_{1,5}  {x_{3,5}, x_{2,3}} → -x_{2,5}  {x_{3,5}, x_{1,4}} → 0       {x_{3,5}, x_{2,4}} → 0
  {x_{4,5}, x_{1,2}} → 0     {x_{4,5}, x_{1,3}} → 0       {x_{4,5}, x_{2,3}} → 0     {x_{4,5}, x_{1,4}} → -x_{1,5}  {x_{4,5}, x_{2,4}} → -x_{2,5}
```

```
In[•]:= Clear[adPower];
adPower[x_, k_][0] = 0;
adPower[x_, k_][c_ * y_] /; MemberQ[$Basis, y] := Expand[c adPower[x, k][y]];
adPower[x_, k_][y_Plus] := adPower[x, k] /@ y;
adPower[_, 0][y_] := y;
adPower[x_, k_][y_] /; MemberQ[$Basis, y] :=
  adPower[x, k][y] = B[adPower[x, k - 1][y], x];
```

```
In[•]:= x0 = ($Basis /. x → ξ).$Basis;
y0 = ($Basis /. x → η).$Basis
```

Out[•]=

$$x_{1,2}\,\eta_{1,2} + x_{1,3}\,\eta_{1,3} + x_{1,4}\,\eta_{1,4} + x_{1,5}\,\eta_{1,5} + x_{2,3}\,\eta_{2,3} + x_{2,4}\,\eta_{2,4} + x_{2,5}\,\eta_{2,5} + x_{3,4}\,\eta_{3,4} + x_{3,5}\,\eta_{3,5} + x_{4,5}\,\eta_{4,5}$$

```
In[•]:= adPower[x0, 0][x1,2]
```

Out[•]=

$$x_{1,2}$$

```
In[•]:= B[x1,2, x0]
```

Out[•]=

$$x_{1,3}\,\xi_{2,3} + x_{1,4}\,\xi_{2,4} + x_{1,5}\,\xi_{2,5}$$

```
In[•]:= adPower[x0, 1][x1,2]
```

Out[•]=

$$x_{1,3}\,\xi_{2,3} + x_{1,4}\,\xi_{2,4} + x_{1,5}\,\xi_{2,5}$$

```
In[•]:= adPower[x0, 2][x1,2]
```

Out[•]=

$$x_{1,4}\,\xi_{2,3}\,\xi_{3,4} + x_{1,5}\,\xi_{2,3}\,\xi_{3,5} + x_{1,5}\,\xi_{2,4}\,\xi_{4,5}$$

```
In[•]:= adPower[x0, 3][x1,2]
```

Out[•]=

$$x_{1,5}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

```
In[•]:= adPower[x0, 2][y0]
```

Out[•]=

$$x_{1,4}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3} + x_{1,5}\,\eta_{3,5}\,\xi_{1,2}\,\xi_{2,3} + x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,4} - 2\,x_{1,4}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,4} + x_{1,5}\,\eta_{4,5}\,\xi_{1,3}\,\xi_{3,4} + x_{1,4}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4} + x_{2,5}\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4} - 2\,x_{1,5}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,5} + x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,5} - 2\,x_{1,5}\,\eta_{2,4}\,\xi_{1,2}\,\xi_{4,5} - 2\,x_{1,5}\,\eta_{3,4}\,\xi_{1,3}\,\xi_{4,5} - 2\,x_{2,5}\,\eta_{3,4}\,\xi_{2,3}\,\xi_{4,5} + x_{1,5}\,\eta_{1,2}\,\xi_{2,4}\,\xi_{4,5} + x_{1,5}\,\eta_{1,3}\,\xi_{3,4}\,\xi_{4,5} + x_{2,5}\,\eta_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

```
In[•]:= adPower[x0, 3][y0]
```

Out[•]=

$$-x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4} + 3\,x_{1,5}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{4,5} - 3\,x_{1,5}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,4}\,\xi_{4,5} + x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

```
In[•]:= adExp[x_][y_] := Expand@Module[{k = 0, s = 0},
          While[adPower[x, k][y] =!= 0,
           s += adPower[x, k][y] / k!;
           ++k
          ];
          s
         ]
```

```
In[•]:= adExp[x0][y0]
```

Out[•]=

$$x_{1,2}\,\eta_{1,2}+x_{1,3}\,\eta_{1,3}+x_{1,4}\,\eta_{1,4}+x_{1,5}\,\eta_{1,5}+x_{2,3}\,\eta_{2,3}+x_{2,4}\,\eta_{2,4}+x_{2,5}\,\eta_{2,5}+x_{3,4}\,\eta_{3,4}+x_{3,5}\,\eta_{3,5}+x_{4,5}\,\eta_{4,5}-x_{1,3}\,\eta_{2,3}\,\xi_{1,2}-x_{1,4}\,\eta_{2,4}\,\xi_{1,2}-x_{1,5}\,\eta_{2,5}\,\xi_{1,2}-x_{1,4}\,\eta_{3,4}\,\xi_{1,3}-x_{1,5}\,\eta_{3,5}\,\xi_{1,3}-x_{1,5}\,\eta_{4,5}\,\xi_{1,4}+x_{1,3}\,\eta_{1,2}\,\xi_{2,3}-x_{2,4}\,\eta_{3,4}\,\xi_{2,3}-x_{2,5}\,\eta_{3,5}\,\xi_{2,3}+\frac{1}{2}\,x_{1,4}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}+\frac{1}{2}\,x_{1,5}\,\eta_{3,5}\,\xi_{1,2}\,\xi_{2,3}+x_{1,4}\,\eta_{1,2}\,\xi_{2,4}-x_{2,5}\,\eta_{4,5}\,\xi_{2,4}+\frac{1}{2}\,x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,4}+x_{1,5}\,\eta_{1,2}\,\xi_{2,5}+x_{1,4}\,\eta_{1,3}\,\xi_{3,4}+x_{2,4}\,\eta_{2,3}\,\xi_{3,4}-x_{3,5}\,\eta_{4,5}\,\xi_{3,4}-x_{1,4}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,4}+\frac{1}{2}\,x_{1,5}\,\eta_{4,5}\,\xi_{1,3}\,\xi_{3,4}+\frac{1}{2}\,x_{1,4}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}+\frac{1}{2}\,x_{2,5}\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4}-\frac{1}{6}\,x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}+x_{1,5}\,\eta_{1,3}\,\xi_{3,5}+x_{2,5}\,\eta_{2,3}\,\xi_{3,5}-x_{1,5}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,5}+\frac{1}{2}\,x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,5}+x_{1,5}\,\eta_{1,4}\,\xi_{4,5}+x_{2,5}\,\eta_{2,4}\,\xi_{4,5}+x_{3,5}\,\eta_{3,4}\,\xi_{4,5}-x_{1,5}\,\eta_{2,4}\,\xi_{1,2}\,\xi_{4,5}-x_{1,5}\,\eta_{3,4}\,\xi_{1,3}\,\xi_{4,5}-x_{2,5}\,\eta_{3,4}\,\xi_{2,3}\,\xi_{4,5}+\frac{1}{2}\,x_{1,5}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{4,5}+\frac{1}{2}\,x_{1,5}\,\eta_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\frac{1}{2}\,x_{1,5}\,\eta_{1,3}\,\xi_{3,4}\,\xi_{4,5}+\frac{1}{2}\,x_{2,5}\,\eta_{2,3}\,\xi_{3,4}\,\xi_{4,5}-\frac{1}{2}\,x_{1,5}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,4}\,\xi_{4,5}+\frac{1}{6}\,x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

```
In[•]:= Coefficient[adExp[x0][y0], x_{1,5}]
```

Out[•]=

$$\eta_{1,5}-\eta_{2,5}\,\xi_{1,2}-\eta_{3,5}\,\xi_{1,3}-\eta_{4,5}\,\xi_{1,4}+\frac{1}{2}\,\eta_{3,5}\,\xi_{1,2}\,\xi_{2,3}+\frac{1}{2}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,4}+\eta_{1,2}\,\xi_{2,5}+\frac{1}{2}\,\eta_{4,5}\,\xi_{1,3}\,\xi_{3,4}-\frac{1}{6}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}+\eta_{1,3}\,\xi_{3,5}-\eta_{2,3}\,\xi_{1,2}\,\xi_{3,5}+\frac{1}{2}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,5}+\eta_{1,4}\,\xi_{4,5}-\eta_{2,4}\,\xi_{1,2}\,\xi_{4,5}-\eta_{3,4}\,\xi_{1,3}\,\xi_{4,5}+\frac{1}{2}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{4,5}+\frac{1}{2}\,\eta_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\frac{1}{2}\,\eta_{1,3}\,\xi_{3,4}\,\xi_{4,5}-\frac{1}{2}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,4}\,\xi_{4,5}+\frac{1}{6}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

```
In[•]:= λTangent[] = 0;
        λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂_λ x;
        λTangent[xs_List] := λTangent @@ xs
```

```
In[•]:= $Basis ($Basis /. x → ξ)
```

Out[•]=

$$\{x_{1,2}\,\xi_{1,2},\ x_{1,3}\,\xi_{1,3},\ x_{2,3}\,\xi_{2,3},\ x_{1,4}\,\xi_{1,4},\ x_{2,4}\,\xi_{2,4},\ x_{3,4}\,\xi_{3,4},\ x_{1,5}\,\xi_{1,5},\ x_{2,5}\,\xi_{2,5},\ x_{3,5}\,\xi_{3,5},\ x_{4,5}\,\xi_{4,5}\}$$

In[•]:=
```
λTangent@(λ$Basis ($Basis /. x → ξ))
```

Out[•]=

$$x_{1,2}\,\xi_{1,2}+x_{1,3}\,\xi_{1,3}+x_{1,4}\,\xi_{1,4}+x_{1,5}\,\xi_{1,5}+x_{2,3}\,\xi_{2,3}+\lambda\,x_{1,3}\,\xi_{1,2}\,\xi_{2,3}+x_{2,4}\,\xi_{2,4}+\lambda\,x_{1,4}\,\xi_{1,2}\,\xi_{2,4}+x_{2,5}\,\xi_{2,5}+\lambda\,x_{1,5}\,\xi_{1,2}\,\xi_{2,5}+x_{3,4}\,\xi_{3,4}+\lambda\,x_{1,4}\,\xi_{1,3}\,\xi_{3,4}+\lambda\,x_{2,4}\,\xi_{2,3}\,\xi_{3,4}+\lambda^2\,x_{1,4}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}+x_{3,5}\,\xi_{3,5}+\lambda\,x_{1,5}\,\xi_{1,3}\,\xi_{3,5}+\lambda\,x_{2,5}\,\xi_{2,3}\,\xi_{3,5}+\lambda^2\,x_{1,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,5}+x_{4,5}\,\xi_{4,5}+\lambda\,x_{1,5}\,\xi_{1,4}\,\xi_{4,5}+\lambda\,x_{2,5}\,\xi_{2,4}\,\xi_{4,5}+\lambda^2\,x_{1,5}\,\xi_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\lambda\,x_{3,5}\,\xi_{3,4}\,\xi_{4,5}+\lambda^2\,x_{1,5}\,\xi_{1,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^2\,x_{2,5}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^3\,x_{1,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

In[•]:=
```
lhs = Join[λ$Basis ($Basis /. x → ξ), λ$Basis ($Basis /. x → η)] // λTangent
```

Out[•]=

$$x_{1,2}\,\eta_{1,2}+x_{1,3}\,\eta_{1,3}+x_{1,4}\,\eta_{1,4}+x_{1,5}\,\eta_{1,5}+x_{2,3}\,\eta_{2,3}+\lambda\,x_{1,3}\,\eta_{1,2}\,\eta_{2,3}+x_{2,4}\,\eta_{2,4}+\lambda\,x_{1,4}\,\eta_{1,2}\,\eta_{2,4}+x_{2,5}\,\eta_{2,5}+\lambda\,x_{1,5}\,\eta_{1,2}\,\eta_{2,5}+x_{3,4}\,\eta_{3,4}+\lambda\,x_{1,4}\,\eta_{1,3}\,\eta_{3,4}+\lambda\,x_{2,4}\,\eta_{2,3}\,\eta_{3,4}+\lambda^2\,x_{1,4}\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,4}+x_{3,5}\,\eta_{3,5}+\lambda\,x_{1,5}\,\eta_{1,3}\,\eta_{3,5}+\lambda\,x_{2,5}\,\eta_{2,3}\,\eta_{3,5}+\lambda^2\,x_{1,5}\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,5}+x_{4,5}\,\eta_{4,5}+\lambda\,x_{1,5}\,\eta_{1,4}\,\eta_{4,5}+\lambda\,x_{2,5}\,\eta_{2,4}\,\eta_{4,5}+\lambda^2\,x_{1,5}\,\eta_{1,2}\,\eta_{2,4}\,\eta_{4,5}+\lambda\,x_{3,5}\,\eta_{3,4}\,\eta_{4,5}+\lambda^2\,x_{1,5}\,\eta_{1,3}\,\eta_{3,4}\,\eta_{4,5}+\lambda^2\,x_{2,5}\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}+\lambda^3\,x_{1,5}\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}+x_{1,2}\,\xi_{1,2}+\lambda\,x_{1,3}\,\eta_{2,3}\,\xi_{1,2}+\lambda\,x_{1,4}\,\eta_{2,4}\,\xi_{1,2}+\lambda\,x_{1,5}\,\eta_{2,5}\,\xi_{1,2}+\lambda^2\,x_{1,4}\,\eta_{2,3}\,\eta_{3,4}\,\xi_{1,2}+\lambda^2\,x_{1,5}\,\eta_{2,3}\,\eta_{3,5}\,\xi_{1,2}+\lambda^2\,x_{1,5}\,\eta_{2,4}\,\eta_{4,5}\,\xi_{1,2}+\lambda^3\,x_{1,5}\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,2}+x_{1,3}\,\xi_{1,3}+\lambda\,x_{1,4}\,\eta_{3,4}\,\xi_{1,3}+\lambda\,x_{1,5}\,\eta_{3,5}\,\xi_{1,3}+\lambda^2\,x_{1,5}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,3}+x_{1,4}\,\xi_{1,4}+\lambda\,x_{1,5}\,\eta_{4,5}\,\xi_{1,4}+x_{1,5}\,\xi_{1,5}+x_{2,3}\,\xi_{2,3}-\lambda\,x_{1,3}\,\eta_{1,2}\,\xi_{2,3}+\lambda\,x_{2,4}\,\eta_{3,4}\,\xi_{2,3}-\lambda^2\,x_{1,4}\,\eta_{1,2}\,\eta_{3,4}\,\xi_{2,3}+\lambda\,x_{2,5}\,\eta_{3,5}\,\xi_{2,3}-\lambda^2\,x_{1,5}\,\eta_{1,2}\,\eta_{3,5}\,\xi_{2,3}+\lambda^2\,x_{2,5}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{2,3}-\lambda^3\,x_{1,5}\,\eta_{1,2}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{2,3}+\lambda\,x_{1,3}\,\xi_{1,2}\,\xi_{2,3}+\lambda^2\,x_{1,4}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}+\lambda^2\,x_{1,5}\,\eta_{3,5}\,\xi_{1,2}\,\xi_{2,3}+\lambda^3\,x_{1,5}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}+x_{2,4}\,\xi_{2,4}-\lambda\,x_{1,4}\,\eta_{1,2}\,\xi_{2,4}+\lambda\,x_{2,5}\,\eta_{4,5}\,\xi_{2,4}-\lambda^2\,x_{1,5}\,\eta_{1,2}\,\eta_{4,5}\,\xi_{2,4}+\lambda\,x_{1,4}\,\xi_{1,2}\,\xi_{2,4}+\lambda^2\,x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,4}+x_{2,5}\,\xi_{2,5}-\lambda\,x_{1,5}\,\eta_{1,2}\,\xi_{2,5}+\lambda\,x_{1,5}\,\xi_{1,2}\,\xi_{2,5}+x_{3,4}\,\xi_{3,4}-\lambda\,x_{1,4}\,\eta_{1,3}\,\xi_{3,4}-\lambda\,x_{2,4}\,\eta_{2,3}\,\xi_{3,4}+\lambda\,x_{3,5}\,\eta_{4,5}\,\xi_{3,4}-\lambda^2\,x_{1,5}\,\eta_{1,3}\,\eta_{4,5}\,\xi_{3,4}-\lambda^2\,x_{2,5}\,\eta_{2,3}\,\eta_{4,5}\,\xi_{3,4}+\lambda\,x_{1,4}\,\xi_{1,3}\,\xi_{3,4}+\lambda^2\,x_{1,5}\,\eta_{4,5}\,\xi_{1,3}\,\xi_{3,4}+\lambda\,x_{2,4}\,\xi_{2,3}\,\xi_{3,4}-\lambda^2\,x_{1,4}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}+\lambda^2\,x_{2,5}\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4}-\lambda^3\,x_{1,5}\,\eta_{1,2}\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4}+\lambda^2\,x_{1,4}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}+\lambda^3\,x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}+x_{3,5}\,\xi_{3,5}-\lambda\,x_{1,5}\,\eta_{1,3}\,\xi_{3,5}-\lambda\,x_{2,5}\,\eta_{2,3}\,\xi_{3,5}+\lambda\,x_{1,5}\,\xi_{1,3}\,\xi_{3,5}+\lambda\,x_{2,5}\,\xi_{2,3}\,\xi_{3,5}-\lambda^2\,x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,5}+\lambda^2\,x_{1,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,5}+x_{4,5}\,\xi_{4,5}-\lambda\,x_{1,5}\,\eta_{1,4}\,\xi_{4,5}-\lambda\,x_{2,5}\,\eta_{2,4}\,\xi_{4,5}-\lambda\,x_{3,5}\,\eta_{3,4}\,\xi_{4,5}+\lambda\,x_{1,5}\,\xi_{1,4}\,\xi_{4,5}+\lambda\,x_{2,5}\,\xi_{2,4}\,\xi_{4,5}-\lambda^2\,x_{1,5}\,\eta_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\lambda^2\,x_{1,5}\,\xi_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\lambda\,x_{3,5}\,\xi_{3,4}\,\xi_{4,5}-\lambda^2\,x_{1,5}\,\eta_{1,3}\,\xi_{3,4}\,\xi_{4,5}-\lambda^2\,x_{2,5}\,\eta_{2,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^2\,x_{1,5}\,\xi_{1,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^2\,x_{2,5}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}-\lambda^3\,x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^3\,x_{1,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

In[•]:=
```
rhs = λTangent@($Basis ($Basis /. x_αβ__ → f_αβ[λ]))
```

Out[•]=

$$x_{1,2}\,f_{1,2}'[\lambda]+x_{1,3}\,f_{2,3}[\lambda]\,f_{1,2}'[\lambda]+x_{1,4}\,f_{2,4}[\lambda]\,f_{1,2}'[\lambda]+x_{1,5}\,f_{2,5}[\lambda]\,f_{1,2}'[\lambda]+x_{1,4}\,f_{2,3}[\lambda]\,f_{3,4}[\lambda]\,f_{1,2}'[\lambda]+x_{1,5}\,f_{2,3}[\lambda]\,f_{3,5}[\lambda]\,f_{1,2}'[\lambda]+x_{1,5}\,f_{2,4}[\lambda]\,f_{4,5}[\lambda]\,f_{1,2}'[\lambda]+x_{1,5}\,f_{2,3}[\lambda]\,f_{3,4}[\lambda]\,f_{4,5}[\lambda]\,f_{1,2}'[\lambda]+x_{1,3}\,f_{1,3}'[\lambda]+x_{1,4}\,f_{3,4}[\lambda]\,f_{1,3}'[\lambda]+x_{1,5}\,f_{3,5}[\lambda]\,f_{1,3}'[\lambda]+x_{1,5}\,f_{3,4}[\lambda]\,f_{4,5}[\lambda]\,f_{1,3}'[\lambda]+x_{1,4}\,f_{1,4}'[\lambda]+x_{1,5}\,f_{4,5}[\lambda]\,f_{1,4}'[\lambda]+x_{1,5}\,f_{1,5}'[\lambda]+x_{2,3}\,f_{2,3}'[\lambda]+x_{2,4}\,f_{3,4}[\lambda]\,f_{2,3}'[\lambda]+x_{2,5}\,f_{3,5}[\lambda]\,f_{2,3}'[\lambda]+x_{2,5}\,f_{3,4}[\lambda]\,f_{4,5}[\lambda]\,f_{2,3}'[\lambda]+x_{2,4}\,f_{2,4}'[\lambda]+x_{2,5}\,f_{4,5}[\lambda]\,f_{2,4}'[\lambda]+x_{2,5}\,f_{2,5}'[\lambda]+x_{3,4}\,f_{3,4}'[\lambda]+x_{3,5}\,f_{4,5}[\lambda]\,f_{3,4}'[\lambda]+x_{3,5}\,f_{3,5}'[\lambda]+x_{4,5}\,f_{4,5}'[\lambda]$$

In[•]:=
```
(Coefficient[lhs - rhs, #] == 0) & /@ $Basis
```

Out[•]=

$$\Big\{\eta_{1,2}+\xi_{1,2}-\mathtt{f}_{1,2}'[\lambda]==0,$$
$$\eta_{1,3}+\lambda\,\eta_{1,2}\,\eta_{2,3}+\lambda\,\eta_{2,3}\,\xi_{1,2}+\xi_{1,3}-\lambda\,\eta_{1,2}\,\xi_{2,3}+\lambda\,\xi_{1,2}\,\xi_{2,3}-\mathtt{f}_{2,3}[\lambda]\,\mathtt{f}_{1,2}'[\lambda]-\mathtt{f}_{1,3}'[\lambda]==0,$$
$$\eta_{2,3}+\xi_{2,3}-\mathtt{f}_{2,3}'[\lambda]==0,$$
$$\eta_{1,4}+\lambda\,\eta_{1,2}\,\eta_{2,4}+\lambda\,\eta_{1,3}\,\eta_{3,4}+\lambda^2\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,4}+\lambda\,\eta_{2,4}\,\xi_{1,2}+\lambda^2\,\eta_{2,3}\,\eta_{3,4}\,\xi_{1,2}+\lambda\,\eta_{3,4}\,\xi_{1,3}+\xi_{1,4}-$$
$$\lambda^2\,\eta_{1,2}\,\eta_{3,4}\,\xi_{2,3}+\lambda^2\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}-\lambda\,\eta_{1,2}\,\xi_{2,4}+\lambda\,\xi_{1,2}\,\xi_{2,4}-\lambda\,\eta_{1,3}\,\xi_{3,4}+\lambda\,\xi_{1,3}\,\xi_{3,4}-\lambda^2\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}+$$
$$\lambda^2\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}-\mathtt{f}_{2,4}[\lambda]\,\mathtt{f}_{1,2}'[\lambda]-\mathtt{f}_{2,3}[\lambda]\,\mathtt{f}_{3,4}[\lambda]\,\mathtt{f}_{1,2}'[\lambda]-\mathtt{f}_{3,4}[\lambda]\,\mathtt{f}_{1,3}'[\lambda]-\mathtt{f}_{1,4}'[\lambda]==0,$$
$$\eta_{2,4}+\lambda\,\eta_{2,3}\,\eta_{3,4}+\lambda\,\eta_{3,4}\,\xi_{2,3}+\xi_{2,4}-\lambda\,\eta_{2,3}\,\xi_{3,4}+\lambda\,\xi_{2,3}\,\xi_{3,4}-\mathtt{f}_{3,4}[\lambda]\,\mathtt{f}_{2,3}'[\lambda]-\mathtt{f}_{2,4}'[\lambda]==0,$$
$$\eta_{3,4}+\xi_{3,4}-\mathtt{f}_{3,4}'[\lambda]==0,$$
$$\eta_{1,5}+\lambda\,\eta_{1,2}\,\eta_{2,5}+\lambda\,\eta_{1,3}\,\eta_{3,5}+\lambda^2\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,5}+\lambda\,\eta_{1,4}\,\eta_{4,5}+\lambda^2\,\eta_{1,2}\,\eta_{2,4}\,\eta_{4,5}+\lambda^2\,\eta_{1,3}\,\eta_{3,4}\,\eta_{4,5}+$$
$$\lambda^3\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}+\lambda\,\eta_{2,5}\,\xi_{1,2}+\lambda^2\,\eta_{2,3}\,\eta_{3,5}\,\xi_{1,2}+\lambda^2\,\eta_{2,4}\,\eta_{4,5}\,\xi_{1,2}+\lambda^3\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,2}+$$
$$\lambda\,\eta_{3,5}\,\xi_{1,3}+\lambda^2\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,3}+\lambda\,\eta_{4,5}\,\xi_{1,4}+\xi_{1,5}-\lambda^2\,\eta_{1,2}\,\eta_{3,5}\,\xi_{2,3}-\lambda^3\,\eta_{1,2}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{2,3}+$$
$$\lambda^2\,\eta_{3,5}\,\xi_{1,2}\,\xi_{2,3}+\lambda^3\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}-\lambda^2\,\eta_{1,2}\,\eta_{4,5}\,\xi_{2,4}+\lambda^2\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,4}-$$
$$\lambda\,\eta_{1,2}\,\xi_{2,5}+\lambda\,\xi_{1,2}\,\xi_{2,5}-\lambda^2\,\eta_{1,3}\,\eta_{4,5}\,\xi_{3,4}+\lambda^2\,\eta_{4,5}\,\xi_{1,3}\,\xi_{3,4}-\lambda^3\,\eta_{1,2}\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4}+$$
$$\lambda^3\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}-\lambda\,\eta_{1,3}\,\xi_{3,5}+\lambda\,\xi_{1,3}\,\xi_{3,5}-\lambda^2\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,5}+\lambda^2\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,5}-$$
$$\lambda\,\eta_{1,4}\,\xi_{4,5}+\lambda\,\xi_{1,4}\,\xi_{4,5}-\lambda^2\,\eta_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\lambda^2\,\xi_{1,2}\,\xi_{2,4}\,\xi_{4,5}-\lambda^2\,\eta_{1,3}\,\xi_{3,4}\,\xi_{4,5}+$$
$$\lambda^2\,\xi_{1,3}\,\xi_{3,4}\,\xi_{4,5}-\lambda^3\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^3\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}-\mathtt{f}_{2,5}[\lambda]\,\mathtt{f}_{1,2}'[\lambda]-$$
$$\mathtt{f}_{2,3}[\lambda]\,\mathtt{f}_{3,5}[\lambda]\,\mathtt{f}_{1,2}'[\lambda]-\mathtt{f}_{2,4}[\lambda]\,\mathtt{f}_{4,5}[\lambda]\,\mathtt{f}_{1,2}'[\lambda]-\mathtt{f}_{2,3}[\lambda]\,\mathtt{f}_{3,4}[\lambda]\,\mathtt{f}_{4,5}[\lambda]\,\mathtt{f}_{1,2}'[\lambda]-$$
$$\mathtt{f}_{3,5}[\lambda]\,\mathtt{f}_{1,3}'[\lambda]-\mathtt{f}_{3,4}[\lambda]\,\mathtt{f}_{4,5}[\lambda]\,\mathtt{f}_{1,3}'[\lambda]-\mathtt{f}_{4,5}[\lambda]\,\mathtt{f}_{1,4}'[\lambda]-\mathtt{f}_{1,5}'[\lambda]==0,$$
$$\eta_{2,5}+\lambda\,\eta_{2,3}\,\eta_{3,5}+\lambda\,\eta_{2,4}\,\eta_{4,5}+\lambda^2\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}+\lambda\,\eta_{3,5}\,\xi_{2,3}+\lambda^2\,\eta_{3,4}\,\eta_{4,5}\,\xi_{2,3}+\lambda\,\eta_{4,5}\,\xi_{2,4}+\xi_{2,5}-$$
$$\lambda^2\,\eta_{2,3}\,\eta_{4,5}\,\xi_{3,4}+\lambda^2\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4}-\lambda\,\eta_{2,3}\,\xi_{3,5}+\lambda\,\xi_{2,3}\,\xi_{3,5}-\lambda\,\eta_{2,4}\,\xi_{4,5}+\lambda\,\xi_{2,4}\,\xi_{4,5}-\lambda^2\,\eta_{2,3}\,\xi_{3,4}\,\xi_{4,5}+$$
$$\lambda^2\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}-\mathtt{f}_{3,5}[\lambda]\,\mathtt{f}_{2,3}'[\lambda]-\mathtt{f}_{3,4}[\lambda]\,\mathtt{f}_{4,5}[\lambda]\,\mathtt{f}_{2,3}'[\lambda]-\mathtt{f}_{4,5}[\lambda]\,\mathtt{f}_{2,4}'[\lambda]-\mathtt{f}_{2,5}'[\lambda]==0,$$
$$\eta_{3,5}+\lambda\,\eta_{3,4}\,\eta_{4,5}+\lambda\,\eta_{4,5}\,\xi_{3,4}+\xi_{3,5}-\lambda\,\eta_{3,4}\,\xi_{4,5}+\lambda\,\xi_{3,4}\,\xi_{4,5}-\mathtt{f}_{4,5}[\lambda]\,\mathtt{f}_{3,4}'[\lambda]-\mathtt{f}_{3,5}'[\lambda]==0,$$
$$\eta_{4,5}+\xi_{4,5}-\mathtt{f}_{4,5}'[\lambda]==0\Big\}$$

In[•]:=
```
{sol} = DSolve[
  Join[
   (Coefficient[lhs - rhs, #] == 0) & /@ $Basis,
   $Basis /. x_αβ__ :> f_αβ[0] == 0
  ],
  $Basis /. x_αβ__ → f_αβ[λ],
  λ
 ]
```

Out[•]=

$$\Big\{\Big\{\mathtt{f}_{1,2}[\lambda]\to\lambda\,\eta_{1,2}+\lambda\,\xi_{1,2},\ \mathtt{f}_{1,3}[\lambda]\to\lambda\,\eta_{1,3}+\lambda\,\xi_{1,3}-\lambda^2\,\eta_{1,2}\,\xi_{2,3},$$
$$\mathtt{f}_{2,3}[\lambda]\to\lambda\,\eta_{2,3}+\lambda\,\xi_{2,3},\ \mathtt{f}_{1,4}[\lambda]\to-\lambda\,(-\eta_{1,4}-\xi_{1,4}+\lambda\,\eta_{1,2}\,\xi_{2,4}+\lambda\,\eta_{1,3}\,\xi_{3,4}),$$
$$\mathtt{f}_{2,4}[\lambda]\to-\lambda\,(-\eta_{2,4}-\xi_{2,4}+\lambda\,\eta_{2,3}\,\xi_{3,4}),\ \mathtt{f}_{3,4}[\lambda]\to\lambda\,(\eta_{3,4}+\xi_{3,4}),\ \mathtt{f}_{4,5}[\lambda]\to\lambda\,\eta_{4,5}+\lambda\,\xi_{4,5},$$
$$\mathtt{f}_{3,5}[\lambda]\to\lambda\,\eta_{3,5}+\lambda\,\xi_{3,5}-\lambda^2\,\eta_{3,4}\,\xi_{4,5},\ \mathtt{f}_{2,5}[\lambda]\to\lambda\,\eta_{2,5}+\lambda\,\xi_{2,5}-\lambda^2\,\eta_{2,3}\,\xi_{3,5}-\lambda^2\,\eta_{2,4}\,\xi_{4,5},$$
$$\mathtt{f}_{1,5}[\lambda]\to\lambda\,\eta_{1,5}+\lambda\,\xi_{1,5}-\lambda^2\,\eta_{1,2}\,\xi_{2,5}-\lambda^2\,\eta_{1,3}\,\xi_{3,5}-\lambda^2\,\eta_{1,4}\,\xi_{4,5}\Big\}\Big\}$$

In[•]:=
```
sol /. λ → 1
```

Out[•]=
$$\{f_{1,2}[1] \to \eta_{1,2} + \xi_{1,2},\ f_{1,3}[1] \to \eta_{1,3} + \xi_{1,3} - \eta_{1,2}\,\xi_{2,3},\ f_{2,3}[1] \to \eta_{2,3} + \xi_{2,3},$$
$$f_{1,4}[1] \to \eta_{1,4} + \xi_{1,4} - \eta_{1,2}\,\xi_{2,4} - \eta_{1,3}\,\xi_{3,4},\ f_{2,4}[1] \to \eta_{2,4} + \xi_{2,4} - \eta_{2,3}\,\xi_{3,4},$$
$$f_{3,4}[1] \to \eta_{3,4} + \xi_{3,4},\ f_{4,5}[1] \to \eta_{4,5} + \xi_{4,5},\ f_{3,5}[1] \to \eta_{3,5} + \xi_{3,5} - \eta_{3,4}\,\xi_{4,5},$$
$$f_{2,5}[1] \to \eta_{2,5} + \xi_{2,5} - \eta_{2,3}\,\xi_{3,5} - \eta_{2,4}\,\xi_{4,5},\ f_{1,5}[1] \to \eta_{1,5} + \xi_{1,5} - \eta_{1,2}\,\xi_{2,5} - \eta_{1,3}\,\xi_{3,5} - \eta_{1,4}\,\xi_{4,5}\}$$

In[•]:=
```
osol = 𝔼{1,2}→{2}[x1,1[2] ξ1,1[1] + x1,1[2] ξ1,1[2] + e^-ξ1,1[2] x1,2[2] ξ1,2[1] + e^-ξ2,2[1] x1,2[2] ξ1,2[2] +
    e^-ξ1,1[2] x1,3[2] ξ1,3[1] + e^-ξ3,3[1] x1,3[2] ξ1,3[2] + e^-ξ1,1[2] x1,4[2] ξ1,4[1] +
    e^-ξ4,4[1] x1,4[2] ξ1,4[2] + e^-ξ1,1[2] x1,5[2] ξ1,5[1] + e^-ξ5,5[1] x1,5[2] ξ1,5[2] +
    x2,2[2] ξ2,2[1] + x2,2[2] ξ2,2[2] + e^-ξ2,2[2] x2,3[2] ξ2,3[1] - x1,3[2] ξ1,2[2] ξ2,3[1] +
    e^-ξ3,3[1] x2,3[2] ξ2,3[2] + e^-ξ2,2[2] x2,4[2] ξ2,4[1] - x1,4[2] ξ1,2[2] ξ2,4[1] +
    e^-ξ4,4[1] x2,4[2] ξ2,4[2] + e^-ξ2,2[2] x2,5[2] ξ2,5[1] - x1,5[2] ξ1,2[2] ξ2,5[1] +
    e^-ξ5,5[1] x2,5[2] ξ2,5[2] + x3,3[2] ξ3,3[1] + x3,3[2] ξ3,3[2] + e^-ξ3,3[2] x3,4[2] ξ3,4[1] -
    x1,4[2] ξ1,3[2] ξ3,4[1] - x2,4[2] ξ2,3[2] ξ3,4[1] + e^-ξ4,4[1] x3,4[2] ξ3,4[2] +
    e^-ξ3,3[2] x3,5[2] ξ3,5[1] - x1,5[2] ξ1,3[2] ξ3,5[1] - x2,5[2] ξ2,3[2] ξ3,5[1] +
    e^-ξ5,5[1] x3,5[2] ξ3,5[2] + x4,4[2] ξ4,4[1] + x4,4[2] ξ4,4[2] + e^-ξ4,4[2] x4,5[2] ξ4,5[1] -
    x1,5[2] ξ1,4[2] ξ4,5[1] - x2,5[2] ξ2,4[2] ξ4,5[1] - x3,5[2] ξ3,4[2] ξ4,5[1] +
    e^-ξ5,5[1] x4,5[2] ξ4,5[2] + x5,5[2] ξ5,5[1] + x5,5[2] ξ5,5[2], 0, 0][[1]] /. ξi_,i_[_] → 0
```

Out[•]=
$$x_{1,2}[2]\,\xi_{1,2}[1] + x_{1,2}[2]\,\xi_{1,2}[2] + x_{1,3}[2]\,\xi_{1,3}[1] + x_{1,3}[2]\,\xi_{1,3}[2] +$$
$$x_{1,4}[2]\,\xi_{1,4}[1] + x_{1,4}[2]\,\xi_{1,4}[2] + x_{1,5}[2]\,\xi_{1,5}[1] + x_{1,5}[2]\,\xi_{1,5}[2] + x_{2,3}[2]\,\xi_{2,3}[1] -$$
$$x_{1,3}[2]\,\xi_{1,2}[2]\,\xi_{2,3}[1] + x_{2,3}[2]\,\xi_{2,3}[2] + x_{2,4}[2]\,\xi_{2,4}[1] - x_{1,4}[2]\,\xi_{1,2}[2]\,\xi_{2,4}[1] +$$
$$x_{2,4}[2]\,\xi_{2,4}[2] + x_{2,5}[2]\,\xi_{2,5}[1] - x_{1,5}[2]\,\xi_{1,2}[2]\,\xi_{2,5}[1] + x_{2,5}[2]\,\xi_{2,5}[2] + x_{3,4}[2]\,\xi_{3,4}[1] -$$
$$x_{1,4}[2]\,\xi_{1,3}[2]\,\xi_{3,4}[1] - x_{2,4}[2]\,\xi_{2,3}[2]\,\xi_{3,4}[1] + x_{3,4}[2]\,\xi_{3,4}[2] + x_{3,5}[2]\,\xi_{3,5}[1] -$$
$$x_{1,5}[2]\,\xi_{1,3}[2]\,\xi_{3,5}[1] - x_{2,5}[2]\,\xi_{2,3}[2]\,\xi_{3,5}[1] + x_{3,5}[2]\,\xi_{3,5}[2] + x_{4,5}[2]\,\xi_{4,5}[1] -$$
$$x_{1,5}[2]\,\xi_{1,4}[2]\,\xi_{4,5}[1] - x_{2,5}[2]\,\xi_{2,4}[2]\,\xi_{4,5}[1] - x_{3,5}[2]\,\xi_{3,4}[2]\,\xi_{4,5}[1] + x_{4,5}[2]\,\xi_{4,5}[2]$$

In[•]:=
```
Expand@Total[sol /. {ξαβ__ :→ ξαβ[1], ηαβ__ :→ ξαβ[2] , Rule → Times, λ → 1, fαβ__[_] :→ xαβ[2]}] -
 osol
```

Out[•]=
```
0
```