

```
In[ ]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW"]
Once[<< KnotTheory`];
```

SetDirectory: Cannot set current directory to C:\drorbn\AcademicPensieve\Projects\SolvablePBW.

```
Out[ ]:=
```

\$Failed

Get: Cannot open KnotTheory`.

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_q(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_h(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

pdf

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

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This version: April 2020. Original version: July 1994.

```
In[ ]:= MakePDF[]
```

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```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[ $\epsilon$ _, op_] := Collect[Normal@Series[ $\epsilon$ , { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[ $\epsilon$ _] := SS[ $\epsilon$ , Together];
```

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```

In[ ]:= $N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x<>i<j; a[i_] := a<>i; A[i_] := A<>i;
ξ[i_, j_] := ξ<>i<j; α[i_] := α<>i; ℳ[i_] := ℳ<>i;
y[i_, j_] := y<>i<j; b[i_] := b<>i; B[i_] := B<>i;
η[i_, j_] := η<>i<j; β[i_] := β<>i;
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_] := τ[s]; ℓ[s_] := ℓ[s];

```

```

In[ ]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm

```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

Greek - Latin duality:

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```

In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]* = ξ[i, j];
  ξ[i, j]* = x[i, j]; y[i, j]* = η[i, j];
  η[i, j]* = y[i, j]]
Do[As[i, 1, $N - 1], A[i]* = ℳ[i]; ℳ[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i];
  b[i]* = β[i]; β[i]* = b[i]]
(vs_List)* := (v ↦ v*) /@ vs;
(u_{αβ})* := (u*)_{αβ};
F[u_i] := F[u_i] = ToExpression["F" <> ToString[u]]_i

```

```

In[ ]:= $N = 9

```

Time: 0.05 seconds Out[]:=

9

```

In[ ]:= τ[α1]

```

Time: 0.00 seconds Out[]:=

α

```

Flatten@Table[{ℳ[j]_{i_}^{p_} → e^{p α[j]_i}, B[j]_{i_}^{p_} → e^{-p ℏ b[j]_i}}, {j, 1, $N - 1}];
Join[{e^{χ_} → e^{Expand@12U@χ_}},
  Flatten@Table[{e^{c_· b[j]_{i_} + d_} → B[j]_{i_}^{-c/ℏ} e^d, e^{c_· α[j]_{i_} + d_} → ℳ[j]_{i_}^c e^d}, {j, 1, $N - 1}]]];

```

```

In[*]:=  $\hbar = 1;$ 
U21[ $\mathcal{E}_-$ ] :=  $\mathcal{E} /. \{ \mathcal{A}1_{i-}^{p-} \mapsto e^{p \alpha_{1i}}, B1_{i-}^{p-} \mapsto e^{-p \hbar b_{1i}}, \mathcal{A}2_{i-}^{p-} \mapsto e^{p \alpha_{2i}}, B2_{i-}^{p-} \mapsto e^{-p \hbar b_{2i}}, \mathcal{A}3_{i-}^{p-} \mapsto e^{p \alpha_{3i}},$ 
 $B3_{i-}^{p-} \mapsto e^{-p \hbar b_{3i}}, \mathcal{A}4_{i-}^{p-} \mapsto e^{p \alpha_{4i}}, B4_{i-}^{p-} \mapsto e^{-p \hbar b_{4i}}, \mathcal{A}5_{i-}^{p-} \mapsto e^{p \alpha_{5i}}, B5_{i-}^{p-} \mapsto e^{-p \hbar b_{5i}}, \mathcal{A}6_{i-}^{p-} \mapsto e^{p \alpha_{6i}},$ 
 $B6_{i-}^{p-} \mapsto e^{-p \hbar b_{6i}}, \mathcal{A}7_{i-}^{p-} \mapsto e^{p \alpha_{7i}}, B7_{i-}^{p-} \mapsto e^{-p \hbar b_{7i}}, \mathcal{A}8_{i-}^{p-} \mapsto e^{p \alpha_{8i}}, B8_{i-}^{p-} \mapsto e^{-p \hbar b_{8i}} \}$ 

12U[ $\mathcal{E}_-$ ] :=  $\mathcal{E} //. \{ e^{d_- + c_- \cdot b_{1i}} \mapsto B1_i^{-\frac{c}{\hbar}} e^d, e^{d_- + c_- \cdot \alpha_{1i}} \mapsto \mathcal{A}1_i^c e^d,$ 
 $e^{d_- + c_- \cdot b_{2i}} \mapsto B2_i^{-\frac{c}{\hbar}} e^d, e^{d_- + c_- \cdot \alpha_{2i}} \mapsto \mathcal{A}2_i^c e^d, e^{d_- + c_- \cdot b_{3i}} \mapsto B3_i^{-\frac{c}{\hbar}} e^d,$ 
 $e^{d_- + c_- \cdot \alpha_{3i}} \mapsto \mathcal{A}3_i^c e^d, e^{d_- + c_- \cdot b_{4i}} \mapsto B4_i^{-\frac{c}{\hbar}} e^d, e^{d_- + c_- \cdot \alpha_{4i}} \mapsto \mathcal{A}4_i^c e^d, e^{d_- + c_- \cdot b_{5i}} \mapsto B5_i^{-\frac{c}{\hbar}} e^d,$ 
 $e^{d_- + c_- \cdot \alpha_{5i}} \mapsto \mathcal{A}5_i^c e^d, e^{d_- + c_- \cdot b_{6i}} \mapsto B6_i^{-\frac{c}{\hbar}} e^d, e^{d_- + c_- \cdot \alpha_{6i}} \mapsto \mathcal{A}6_i^c e^d, e^{d_- + c_- \cdot b_{7i}} \mapsto B7_i^{-\frac{c}{\hbar}} e^d,$ 
 $e^{d_- + c_- \cdot \alpha_{7i}} \mapsto \mathcal{A}7_i^c e^d, e^{d_- + c_- \cdot b_{8i}} \mapsto B8_i^{-\frac{c}{\hbar}} e^d, e^{d_- + c_- \cdot \alpha_{8i}} \mapsto \mathcal{A}8_i^c e^d, e^{\mathcal{X}_-} \mapsto \text{Expand}[12U[\mathcal{X}_-]] \}$ 

12U[r_Rule] := Module[{U = r[[1]] /. {b  $\rightarrow$  B, t  $\rightarrow$  T,  $\alpha \rightarrow \mathcal{A}$ }}, U  $\rightarrow$  12U[U21[U] /. r]];
AlsoUpper[rs_List] := rs  $\cup$  (12U/@rs);

```

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Derivatives in the presence of exponentiated variables:

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```

In[*]:= Db1i[f_] :=  $\partial_{b_{1i}} f - \hbar B_{1i} \partial_{B_{1i}} f;$ 
Db2i[f_] :=  $\partial_{b_{2i}} f - \hbar B_{2i} \partial_{B_{2i}} f;$ 
Db3i[f_] :=  $\partial_{b_{3i}} f - \hbar B_{3i} \partial_{B_{3i}} f;$ 
Db4i[f_] :=  $\partial_{b_{4i}} f - \hbar B_{4i} \partial_{B_{4i}} f;$ 
Db5i[f_] :=  $\partial_{b_{5i}} f - \hbar B_{5i} \partial_{B_{5i}} f;$ 
Db6i[f_] :=  $\partial_{b_{6i}} f - \hbar B_{6i} \partial_{B_{6i}} f;$ 
Db7i[f_] :=  $\partial_{b_{7i}} f - \hbar B_{7i} \partial_{B_{7i}} f;$ 
Db8i[f_] :=  $\partial_{b_{8i}} f - \hbar B_{8i} \partial_{B_{8i}} f;$ 
D $\alpha_{1i}$ [f_] :=  $\partial_{\alpha_{1i}} f + \mathcal{A}_{1i} \partial_{\mathcal{A}_{1i}} f;$ 
D $\alpha_{2i}$ [f_] :=  $\partial_{\alpha_{2i}} f + \mathcal{A}_{2i} \partial_{\mathcal{A}_{2i}} f;$ 
D $\alpha_{3i}$ [f_] :=  $\partial_{\alpha_{3i}} f + \mathcal{A}_{3i} \partial_{\mathcal{A}_{3i}} f;$ 
D $\alpha_{4i}$ [f_] :=  $\partial_{\alpha_{4i}} f + \mathcal{A}_{4i} \partial_{\mathcal{A}_{4i}} f;$ 
D $\alpha_{5i}$ [f_] :=  $\partial_{\alpha_{5i}} f + \mathcal{A}_{5i} \partial_{\mathcal{A}_{5i}} f;$ 
D $\alpha_{6i}$ [f_] :=  $\partial_{\alpha_{6i}} f + \mathcal{A}_{6i} \partial_{\mathcal{A}_{6i}} f;$ 
D $\alpha_{7i}$ [f_] :=  $\partial_{\alpha_{7i}} f + \mathcal{A}_{7i} \partial_{\mathcal{A}_{7i}} f;$ 
D $\alpha_{8i}$ [f_] :=  $\partial_{\alpha_{8i}} f + \mathcal{A}_{8i} \partial_{\mathcal{A}_{8i}} f;$ 
Dv[f_] :=  $\partial_v f;$ 

```

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```

In[*]:= LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
  Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars  $\cup$  GreekVars))_;

```

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```

In[ ]:= CF[ $\mathcal{E}$ _] := PPCF@Module[{s1},
  s1 = If[$k ==  $\infty$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor}^p}$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor}^p}$  /.  $q^{n-} \rightarrow \sum_{k=0}^{k} \frac{(\hbar \in \gamma n)^k}{k!}$  /.  $e^{k-}$  /.  $k > $k \rightarrow 0$ 
  ]
  (*Expand@Collect[s1,Cases[s1,VarPattern, $\infty$ ],CCF]/.CCF→Factor*)
]

```

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Weights:

```

In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N],
  Wt[x[i, j]] = {1, i - j}; Wt[y[i, j]] = {0, -(i - j)};
  Wt[ $\xi$ [i, j]] = {0, -(i - j)}; Wt[ $\eta$ [i, j]] = {1, (i - j)};
];
Do[As[i, 1, $N - 1],
  Wt[a[i]] = {1, 0}; Wt[b[i]] = {0, 0};
  Wt[ $\alpha$ [i]] = {0, 0}; Wt[ $\beta$ [i]] = {1, 0};
];
Wt[u_] := Wt[u];

```

```

In[ ]:= $PBWGens

```

Time: 0.03 seconds Out[]:=

```
{x12, x13, x23, a1, a2}
```

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```

In[ ]:= (* q~e $\hbar \gamma$   $\epsilon$ ; Ai[n]~e $\hbar$   $\epsilon$  ai; *)

```

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```

In[ ]:= $n = 3; $k =  $\infty$ ; $E := {$n, $k}

```

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```

In[ ]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
  Table[As[i, 1, $n - 1], {a[i], A[i][_]}]
}
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y}] /. $EPBWRule; x_ < y_ := ! OrderedQ[{y, x}] /. $EPBWRule;
$PBWGens := DeleteCases[$EPBWGens, _[_]]

```

```

In[ ]:= $PBWGens*

```

Time: 0.01 seconds Out[]:=

```
{ $\xi$ 12,  $\xi$ 13,  $\xi$ 14,  $\xi$ 15,  $\xi$ 23,  $\xi$ 24,  $\xi$ 25,  $\xi$ 34,  $\xi$ 35,  $\xi$ 45,  $\alpha$ 1,  $\alpha$ 2,  $\alpha$ 3,  $\alpha$ 4}
```

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```

In[*]:= UCF[ $\mathcal{E}_-$ ] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a@@L[AB], k], r]
  }]],
  U[____], F
] /. F → CF]

```

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```

In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j[_] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}_-$ ] := ms[UCF[ $\mathcal{E}$ ]];

```

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```

In[*]:= Supp[ $\mathcal{E}_-$ ] := Union@Cases[{ $\mathcal{E}$ }, Ui[____] ⇒ i, ∞];

```

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```

In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PPNCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /.  $\epsilon^{k-}$  /.  $k > $k \Rightarrow 0$ ];
  Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];

```

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```

In[*]:= Δi→j,k[_] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk,k1→k@mj,j1→j[Δj,k[f] Δi→j1,k1[Ui[fs]]]
}];
Δi→j,k,l[_] :=  $\mathcal{E}$  /. Δi→j,k /. Δk→k,l

```

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```
In[*]:= Si[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i→i[Si[f] Si1[Ui1[fs]]]
}];
S̄i[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i→i[S̄i[f] S̄i1[Ui1[fs]]]
}];
```

```
In[*]:= Δ1,2→3[U1[x12]]
```

```
Time: 0.02 seconds Out[ ]=
```

```
Δ1,2→3[U1[x12]]
```

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The $\mathbb{A}$ -algebra

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```
In[*]:= RRk[ai_, aj_] /; τ[ai] == τ[aj] == "a" := Uk[aj, ai];
RRk[ai_, Aj_[n_]] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[Aj[n], ai];
RRk[Aj_[n_], ai_] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[ai, Aj[n]];
RRk[Ai_[m_], Aj_[n_]] /; τ[Ai] == τ[Aj] == "A" := Uk[Aj[n], Ai[m]];
RRk[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@am;
    Uk[xij, am] + γ (δm,i - δm+1,i - δm,j + δm+1,j) Uk[xij] / 2
];
RRk[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@Am; q-n (δn,i - δn+1,i - δn,j + δn+1,j) Uk[xij, Am[n]]
]
```

```
In[*]:= $k
```

```
Time: 0.03 seconds Out[ ]=
```

```
1
```

```
In[*]:= RR0[a[1], x[1, 2]]
```

```
Time: 0.05 seconds Out[ ]=
```

```
γ U0[x12] + U0[x12, a1]
```

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We take the reduction rules RR for xij's from Yamane page

509:

- (I) $i = m < j < n$, (II) $i < m < n < j$, (III) $i < m < j = n$,
 $\frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n}$
 (IV) $i < m < j < n$, (V) $i < j = m < n$, (VI) $i < j < m < n$.
 $\frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n}$

$$\begin{aligned}
q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} &= 0 && \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\
[e_{ij}, e_{mn}] &= 0 && \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\
[e_{ij}, e_{mn}] &= (q^2 - q^{-2})e_{in}e_{mj} && \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\
q^2e_{ij}e_{mn} - e_{mn}e_{ij} &= qe_{in} && \text{if } ((i, j), (m, n)) \in C_{(V)}.
\end{aligned}$$

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```

In[*]:= RR_k[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 U_k[xij, xmn], (* I∨III *)
    (i < m < j ∧ n < j) ∨ (j < m), U_k[xij, xmn], (* II∨VI *)
    i < m < j ∧ n > j, U_k[xij, xmn] + (q-2 - q2) U_k[x[i, n], x[m, j]], (* IV *)
    j == m, q2 U_k[xij, xmn] - q U_k[x[i, n]] (* V *)
  ]
]

In[*]:= U_1[x34] U_2[x23] U_3[x12] // m1,2→2
Time: 0.06 seconds Out:=
-q U_2[x24] U_3[x12] + q2 U_2[x23, x34] U_3[x12]

In[*]:= U_1[x34] U_2[x23] U_3[x12] // m1,2→2 // m2,3→0
Time: 0.05 seconds Out:=
q2 U_0[x14] - q3 U_0[x12, x24] - q3 U_0[x13, x34] + q4 U_0[x12, x23, x34]

In[*]:= Block[{$k = ∞}, Module[{Bas = $PBWgens, P},
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U_1[X] U_2[Y] U_3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
Time: 1.05 seconds Out:=
{}

In[*]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U_1[X] U_2[Y] U_3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[*]=
{}

```

```

Module[{Bas = $PBWGens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3 ->2 // m2,4 ->2 // m1,2 ->1) ==
      (P // m3,4 ->3 // m2,3 ->2 // m1,2 ->1) == (P // m1,2 ->1 // m3,4 ->3 // m1,3 ->1) ==
      (P // m1,2 ->1 // m1,3 ->1 // m1,4 ->1) == (P // m2,3 ->2 // m1,2 ->1 // m1,4 ->1)
  ]
]
» {x12, x13, x23, a1, a2}

Out[*]=
{ }

```

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The Co-Product, Co-Associativity, and the Hopf Axiom

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According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

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```

In[*]:= Do[As[i, 1, $n - 1],
  Δj,k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj,k[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj,k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj,k[x[i, n]] = mk,k1 ->k[mj,j1 ->j[
    q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
  ]
]

```

```

In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P -> Δi ->j,k[P]
]

```

Time: 0.03 seconds Out[*]=

```

Ui[x12] -> Uj[x12] Uk[] - 2 ∈ ħ Uj[x12] Uk[a1] + Uj[] Uk[x12] + 2 ∈ ħ Uj[a1] Uk[x12]
Ui[x13] -> Uj[x13] Uk[] - 2 ∈ ħ Uj[x13] Uk[a1] - 2 ∈ ħ Uj[x13] Uk[a2] +
  Uj[] Uk[x13] + 2 ∈ ħ Uj[a1] Uk[x13] + 2 ∈ ħ Uj[a2] Uk[x13] + 4 ∇ ∈ ħ Uj[x12] Uk[x23]
Ui[x23] -> Uj[x23] Uk[] - 2 ∈ ħ Uj[x23] Uk[a2] + Uj[] Uk[x23] + 2 ∈ ħ Uj[a2] Uk[x23]
Ui[a1] -> Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] -> Uj[a2] Uk[] + Uj[] Uk[a2]

```

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→1,2 // Δ2→2,3) = (P // Δ1→1,3 // Δ1→1,2)
  ]]
]
» {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}
Out[ ]:=
{ }

```

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0][Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]]
]
» {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}
Out[ ]:=
{ }

```

Is Δ a quantization of the co-bracket?

```

In[ ]:= $k = 1
Time: 0.02 seconds Out[ ]:=
1
In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1/@Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[_xs_, A[_n_], as_] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[ ]:=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 γ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 γ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0

```

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The Antipode

pdf

In[]:=

```

Do[As[i, 1, $n - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj-[x[i, n]] = mj1, j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]

Do[As[i, 1, $n - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj-[x[i, n]] = mj1, j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]

```

In[]:=

```

Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
» {x12, x13, x23, a1, a2, A2[-1], A2[-1], A2[1], A1[2], A1[-2], A1[-1], A1[-2], A1[1], A1[0], A1[-2]}

```

Out[]:=

```

{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A2[-1] → U1[],
 A2[-1] → U1[], A2[1] → U1[], A1[2] → U1[], A1[-2] → U1[],
 A1[-1] → U1[], A1[-2] → U1[], A1[1] → U1[], A1[0] → U1[], A1[-2] → U1[]}

```

```

In[*]:= $k = ∞
Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> UCF[(P // S1 // S1) - P]
  ]
]
Time: 0.05 seconds Out|=
∞

» {x12, x13, x23, a1, a2, A2[-1], A1[-2], A2[0], A1[-2], A1[2], A1[0], A1[0], A2[2], A1[0], A2[-1]}
Out[*]=
{}

```

pdf

The Generating Functions for the $\mathbb{A}$ operations

```

In[*]:= BeginProfile[]
Time: 0.03 seconds Out|=
BeginProfile[]

```

```

In[*]:= $k = 1;

```

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Ad, TT, Seepage

pdf

```

In[*]:= PotQ[0] = True;
PotQ[c_. Ui[aj_] (EUs : U-[...] ..) ] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);

```

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```

In[*]:= Ad[x_] [y_] /; PotQ[x] ∧ (Head[y] != Plus) := PPAdPoty [If[y === 0, 0, eUB[y,x]/y y]];
Ad[x_] [y_Plus] /; PotQ[x] := PPAdPotPlus [Ad[x] /@ y];

```

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```

In[*]:= Ad[x_] [y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil @Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]

```

In[]:= **Ad**[7 $\hbar$ U_j [**x12**] U_k [**a3**]] [U_j [**x23**] U_k [**a4**]]

Time: 0.09 seconds Out[]:=

$$U_j[x23] U_k[a4] + \left(-7 \hbar - 7 \gamma \in \hbar^2 - \frac{7}{2} \gamma^2 \epsilon^2 \hbar^3 \right) U_j[x13] U_k[a3, a4] + \\ \left(14 \gamma \in \hbar^2 + 14 \gamma^2 \epsilon^2 \hbar^3 \right) U_j[x12, x23] U_k[a3, a4] - 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x12, x13] U_k[a3, a3, a4] + \\ 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x12, x12, x23] U_k[a3, a3, a4] - \frac{686}{3} \gamma^2 \epsilon^2 \hbar^5 U_j[x12, x12, x13] U_k[a3, a3, a3, a4]$$

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In[]:= **TT** _{λ} [] = 0; (* Translated Tangent *)
TT _{λ} [**xs**____, **x**_] := **PP****TT**@**UCF**[**Ad**[**x**][**TT** _{λ} [**xs**]] + ∂_{λ} **x**];
TT _{λ} [**xs_List**] := **TT** _{λ} @@**xs**

In[]:= **TT** _{λ} [λ ξ 12 U_0 [**x12**] U_1 [], λ ξ 23 U_0 [**x23**] U_1 []]

Time: 0.01 seconds Out[]:=

$$\xi 12 U_0[x12] U_1[] + \lambda \xi 12 \xi 23 U_0[x13] U_1[] + \xi 23 U_0[x23] U_1[]$$

pdf

In[]:= **O** _{i} [**poly**_] := **Total**[
CoefficientRules[**poly**, # _{i} & /@ **\$PBWGens**] /.
(**p** → **c**) => **c** **U** _{i} @@**Flatten**@**MapThread**[**ConstantArray**, {**\$PBWGens**, **p**}]
];
O _{i} ,**is**_ _{i} [**poly**_] := **O** _{i} [**O** _{i} ,**is**_ _{i} [**poly**]]];
P _{i} [**e**_] := **e** /. **U** _{i} [**xs**____] => **Times**@@ (# _{i} & /@ {**xs**});
P _{i} ,**is**_ _{i} [**poly**_] := **P** _{i} [**P** _{i} ,**is**_ _{i} [**poly**]]];

In[]:= **O** _{i} [**x12** _{i} ² **x13** _{i} + 7 **x23** _{i} ¹⁷]

Time: 0.06 seconds Out[]:=

$$U_i[x12, x12, x13] + \\ 7 U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]$$

In[]:= **O** _{i,j} [**x12** _{i} ² **x13** _{i} **x24** _{j} + 7 **x23** _{i} ¹⁷ **x25** _{j} ³]

Time: 0.03 seconds Out[]:=

$$U_i[x12, x12, x13] U_j[x24] + \\ 7 U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23] \\ U_j[x25, x25, x25]$$

In[]:= **P** _{i,j} @**O** _{i,j} [**x12** _{i} ² **x13** _{i} **x24** _{j} + 7 **x23** _{i} ¹⁷ **x25** _{j} ³]

Time: 0.06 seconds Out[]:=

$$x12_i^2 x13_i x24_j + 7 x23_i^{17} x25_j^3$$

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```

In[ ]:= Seepage_i_[{ }, c_] = c;
Seepage_i_[ss_List, P_] := Module[{lx},
  lx = $PBWgens[Length@ss];
  CF@Collect[P, lx_i, P_i[Ad[Last[ss] U_i[lx]] [0_i[Seepage_i[Most@ss, #]]]]
    &]
];
Seepage_i_,is_[ss_List, ssss_List, P_] := Seepage_i_[ss, Seepage_is[ssss, P]]

```

pdf

```

In[ ]:= Invert_is_[s_] := Module[{A, B, U1, j}, (* s=A U[] + e B *)
  U1 = Times@@ (U_[] & /@ {is});
  A = Coefficient[s, U1] /. e -> 0;
  B = UCF[(s - A U1)];
  UCF[
    A^-1 (U1 + Sum[As[j, 1, $k], (-1)^j NonCommutativeMultiply@@ ConstantArray[B/A, j]])]
]

```

```

In[ ]:= $k = 3;
Invert_i,j[7 U_i[] U_j[] + e U_i[x12] U_j[x23]]

```

Time: 0.09 seconds Out[]:=

$$\begin{aligned}
& \frac{1}{7} U_i[] U_j[] - \frac{1}{49} e U_i[x12] U_j[x23] + \\
& \frac{1}{343} e^2 U_i[x12, x12] U_j[x23, x23] - \frac{e^3 U_i[x12, x12, x12] U_j[x23, x23, x23]}{2401}
\end{aligned}$$

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```

In[ ]:= TriangularDSolve[eqns_, inits_, fs_, l_, CF_] :=
  PPTriangularDSolve@Module[{ss = {}, reqns = eqns, rinits = inits, rfs = fs, s, p},
    While[rfs != {},
      {p} = FirstPosition[Count[#, Alternatives@@rfs, ∞, Heads -> True] & /@ reqns, 1];
      {{s}} = DSolve[reqns[[p]] ^ rinits[[p]], rfs[[p], l];
      AppendTo[ss, s = s /. F_Function -> ReplacePart[F, 2 -> CF[F[[2]]]]];
      reqns = Delete[reqns, p] /. s /. e_ == 0 -> CF[e] == 0;
      rinits = Delete[rinits, p];
      rfs = Delete[rfs, p];
    ];
    ss
  ]

```

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The Generating Functions Engine

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Canonical Forms:

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```
In[*]:= LogReduce[ $\mathcal{E}$ _] :=
   $\mathcal{E}$  /.  $c$ _ * Log[ $a$ _]  $\Rightarrow$  Log@Factor[ $a^c$ ] /. Log[ $a$ _] + Log[ $b$ _]  $\Rightarrow$  Log@Factor[ $a b$ ];
CF[ $\mathcal{E}$ _E] := CF /@ MapAt[LogReduce,  $\mathcal{E}$ , 1];
CF[ $\mathcal{E}$  : E_[___]] := CF /@  $\mathcal{E}$ ;
```

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E operations:

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```
In[*]:=  $\mathcal{E}$ _E[$_] := Length[ $\mathcal{E}$ ] - 1; E_[ $\mathcal{E}$ s___] [$_] := E[ $\mathcal{E}$ s] [$];
 $\mathcal{E}$ _E[k_Integer] :=  $\mathcal{E}$ [[k+1]]; E_[ $\mathcal{E}$ s___][k_Integer] := { $\mathcal{E}$ s}[[k+1]];
E /:  $\mathcal{E}1$ _E  $\equiv$   $\mathcal{E}2$ _E := Inner[CF@#1 == CF@#2 &,  $\mathcal{E}1$ ,  $\mathcal{E}2$ , And];
 $\mathbb{E}_{d1 \rightarrow r1}$ [ $\mathcal{E}1$ s___]  $\equiv$   $\mathbb{E}_{d2 \rightarrow r2}$ [ $\mathcal{E}2$ s___]  $\wedge$  := ( $d1 == d2$ )  $\wedge$  ( $r1 == r2$ )  $\wedge$  ( $\mathbb{E}[\mathcal{E}1$ s]  $\equiv$   $\mathbb{E}[\mathcal{E}2$ s]);
E /:  $\mathcal{E}1$ _E *  $\mathcal{E}2$ _E := E @@ Table[CF[ $\mathcal{E}1$ [kk] +  $\mathcal{E}2$ [kk]], {kk, 0, Min[ $\mathcal{E}1$ [$],  $\mathcal{E}2$ [$]]};
 $\mathbb{E}_{d1 \rightarrow r1}$ [ $\mathcal{E}1$ s___]  $\mathbb{E}_{d2 \rightarrow r2}$ [ $\mathcal{E}2$ s___]  $\wedge$  :=  $\mathbb{E}_{(d1 \cup d2) \rightarrow (r1 \cup r2)}$  @@ ( $\mathbb{E}[\mathcal{E}1$ s]  $\mathbb{E}[\mathcal{E}2$ s]);
```

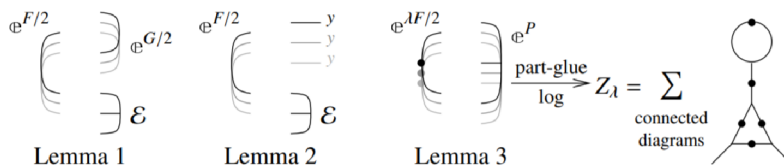
pdf

```
In[*]:=  $\mathbb{E}_{d1 \rightarrow r1}$ [ $\mathcal{E}1$ s___] //  $\mathbb{E}_{d2 \rightarrow r2}$ [ $\mathcal{E}2$ s___] := Module[{is =  $r1 \cap d2$ , lvs, llvs},
  lvs = Cases[{ $\mathcal{E}1$ s}, (Alternatives @@ LatinVars)_i_ /; MemberQ[is, i],  $\infty$ ]  $\cup$ 
  Cases[{ $\mathcal{E}2$ s}, (Alternatives @@ GreekVars)_i_ /; MemberQ[is, i],  $\infty$ ]*;
  llvs = lvs /.  $u_{i\_}$   $\Rightarrow$   $u_{\$i}$ ;
   $\mathbb{E}_{(d1 \cup \text{Complement}[d2, is]) \rightarrow (r2 \cup \text{Complement}[r1, is])}$  @@ (Zip[llvs, llvs*][{(F /@ llvs*), (F /@ llvs)}, Times[
     $\mathbb{E}[\mathcal{E}1$ s] /. Thread[lvs  $\rightarrow$  llvs],
     $\mathbb{E}[\mathcal{E}2$ s] /. Thread[lvs*  $\rightarrow$  llvs*]
  ]])
]
```

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Zippping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii} ’s and the y_{ii} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

pdf

```

In[*]:= Zipvs [{ $\mathcal{F}$ _,  $\mathcal{E}$ _}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0}  $\vee$  Wt[#] == {1, 0}) &]},
  { $\mathcal{F}$ _,  $\mathcal{E}$ _} // Zip1vs (* // Zip2Complement[vs,dvs] *) // Zip2dvs //
  EZip3Complement[vs,dvs] // Zip3dvs
] // Last

```

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Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F: \mathcal{E} \otimes^{\frac{1}{2}} \sum_{i,j \in B} G_{ij} z_i z_j \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1}: \mathcal{E} \right\rangle_B$$

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```

In[*]:= Zip1{} = Identity;
Zip1vs @ { $\mathcal{F}$ _,  $\mathbb{E}[Q_, P\_]$ } := Module[{ $\mathcal{I}$ , F, G, u, v},
  (*Echo[{vs, $\mathcal{F}$ ,Q,P}];*)
   $\mathcal{I}$  = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{\mathcal{F}[u], \mathcal{F}[v]} \mathcal{F}$ , 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v} Q$ , 0], {u, vs}, {v, vs}];
  (*Echo@MatrixForm[ $\mathcal{I} - G.F$ ];*)
  {CF[(F /@ vs) . (F.Inverse[ $\mathcal{I} - G.F$ ]) . (F /@ vs) / 2],
    $\mathbb{E}$ [CF[Q - PowerExpand@Log[Det[ $\mathcal{I} - G.F$ ]] / 2 - vs.G.vs / 2], P]}
]

```

```

In[*]:= Wt /@ {x12$[2], y12$[1],  $\eta$ 12$[1],  $\xi$ 12$[2]}

```

Time: 0.00 seconds Out[*]=

```
{ {1, -1}, {0, 1}, {1, -1}, {0, 1} }
```

```

In[*]:=  $\mathbb{E}_{\{\} \rightarrow \{1,2\}}$  [x122 y121] //  $\mathbb{E}_{\{1,2\} \rightarrow \{\}}$  [(1 - eb11)  $\eta$ 121  $\xi$ 122]

```

Time: 0.02 seconds Out[*]=

```
 $\mathbb{E}_{\{\} \rightarrow \{\}}$  [-b11]
```

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Getting rid of linear terms.

Lemma 2. $\left\langle F: \mathcal{E} \otimes^{\frac{1}{2}} \sum_{i,j \in B} F_{ij} y_i y_j \right\rangle_B = \otimes^{\frac{1}{2}} \sum_{i,j \in B} F_{ij} y_i y_j \left\langle F: \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

pdf

```

In[*]:= Zip2{} = Identity;
Zip2vs @ { $\mathcal{F}$ _,  $\mathbb{E}[Q_, P\_]$ } := Module[{F, Y, u, v},
  (*EchoLabel["Zip2"] [{vs, $\mathcal{F}$ ,Q,P}];*)
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[ $\partial_{\mathcal{F}[u], \mathcal{F}[v]} \mathcal{F}$ ], 0], {u, vs}, {v, vs}];
  Y = Table[ $\partial_v Q$ , {v, vs}] /. (*AlsoUpper@*) Table[v  $\rightarrow$  0, {v, vs}];
  CF /@ ({ $\mathcal{F}$ _,  $\mathbb{E}[Q - Y.vs + Y.F.Y / 2$ , P]} /. (*AlsoUpper@*) Thread[vs  $\rightarrow$  vs + F.Y])
]

```

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Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathbb{P}]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} (\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

pdf

In[*]:=

```
Zip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := Module[
  {F, u, v, Z, kk, jj, $m = 0, m, n},
  (*Echo[{"Zip3", vs,  $\mathcal{F}$ ,  $\mathcal{E}$ }]*)
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0,  $\mathcal{E}$ @$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0},  $\partial_{F[u], F[v]} \mathcal{F}$ , 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤  $\mathcal{E}$ @$, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2(m+1)}$  (Z[m, kk, u, v] +
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1];
  ];
  CF /@ (
     $\mathcal{F}$  - Sum[F[u, v] * F[u] * F[v] / 2, {u, vs}, {v, vs}],
    E@@Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0,  $\mathcal{E}$ @$}]
  ) /. (*AlsoUpper@*) Table[v → 0, {v, vs}]
]
```

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In[*]:=

```
EZip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := PPEZip3@Module[
  {n $\mathcal{E}$ , n $\mathcal{F}$ , rc, ps, rr = {(*release rules*)}, Fvs},
  rc = 0; n $\mathcal{E}$  = Total[
    CoefficientRules[#, vs] /.
      (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{E}$ [++rc] → c]; c $\mathcal{E}$ [rc] (Times@@vsps))
  ] & /@  $\mathcal{E}$ ;
  rc = 0; Fvs = F /@ vs;
  n $\mathcal{F}$  = Total[CoefficientRules[ $\mathcal{F}$ , Fvs] /.
    (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{F}$ [++rc] → c]; c $\mathcal{F}$ [rc] (Times@@Fvsps))];
  CF[Expand[{n $\mathcal{F}$ , n $\mathcal{E}$ } // Zip3vs] /. rr]
]
```

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“Define” Code

pdf

```

In[ ]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs]);
Define[op_is__ = ε_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[op_nisp, $E_, PPBoot@Block[{i, j, k}, op_isp, $E = ε; op_nis, $E]];
    SD[op_isp, op_{is}, $E]; SD[op_sis__, op_{sis}];
  ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
  } ] ]

```

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$$L_\lambda = \bigoplus_{i,j} \langle \lambda_{ij}^1 x \rangle \bigoplus_{i,j} \langle \lambda_{ij}^2 x \rangle \quad \text{Seepage is defined by: } \bigoplus_{i,j} \langle f_i x_i \rangle \bigoplus_{i,j} \langle \text{Seepage}(f_i, A) \rangle = \bigoplus_{i,j} \langle e^{f_i x_i} A \rangle$$

$$TT(L_\lambda) = L' \partial_\lambda L_\lambda \quad S(B) = \text{Seepage}(\{f_i\}, B).$$

$$R_\lambda = \bigoplus_{i,j} (f + g), \quad f = f_i x_i, \quad g = \sum_{i,j} g_{ij} w, \quad e^f = F, \quad e^g = G, \quad R_\lambda = \bigoplus (FG), \quad \partial_\lambda F = F \partial_\lambda f$$

$$TT(R_\lambda) = R'_\lambda \partial_\lambda R_\lambda = \bigoplus (FG)' \bigoplus (\partial_\lambda FG) = S(G)' \bigoplus (F)' \bigoplus (F(\partial_\lambda f G + \partial_\lambda G)) = S(G)' S(\partial_\lambda f G + \partial_\lambda G)$$

pdf

In[*]:=

```

Define[amti,j→k = PPamt[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ES[s_][ε_] := ε;
  TTL = ES["TTL"]@PPTTL@TTλ@
    Join[l1 = Table[If[τ[g] == "x", λ, 1] (g*)i Uk[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
  ff = Table[
    Sum[εp fp,uk[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp-,u-[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
  fs = Flatten[Table[
    fp,ℙk[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙk[u][λ] ℙk[u],
    {p, 0, $k}, {u, Complement[mons[p], Uk /@ $PBWGens]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Ok[Seepagek[ff, SS@UCF@B]]];
  SG = ES["SG"]@PPSG@S[G];
  TTR = ES["TTR"]@PPTTR@UCF[Invertk[SG] ⊗ S[(∂λ(ff.(#k & /@ $PBWGens)) G + ∂λG)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i,j}→{k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], (α[m]i + α[m]j) a[m]k], 0],
    Total[Cases[fs, fp,-]] /. fp,- => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]

```

pdf

In[]:=

```

Define[aΔti→j,k = PPaΔt[$k,$n] @Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
TTL = ES["TTL"] @
  TTλ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
mons[-1] = {};
mons[p_] := mons[p-1] ∪
  Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] => Uj[x] Uk[y], ∞];
ffj = Table[
  Sum[εp fp,uj[λ], {p, 0, $k}],
  {u, $PBWGens}
] /. fp,uj[λ] /; τ[u] == "a" => If[p == 0, (u*)i, 0];
ffk = ffj /. j -> k;
fs = Flatten[Table[
  fp,ℙj,k[m],
  {p, 0, $k}, {m, mons[p]}
]];
gg = Sum[
  εp fp,ℙj,k[u][λ] ℙj,k[u],
  {p, 0, $k}, {u, Complement[mons[p], Uj[] (Uk /@ $PBWGens) ∪ Uk[] (Uj /@ $PBWGens)]}
];
G = Series[egg, {ε, 0, $k}];
S[B_] := UCF[SS@Oj,k[Seepagej,k[ffj, ffk, SS@UCF@B]]];
SG = ES["SG"] @ PPSG @ S[G];
TTR = ES["TTR"] @ PPTTR @ UCF[
  Inverttj,k[SG] ⊗ S[(∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λ G)]];
diff = UCF[TTL - TTR];
eqns = Flatten[Table[
  Coefficient[Coefficient[diff, m], ε, p] == 0,
  {p, 0, $k}, {m, mons[p]}
]];
inits = (#[0] == 0) & /@ fs;
ss = TriangularDSolve[eqns, inits, fs, λ, CF];
ℰ{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
  If[p == 0, Sum[As[m, 1, $n-1], α[m]i (a[m]j + a[m]k)], 0],
  Total[Cases[fs, fp,-]] /. fp,u => u (fp,u /. ss)[λ] /. λ | ħ | γ -> 1]
]
]]

```

In[*]:=

```

Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] :=
    mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c_. Ui[x__] => Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,ui[λ] /; τ[u] == "a" => If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp,ℙi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙi[u][λ] ℙi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⓧ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], -α[m]i a[m]i], 0],
    Total[Cases[fs, fp,u]] /. fp,u => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

```

In[*]:= Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] :=
    mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c_. Ui[x___] ⇒ Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,ui[λ] /; τ[u] == "a" ⇒ If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp,ℙi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙi[u][λ] ℙi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[εgg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⌗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], -α[m]i a[m]i], 0],
    Total[Cases[fs, fp,-]] /. fp,- ⇒ u (fp,u /. ss)[λ] /. λ | ℏ | γ → 1]
  ]
]]

```

```

In[*]:= $k = 1; $n = 3;

```

```

In[*]:= aSt1 // aSt1

```

Time: 5.94 seconds Out[] =

```

E{1}→{1} [a11 α11 + a21 α21 + x121 ξ121 + x131 ξ131 + x231 ξ231, 0]

```

```
In[*]:= (*Splitter into B tensor A*)
SYi→j, k, l, m := E{i}→{j, k, l, m} [
  Sum[η[r, s]i y[r, s]j + ξ[r, s]i x[r, s]m, {r, 1, $n - 1}, {s, r + 1, $n}] +
  Sum[α[r]i a[r]l + β[r]i b[r]k, {r, 1, $n - 1}], Sequence @@ ConstantArray[0, $k]
]
```

```
In[*]:= SY1→2, 3, 4, 5
```

```
Time: 0.00 seconds Out[ ]=
```

$$E_{\{1\} \rightarrow \{2, 3, 4, 5\}} [a_{14} \alpha_{11} + a_{24} \alpha_{21} + b_{13} \beta_{11} + b_{23} \beta_{21} + y_{112} \eta_{111} + y_{122} \eta_{121} + y_{132} \eta_{131} + y_{222} \eta_{221} + y_{232} \eta_{231} + x_{115} \xi_{111} + x_{125} \xi_{121} + x_{135} \xi_{131} + x_{225} \xi_{221} + x_{235} \xi_{231}, 0]$$

P: A tensor B → k
R in B tensor A

```
In[*]:= Pi, j := E{i, j}→{} @@ PadRight [
  {Sum[η[r, s]j ξ[r, s]i, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[β[r]j α[r]i, {r, 1, $n - 1}]},
  $k + 1]
Ri, j := E{ }→{i, j} @@ PadRight [
  {Sum[y[r, s]i x[r, s]j, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[b[r]i a[r]j, {r, 1, $n - 1}]},
  $k + 1]
```

```
In[*]:= R1, 2 // P2, 3
R1, 2 // P5, 1
```

```
Time: 0.05 seconds Out[ ]=
```

$$E_{\{3\} \rightarrow \{1\}} [b_{11} \beta_{13} + b_{21} \beta_{23} + y_{121} \eta_{123} + y_{131} \eta_{133} + y_{231} \eta_{233}, 0]$$

```
Out[ ]=
```

$$E_{\{5\} \rightarrow \{2\}} [a_{12} \alpha_{15} + a_{22} \alpha_{25} + x_{122} \xi_{125} + x_{132} \xi_{135} + x_{232} \xi_{235}, 0]$$

```
In[*]:= Define[bmtj, k→i = Module[{i0, i1, j1, k1}, (Ri0, i1 // aΔti1→j1, k1 // Pj1, j // Pk1, k) /. i0 → i],
  bΔtk→i, j = Module[{i1, j1, k1, k2}, ((Ri, i1 Rj, j1 // amti1, j1→k1) // Pk1, k2) /. k2 → k],
  bSti = Module[{i1, i2}, (Ri, i1 // aSti1 // Pi1, i2) /. i2 → i],
  bSti = Module[{i1, i2}, (Ri, i1 // aSti1 // Pi1, i2) /. i2 → i]]
Define[Ri, j = Ri, j // bSti]
```

```
In[*]:= amt1, 2→3 // l2U
```

```
Time: 0.03 seconds Out[ ]=
```

$$E_{\{1, 2\} \rightarrow \{3\}} \left[a_{13} (\alpha_{11} + \alpha_{12}) + a_{23} (\alpha_{21} + \alpha_{22}) + x_{123} \left(\xi_{121} + e^{\alpha_{11} - \frac{\alpha_{21}}{2}} \xi_{122} \right) + x_{133} \left(\xi_{131} + e^{\frac{\alpha_{11}}{2} + \frac{\alpha_{21}}{2}} \xi_{132} - e^{\alpha_{11} - \frac{\alpha_{21}}{2}} \xi_{122} \xi_{231} \right) + x_{233} \left(\xi_{231} + e^{-\frac{\alpha_{11}}{2} + \alpha_{21}} \xi_{232} \right), \right. \\ \left. - 2 e^{\alpha_{11} - \frac{\alpha_{21}}{2}} x_{123} x_{133} \xi_{122} \xi_{131} - e^{\alpha_{11} - \frac{\alpha_{21}}{2}} x_{133} \xi_{122} \xi_{231} + 2 e^{\alpha_{11} - \frac{\alpha_{21}}{2}} x_{123} x_{233} \xi_{122} \xi_{231} - 2 e^{\frac{\alpha_{11}}{2} + \frac{\alpha_{21}}{2}} x_{133} x_{233} \xi_{132} \xi_{231} \right]$$

```
In[*]:= $k = 1; $n = 2;
```

In[*]:= **bmt**_{1,2→3}

$$\begin{aligned}
 & \gg \text{TTL } e^{-\gamma \alpha_{1i}} \xi_{12i} U_j[x_{12}] U_k[] - 2 e^{-\gamma \alpha_{1i}} \in \xi_{12i} U_j[x_{12}] U_k[a_1] + \\
 & \quad e^{-\gamma \alpha_{1i}} \xi_{12i} U_j[] U_k[x_{12}] + 2 e^{-\gamma \alpha_{1i}} \in \xi_{12i} U_j[a_1] U_k[x_{12}] \\
 & \gg \text{SG } U_j[] U_k[] + e^{-\gamma \alpha_{1i}} \in U_j[x_{12}] U_k[a_1] f_{1,a_{1k} x_{12j}}[\lambda] + e^{-\gamma \alpha_{1i}} \in U_j[a_1] U_k[x_{12}] f_{1,a_{1j} x_{12k}}[\lambda] \\
 & \gg \text{TTR } U_j[x_{12}] U_k[x_{12}] \left(-e^{-2\gamma \alpha_{1i}} \gamma \in f_{1,a_{1j} x_{12k}}[\lambda] f_{\theta, x_{12j}}'[\lambda] - e^{-2\gamma \alpha_{1i}} \gamma \in f_{1,a_{1k} x_{12j}}[\lambda] f_{\theta, x_{12k}}'[\lambda] \right) + \\
 & \quad \ll 3 \gg + e^{-\gamma \alpha_{1i}} \in U_j[a_1] U_k[x_{12}] f_{1,a_{1j} x_{12k}}'[\lambda]
 \end{aligned}$$

Out[*]=

$$E_{\{1,2\} \rightarrow \{3\}} [b_{13} \beta_{11} + b_{13} \beta_{12} + y_{123} \eta_{121} + y_{123} \eta_{122}, -2 y_{123} \beta_{12} \eta_{121} + 2 y_{123} \beta_{11} \eta_{122}]$$

Testing **A** and **B** dually paired Hopf algebras

co-associativity

In[*]:= { (**aΔt**_{1→1,2} // **aΔt**_{2→2,3}) ≡ (**aΔt**_{1→2,3} // **aΔt**_{2→1,2}) , (**bΔt**_{1→1,2} // **bΔt**_{2→2,3}) ≡ (**bΔt**_{1→2,3} // **bΔt**_{2→1,2}) }

$$\begin{aligned}
 & \gg \text{TTL } e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \xi_{12i} U_j[x_{12}] U_k[] + \\
 & \quad \left(e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i})} \xi_{13i} + e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i})} \lambda \xi_{12i} \xi_{23i} + e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i})} \gamma \in \lambda \xi_{12i} \xi_{23i} \right) U_j[x_{13}] U_k[] + \\
 & \quad \ll 31 \gg + 2 e^{-\frac{3 \gamma \alpha_{2i}}{2}} \gamma \in \lambda \xi_{13i} \xi_{23i} U_j[] U_k[x_{13}, x_{23}] \\
 & \gg \text{SG } U_j[] U_k[] + e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \in U_j[x_{12}] U_k[a_1] f_{1,a_{1k} x_{12j}}[\lambda] + \ll 21 \gg + \\
 & \quad U_j[] U_k[x_{13}, x_{23}] \left(e^{-\frac{3 \gamma \alpha_{2i}}{2}} \in f_{\theta, x_{23k}}[\lambda] f_{1, x_{12k} x_{23k}}[\lambda] + e^{-\frac{3 \gamma \alpha_{2i}}{2}} \in f_{1, x_{13k} x_{23k}}[\lambda] \right) \\
 & \gg \text{TTR } \ll 1 \gg
 \end{aligned}$$

Out[*]=

{True, True}

algebra morphism

In[*]:= { (**aΔt**_{i→1,2} **aΔt**_{j→3,4} // **amt**_{1,3→i} // **amt**_{2,4→j}) ≡ (**amt**_{i,j→k} // **aΔt**_{k→i,j}) ,
 (**bΔt**_{i→1,2} **bΔt**_{j→3,4} // **bmt**_{1,3→i} // **bmt**_{2,4→j}) ≡ (**bmt**_{i,j→k} // **bΔt**_{k→i,j}) }

Time: 0.45 seconds Out[*]=

{True, True}

In[*]:= (aΔt_{i→1,2} aΔt_{j→3,4} // amt_{1,3→i} // amt_{2,4→j})

Time: 0.22 seconds Out[]=

$$\begin{aligned}
& E_{\{i,j\} \rightarrow \{i,j\}} \left[a_{1i} \alpha_{1i} + a_{1j} \alpha_{1i} + a_{1i} \alpha_{1j} + a_{1j} \alpha_{1j} + a_{2i} \alpha_{2i} + a_{2j} \alpha_{2i} + a_{2i} \alpha_{2j} + a_{2j} \alpha_{2j} + \right. \\
& x_{12i} \xi_{12i} + x_{12j} \xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12i} \xi_{12j} + e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12j} \xi_{12j} + x_{13i} \xi_{13i} + x_{13j} \xi_{13i} + \\
& e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{13i} \xi_{13j} + e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{13j} \xi_{13j} + x_{23i} \xi_{23i} + x_{23j} \xi_{23i} - e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{13i} \xi_{12j} \xi_{23i} - \\
& e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{13j} \xi_{12j} \xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i}+2\alpha_{2i})} x_{23i} \xi_{23j} + e^{\frac{1}{2} (-\alpha_{1i}+2\alpha_{2i})} x_{23j} \xi_{23j}, \\
& -2 a_{1j} x_{12i} \xi_{12i} + 2 a_{1i} x_{12j} \xi_{12i} - 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} a_{1j} x_{12i} \xi_{12j} + 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} a_{1i} x_{12j} \xi_{12j} - \\
& 2 a_{1j} x_{13i} \xi_{13i} - 2 a_{2j} x_{13i} \xi_{13i} + 2 a_{1i} x_{13j} \xi_{13i} + 2 a_{2i} x_{13j} \xi_{13i} + 4 x_{12i} x_{23j} \xi_{13i} + \\
& x_{12j} x_{13i} \xi_{12i} \xi_{13i} - x_{12i} x_{13j} \xi_{12i} \xi_{13i} - 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12i} x_{13i} \xi_{12j} \xi_{13i} - \\
& e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12j} x_{13i} \xi_{12j} \xi_{13i} - 3 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12i} x_{13j} \xi_{12j} \xi_{13i} - \\
& 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12j} x_{13j} \xi_{12j} \xi_{13i} - 2 e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} a_{1j} x_{13i} \xi_{13j} - 2 e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} a_{2j} x_{13i} \xi_{13j} + \\
& 2 e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} a_{1i} x_{13j} \xi_{13j} + 2 e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} a_{2i} x_{13j} \xi_{13j} + 4 e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{12i} x_{23j} \xi_{13j} + \\
& e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{12j} x_{13i} \xi_{12i} \xi_{13j} - e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{12i} x_{13j} \xi_{12i} \xi_{13j} + e^{\frac{3\alpha_{1i}}{2}} x_{12j} x_{13i} \xi_{12j} \xi_{13j} - \\
& e^{\frac{3\alpha_{1i}}{2}} x_{12i} x_{13j} \xi_{12j} \xi_{13j} - 2 a_{2j} x_{23i} \xi_{23i} + 2 a_{2i} x_{23j} \xi_{23i} - x_{12j} x_{23i} \xi_{12i} \xi_{23i} + \\
& x_{12i} x_{23j} \xi_{12i} \xi_{23i} - e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{13i} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} a_{1j} x_{13i} \xi_{12j} \xi_{23i} + \\
& 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} a_{2j} x_{13i} \xi_{12j} \xi_{23i} - e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{13j} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} a_{1i} x_{13j} \xi_{12j} \xi_{23i} - \\
& 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} a_{2i} x_{13j} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12i} x_{23i} \xi_{12j} \xi_{23i} + \\
& e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12j} x_{23i} \xi_{12j} \xi_{23i} - e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12i} x_{23j} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12j} x_{23j} \xi_{12j} \xi_{23i} - \\
& e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12j} x_{13i} \xi_{12i} \xi_{12j} \xi_{23i} + e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{12i} x_{13j} \xi_{12i} \xi_{12j} \xi_{23i} - \\
& e^{2\alpha_{1i}-\alpha_{2i}} x_{12j} x_{13i} \xi_{12j}^2 \xi_{23i} + e^{2\alpha_{1i}-\alpha_{2i}} x_{12i} x_{13j} \xi_{12j}^2 \xi_{23i} + x_{13j} x_{23i} \xi_{13i} \xi_{23i} - \\
& x_{13i} x_{23j} \xi_{13i} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{13i} x_{23i} \xi_{13j} \xi_{23i} - e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{13j} x_{23i} \xi_{13j} \xi_{23i} - \\
& 3 e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{13i} x_{23j} \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{13j} x_{23j} \xi_{13j} \xi_{23i} - e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{13j} x_{23i} \xi_{12j} \xi_{23i}^2 + \\
& e^{\frac{1}{2} (2\alpha_{1i}-\alpha_{2i})} x_{13i} x_{23j} \xi_{12j} \xi_{23i}^2 - 2 e^{\frac{1}{2} (-\alpha_{1i}+2\alpha_{2i})} a_{2j} x_{23i} \xi_{23j} + 2 e^{\frac{1}{2} (-\alpha_{1i}+2\alpha_{2i})} a_{2i} x_{23j} \xi_{23j} - \\
& e^{\frac{1}{2} (-\alpha_{1i}+2\alpha_{2i})} x_{12j} x_{23i} \xi_{12i} \xi_{23j} + e^{\frac{1}{2} (-\alpha_{1i}+2\alpha_{2i})} x_{12i} x_{23j} \xi_{12i} \xi_{23j} - \\
& e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{12j} x_{23i} \xi_{12j} \xi_{23j} + e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{12i} x_{23j} \xi_{12j} \xi_{23j} + e^{\frac{1}{2} (-\alpha_{1i}+2\alpha_{2i})} x_{13j} x_{23i} \xi_{13i} \xi_{23j} - \\
& e^{\frac{1}{2} (-\alpha_{1i}+2\alpha_{2i})} x_{13i} x_{23j} \xi_{13i} \xi_{23j} + e^{\frac{3\alpha_{2i}}{2}} x_{13j} x_{23i} \xi_{13j} \xi_{23j} - e^{\frac{3\alpha_{2i}}{2}} x_{13i} x_{23j} \xi_{13j} \xi_{23j} - \\
& e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{13j} x_{23i} \xi_{12j} \xi_{23i} \xi_{23j} + e^{\frac{1}{2} (\alpha_{1i}+\alpha_{2i})} x_{13i} x_{23j} \xi_{12j} \xi_{23i} \xi_{23j} \left. \right]
\end{aligned}$$

associativity

In[*]:= { (amt_{1,2→k} // amt_{k,3→k}) ≡ (amt_{2,3→k} // amt_{1,k→k}) ,
 (bmt_{1,2→k} // bmt_{k,3→k}) ≡ (bmt_{2,3→k} // bmt_{1,k→k}) } // Simplify

Time: 0.11 seconds Out[]=

{True, True}

antipode

```
In[*]:= {aΔti→1,2 // aSt1 // amt1,2→1, aΔti→1,2 // aSt2 // amt1,2→1,
          bΔti→1,2 // bSt1 // bmt1,2→1, bΔti→1,2 // bSt2 // bmt1,2→1}
```

Time: 3.62 seconds Out:=

```
{E{i}→{1}[0, 0], E{i}→{1}[0, 0], E{i}→{1}[0, 0], E{i}→{1}[0, 0]}
```

inverse antipode

```
In[*]:= {aSti // aSti, bSti // bSti}}
```

Time: 3.12 seconds Out:=

```
{E{i}→{i}[a1i α1i + a2i α2i + x12i ξ12i + x13i ξ13i + x23i ξ23i, 0],
 E{i}→{i}[b1i β1i + b2i β2i + y12i η12i + y13i η13i + y23i η23i, 0]}
```

Pairing

```
In[*]:= {(amt1,2→3 // P3,-3) ≡ (bΔt-3→-1,-2 // P1,-1 // P2,-2),
          (bmt1,2→3 // P-3,3) ≡ (aΔt-3→-1,-2 // P-1,1 // P-2,2),
          (aSt1 // P1,-1) ≡ (bSt-1 // P1,-1), (aSt1 // P1,-1) ≡ (bSt-1 // P1,-1)}
```

Time: 0.25 seconds Out:=

```
{True, True, True, True}
```

```
In[*]:= P1,2
```

Time: 0.02 seconds Out:=

```
E{1,2}→{}[α11 β12 + η122 ξ121]
```

```
In[*]:= Define[dmi,j→k = Module[{i1, i2, i3, i4},
  ((sYi→i4,i4,i1,i1 // aΔti1→i1,i2 // aΔti2→i2,i3 // aSti3) (sYj→-i1,-i1,-i4,-i4 // bΔt-i1→-i1,-i2 //
    bΔt-i2→-i2,-i3)) // (Pi1,-i3 Pi3,-i1 amti2,-i4→k bmti4,-i2→k)]
Define[dSi = sYi→1,1,2,2 // (bSt1 aSt2) // dm2,1→i,
  dSi = sYi→1,1,2,2 // (bSt1 aSt2) // dm2,1→i,
  dΔi→j,k = (bΔti→1,3 aΔti→4,2) // (dm3,4→k dm1,2→j)]
(*Notice we take co-opposite on the A-side! to make the double*)
```

Testing Double

```
In[*]:= $k = 0; $n = 2;
```

```
In[*]:= dm1,2→3
```

Time: 1.58 seconds Out:=

```
E{1,2}→{3}[a13 α11 + a13 α12 + a23 α21 + a23 α22 + b13 β11 + b13 β12 + b23 β21 + b23 β22 +
  y123 η121 + e $\frac{1}{2}(-2\alpha_1+\alpha_2)$  y123 η122 + y133 η131 + e $\frac{1}{2}(-\alpha_1-\alpha_2)$  y133 η132 + y233 η231 +
  e $\frac{1}{2}(\alpha_1-2\alpha_2)$  y233 η232 + x123 ξ121 - e $\frac{1}{2}(-2\alpha_1+\alpha_2)$  η122 ξ121 + e $\frac{1}{2}(2b_1-b_2-2\alpha_1+\alpha_2)$  η122 ξ121 -
  e $\frac{1}{2}(2b_1-b_2-\alpha_1-\alpha_2)$  y233 η132 ξ121 + e $\frac{1}{2}(2\alpha_1-\alpha_2)$  x123 ξ122 + x133 ξ131 - e $\frac{1}{2}(-\alpha_1-\alpha_2)$  η132 ξ131 +
  e $\frac{1}{2}(b_1+b_2-\alpha_1-\alpha_2)$  η132 ξ131 + e $\frac{1}{2}(\alpha_1+\alpha_2)$  x133 ξ132 + x233 ξ231 + e $\frac{1}{2}(-\alpha_1-\alpha_2)$  y123 η132 ξ231 -
  e $\frac{1}{2}(\alpha_1-2\alpha_2)$  η232 ξ231 + e $\frac{1}{2}(-b_1+2b_2+\alpha_1-2\alpha_2)$  η232 ξ231 - e $\frac{1}{2}(-\alpha_1-\alpha_2)$  η132 ξ121 ξ231 +
  e $\frac{1}{2}(2b_1-b_2-\alpha_1-\alpha_2)$  η132 ξ121 ξ231 - e $\frac{1}{2}(2\alpha_1-\alpha_2)$  x133 ξ122 ξ231 + e $\frac{1}{2}(-\alpha_1+2\alpha_2)$  x233 ξ232,
  -2 y123 β12 η121 + 2 e $\frac{1}{2}(-2\alpha_1+\alpha_2)$  y123 β11 η122 - 2 y133 β12 η131 - 2 y133 β22 η131 +
```

$$\begin{aligned}
& e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} y_{12_3} y_{13_3} \eta_{12_2} \eta_{13_1} + 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} y_{13_3} \beta_{1_1} \eta_{13_2} + 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} y_{13_3} \beta_{2_1} \eta_{13_2} - \\
& e^{\frac{1}{2}(-\alpha_1-\alpha_2)} y_{12_3} y_{13_3} \eta_{12_1} \eta_{13_2} - 2 y_{23_3} \beta_{2_2} \eta_{23_1} - e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} y_{12_3} y_{23_3} \eta_{12_2} \eta_{23_1} + \\
& e^{\frac{1}{2}(-\alpha_1-\alpha_2)} y_{13_3} y_{23_3} \eta_{13_2} \eta_{23_1} + 2 e^{\frac{1}{2}(\alpha_1-2\alpha_2)} y_{23_3} \beta_{2_1} \eta_{23_2} + 4 e^{\frac{1}{2}(\alpha_1-2\alpha_2)} y_{13_3} \eta_{12_1} \eta_{23_2} + \\
& e^{\frac{1}{2}(\alpha_1-2\alpha_2)} y_{12_3} y_{23_3} \eta_{12_1} \eta_{23_2} - e^{\frac{1}{2}(\alpha_1-2\alpha_2)} y_{13_3} y_{23_3} \eta_{13_1} \eta_{23_2} + 4 x_{12_3} \beta_{1_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} \eta_{12_2} \xi_{12_1} + 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} \eta_{12_2} \xi_{12_1} - 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} a_{1_3} \eta_{12_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} a_{1_3} \eta_{12_2} \xi_{12_1} + 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{12_3} y_{12_3} \eta_{12_2} \xi_{12_1} - 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} \beta_{1_2} \eta_{12_2} \xi_{12_1} + \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} \beta_{1_2} \eta_{12_2} \xi_{12_1} - 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} y_{12_3} \eta_{12_1} \eta_{12_2} \xi_{12_1} + \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-4\alpha_1+2\alpha_2)} y_{12_3} \eta_{12_2}^2 \xi_{12_1} - e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} y_{13_3} \eta_{12_2} \eta_{13_1} \xi_{12_1} + \\
& e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{12_3} y_{13_3} \eta_{13_2} \xi_{12_1} - 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \eta_{13_2} \xi_{12_1} + \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} a_{1_3} y_{23_3} \eta_{13_2} \xi_{12_1} - 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \beta_{1_2} \eta_{13_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \beta_{2_1} \eta_{13_2} \xi_{12_1} - 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{13_3} \eta_{12_1} \eta_{13_2} \xi_{12_1} + \\
& e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{12_3} y_{23_3} \eta_{12_1} \eta_{13_2} \xi_{12_1} - e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_1)} y_{13_3} \eta_{12_2} \eta_{13_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_1)} y_{12_3} y_{23_3} \eta_{12_2} \eta_{13_2} \xi_{12_1} + 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{13_3} y_{23_3} \eta_{13_1} \eta_{13_2} \xi_{12_1} - \\
& e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1-2\alpha_2)} y_{13_3} y_{23_3} \eta_{13_2}^2 \xi_{12_1} + e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} y_{23_3} \eta_{12_2} \eta_{23_1} \xi_{12_1} - \\
& e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3}^2 \eta_{13_2} \eta_{23_1} \xi_{12_1} - e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{12_3} y_{23_3} \eta_{23_2} \xi_{12_1} + \\
& e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \eta_{12_2} \eta_{23_2} \xi_{12_1} + e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_2)} y_{23_3}^2 \eta_{13_2} \eta_{23_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{12_3} \eta_{12_2} \xi_{12_1}^2 + 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} x_{12_3} \eta_{12_2} \xi_{12_1}^2 + 2 e^{2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2} \eta_{12_2}^2 \xi_{12_1}^2 - \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-4\alpha_1+2\alpha_2)} \eta_{12_2}^2 \xi_{12_1}^2 - e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} x_{12_3} y_{23_3} \eta_{13_2} \xi_{12_1}^2 - \\
& 3 e^{\frac{1}{2}(4b_{1_3}-2b_{2_3}-3\alpha_1)} y_{23_3} \eta_{12_2} \eta_{13_2} \xi_{12_1}^2 + 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_1)} y_{23_3} \eta_{12_2} \eta_{13_2} \xi_{12_1}^2 + \\
& e^{2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2} y_{23_3}^2 \eta_{13_2}^2 \xi_{12_1}^2 - 2 x_{12_3} \eta_{12_2} \xi_{12_1} \xi_{12_2} - 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3})} x_{12_3} \eta_{12_2} \xi_{12_1} \xi_{12_2} + \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}+\alpha_1-2\alpha_2)} x_{12_3} y_{23_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} + 4 x_{13_3} \beta_{1_2} \xi_{13_1} + 4 x_{13_3} \beta_{2_2} \xi_{13_1} + \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} x_{23_3} \eta_{12_2} \xi_{13_1} + e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{13_3} y_{12_3} \eta_{12_2} \xi_{13_1} - 6 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} \eta_{13_2} \xi_{13_1} + \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} \eta_{13_2} \xi_{13_1} + 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} a_{1_3} \eta_{13_2} \xi_{13_1} - \\
& 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} a_{1_3} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} a_{2_3} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} a_{2_3} \eta_{13_2} \xi_{13_1} + \\
& 4 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{12_3} y_{12_3} \eta_{13_2} \xi_{13_1} + 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{13_3} y_{13_3} \eta_{13_2} \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} x_{23_3} y_{23_3} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} \beta_{1_2} \eta_{13_2} \xi_{13_1} + \\
& 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} \beta_{1_2} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} \beta_{2_2} \eta_{13_2} \xi_{13_1} + 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} \beta_{2_2} \eta_{13_2} \xi_{13_1} - \\
& e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} y_{12_3} \eta_{12_1} \eta_{13_2} \xi_{13_1} + 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_1)} y_{12_3} \eta_{12_2} \eta_{13_2} \xi_{13_1} + \\
& e^{\frac{1}{2}(b_{1_3}+b_{2_3}-3\alpha_1)} y_{12_3} \eta_{12_2} \eta_{13_2} \xi_{13_1} - 2 e^{-\frac{3\alpha_1}{2}} y_{12_3} \eta_{12_2} \eta_{13_2} \xi_{13_1} - \\
& 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} y_{13_3} \eta_{13_1} \eta_{13_2} \xi_{13_1} + 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-2\alpha_1-2\alpha_2)} y_{13_3} \eta_{13_2}^2 \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1-2\alpha_2)} y_{12_3} y_{23_3} \eta_{13_2}^2 \xi_{13_1} - e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \eta_{13_2} \eta_{23_1} \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{12_3} \eta_{23_2} \xi_{13_1} + e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{13_3} y_{23_3} \eta_{23_2} \xi_{13_1} + 4 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} \eta_{12_2} \eta_{23_2} \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} \eta_{12_2} \eta_{23_2} \xi_{13_1} + 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_2)} y_{23_3} \eta_{13_2} \eta_{23_2} \xi_{13_1} - \\
& e^{\frac{1}{2}(b_{1_3}+b_{2_3}-3\alpha_2)} y_{23_3} \eta_{13_2} \eta_{23_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{13_3} \eta_{12_2} \xi_{12_1} \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} + 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} + \\
& 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} - e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} x_{13_3} y_{23_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} +
\end{aligned}$$

$$\begin{aligned}
& 4 e^{\frac{1}{2} (4 b_1 - 2 b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12} \xi_{13} - 6 e^{\frac{1}{2} (2 b_1 - b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12} \xi_{13} - \\
& 2 e^{\frac{1}{2} (b_1 + b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12} \xi_{13} + 2 e^{\frac{3}{2} (b_1 - a_1)} \eta_{12} \eta_{13} \xi_{12} \xi_{13} + 2 e^{-\frac{3 a_1}{2}} \eta_{12} \eta_{13} \xi_{12} \xi_{13} - \\
& 3 e^{\frac{1}{2} (3 b_1 - 2 a_1 - 2 a_2)} y_{23} \eta_{13}^2 \xi_{12} \xi_{13} + 4 e^{\frac{1}{2} (2 b_1 - b_2 - 2 a_1 - 2 a_2)} y_{23} \eta_{13}^2 \xi_{12} \xi_{13} - \\
& 4 e^{2 b_1 - b_2 - a_1 - a_2} y_{23} \eta_{13}^2 \xi_{12} \xi_{13} - 2 e^{\frac{1}{2} (2 a_1 - a_2)} x_{12} x_{13} \xi_{12} \xi_{13} - \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2)} x_{13} \eta_{12} \xi_{12} \xi_{13} - e^{\frac{1}{2} (a_1 - 2 a_2)} x_{12} \eta_{13} \xi_{12} \xi_{13} - \\
& e^{\frac{1}{2} (b_1 + b_2 + a_1 - 2 a_2)} x_{12} \eta_{13} \xi_{12} \xi_{13} + 4 e^{\frac{1}{2} (2 b_1 - b_2 + a_1 - 2 a_2)} x_{13} y_{23} \eta_{13} \xi_{12} \xi_{13} - \\
& 2 e^{\frac{1}{2} (-a_1 - a_2)} x_{13} \eta_{13} \xi_{13}^2 + 2 e^{\frac{1}{2} (b_1 + b_2 - a_1 - a_2)} x_{13} \eta_{13} \xi_{13}^2 - \\
& 2 e^{\frac{1}{2} (b_1 + b_2 - 2 a_1 - 2 a_2)} \eta_{13}^2 \xi_{13}^2 + 2 e^{b_1 + b_2 - a_1 - a_2} \eta_{13}^2 \xi_{13}^2 - e^{\frac{1}{2} (-a_1 + 2 a_2)} x_{13} \eta_{12} \xi_{12} \xi_{13} - \\
& e^{\frac{1}{2} (2 b_1 - b_2 - a_1 + 2 a_2)} x_{13} \eta_{12} \xi_{12} \xi_{13} + e^{\frac{1}{2} (2 b_1 - b_2)} x_{13} y_{23} \eta_{13} \xi_{12} \xi_{13} - \\
& 2 x_{13} \eta_{13} \xi_{13} \xi_{13} - 2 e^{\frac{1}{2} (b_1 + b_2)} x_{13} \eta_{13} \xi_{13} \xi_{13} + 4 x_{23} \beta_2 \xi_{23} - \\
& e^{\frac{1}{2} (-2 a_1 + a_2)} x_{23} y_{12} \eta_{12} \xi_{23} + 3 e^{\frac{1}{2} (-a_1 - a_2)} y_{12} \eta_{13} \xi_{23} + 2 e^{\frac{1}{2} (-a_1 - a_2)} a_2 y_{12} \eta_{13} \xi_{23} + \\
& e^{\frac{1}{2} (-a_1 - a_2)} x_{23} y_{13} \eta_{13} \xi_{23} + 2 e^{\frac{1}{2} (-a_1 - a_2)} y_{12} \beta_1 \eta_{13} \xi_{23} + 2 e^{\frac{1}{2} (-a_1 - a_2)} y_{12} \beta_2 \eta_{13} \xi_{23} + \\
& e^{\frac{1}{2} (-a_1 - a_2)} y_{12} y_{13} \eta_{13} \eta_{13} \xi_{23} - e^{\frac{1}{2} (-a_1 - a_2)} y_{12} y_{23} \eta_{13} \eta_{23} \xi_{23} - \\
& 2 e^{\frac{1}{2} (a_1 - 2 a_2)} \eta_{23} \xi_{23} + 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} \eta_{23} \xi_{23} - 2 e^{\frac{1}{2} (a_1 - 2 a_2)} a_2 \eta_{23} \xi_{23} - \\
& 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} a_2 \eta_{23} \xi_{23} + 2 e^{\frac{1}{2} (a_1 - 2 a_2)} x_{23} y_{23} \eta_{23} \xi_{23} - 2 e^{\frac{1}{2} (a_1 - 2 a_2)} \beta_2 \eta_{23} \xi_{23} + \\
& 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} \beta_2 \eta_{23} \xi_{23} + e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} y_{12} \eta_{12} \eta_{23} \xi_{23} - \\
& 2 e^{\frac{1}{2} (-a_1 - a_2)} y_{12} \eta_{12} \eta_{23} \xi_{23} - e^{\frac{1}{2} (-b_1 + 2 b_2 - a_1 - a_2)} y_{12} \eta_{12} \eta_{23} \xi_{23} - \\
& e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} y_{13} \eta_{13} \eta_{23} \xi_{23} + e^{\frac{1}{2} (-b_1 + 2 b_2 - 3 a_2)} y_{13} \eta_{13} \eta_{23} \xi_{23} + \\
& 2 e^{-\frac{3 a_2}{2}} y_{13} \eta_{13} \eta_{23} \xi_{23} - 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} y_{23} \eta_{23} \eta_{23} \xi_{23} + \\
& 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + 2 a_1 - 4 a_2)} y_{23} \eta_{23}^2 \xi_{23} + 2 e^{\frac{1}{2} (-2 a_1 + a_2)} x_{23} \eta_{12} \xi_{12} \xi_{23} - \\
& 5 e^{\frac{1}{2} (-a_1 - a_2)} \eta_{13} \xi_{12} \xi_{23} + 5 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} a_1 \eta_{13} \xi_{12} \xi_{23} - \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} a_1 \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} a_2 \eta_{13} \xi_{12} \xi_{23} + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} a_2 \eta_{13} \xi_{12} \xi_{23} + 2 e^{\frac{1}{2} (-a_1 - a_2)} x_{12} y_{12} \eta_{13} \xi_{12} \xi_{23} - \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} x_{23} y_{23} \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} \beta_1 \eta_{13} \xi_{12} \xi_{23} + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} \beta_1 \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} \beta_2 \eta_{13} \xi_{12} \xi_{23} + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} \beta_2 \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} y_{12} \eta_{12} \eta_{13} \xi_{12} \xi_{23} + \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2 - 3 a_1)} y_{12} \eta_{12} \eta_{13} \xi_{12} \xi_{23} - e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} y_{13} \eta_{13} \eta_{13} \xi_{12} \xi_{23} - \\
& e^{\frac{1}{2} (2 b_1 - b_2 - 2 a_1 - 2 a_2)} y_{13} \eta_{13}^2 \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (2 b_1 - b_2 - 2 a_1 - 2 a_2)} y_{12} y_{23} \eta_{13}^2 \xi_{12} \xi_{23} + \\
& e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} y_{23} \eta_{13} \eta_{23} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} x_{12} \eta_{23} \xi_{12} \xi_{23} + \\
& 2 e^{\frac{1}{2} (-a_1 - a_2)} \eta_{12} \eta_{23} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} \eta_{12} \eta_{23} \xi_{12} \xi_{23} - \\
& 2 e^{\frac{1}{2} (b_1 + b_2 - a_1 - a_2)} \eta_{12} \eta_{23} \xi_{12} \xi_{23} + 2 e^{\frac{1}{2} (-b_1 + 2 b_2 - a_1 - a_2)} \eta_{12} \eta_{23} \xi_{12} \xi_{23} - \\
& e^{\frac{1}{2} (2 b_1 - b_2 - 3 a_2)} y_{23} \eta_{13} \eta_{23} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} x_{12} \eta_{13} \xi_{12}^2 \xi_{23} + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} x_{12} \eta_{13} \xi_{12}^2 \xi_{23} + 4 e^{\frac{1}{2} (4 b_1 - 2 b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12}^2 \xi_{23} - \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12}^2 \xi_{23} + 2 e^{\frac{1}{2} (2 b_1 - b_2 - 2 a_1 - 2 a_2)} y_{23} \eta_{13}^2 \xi_{12}^2 \xi_{23} - \\
& 3 e^{2 b_1 - b_2 - a_1 - a_2} y_{23} \eta_{13}^2 \xi_{12}^2 \xi_{23} - e^{\frac{1}{2} (2 a_1 - a_2)} x_{13} \xi_{12} \xi_{23} + 2 e^{\frac{1}{2} (2 a_1 - a_2)} x_{12} x_{23} \xi_{12} \xi_{23} - \\
& 4 e^{\frac{1}{2} (2 a_1 - a_2)} x_{13} \beta_2 \xi_{12} \xi_{23} + x_{13} y_{12} \eta_{12} \xi_{12} \xi_{23} - e^{\frac{1}{2} (a_1 - 2 a_2)} x_{12} y_{12} \eta_{13} \xi_{12} \xi_{23} -
\end{aligned}$$

$$\begin{aligned}
& e^{\frac{1}{2}(\alpha_1-2\alpha_{21})} x_{13_3} y_{13_3} \eta_{13_2} \xi_{12_2} \xi_{23_1} + e^{\frac{1}{2}(-b_{13}+2b_{23}+3\alpha_1-3\alpha_{21})} x_{12_3} \eta_{23_2} \xi_{12_2} \xi_{23_1} + \\
& e^{\frac{3}{2}(\alpha_1-\alpha_{21})} x_{12_3} \eta_{23_2} \xi_{12_2} \xi_{23_1} - 2 e^{\frac{3}{2}(\alpha_1-\alpha_{21})} x_{13_3} y_{23_3} \eta_{23_2} \xi_{12_2} \xi_{23_1} + \\
& 2 e^{\frac{1}{2}(2b_{13}-b_{23})} x_{13_3} \eta_{12_2} \xi_{12_1} \xi_{12_2} \xi_{23_1} - e^{\frac{1}{2}(\alpha_1-2\alpha_{21})} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} \xi_{23_1} - \\
& 3 e^{\frac{1}{2}(2b_{13}-b_{23}+\alpha_1-2\alpha_{21})} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} \xi_{23_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_{21})} x_{23_3} \eta_{13_2} \xi_{13_1} \xi_{23_1} + \\
& 4 e^{\frac{1}{2}(2b_{13}-b_{23}-\alpha_1-\alpha_{21})} x_{23_3} \eta_{13_2} \xi_{13_1} \xi_{23_1} + e^{\frac{1}{2}(-\alpha_1-\alpha_{21})} x_{13_3} y_{12_3} \eta_{13_2} \xi_{13_1} \xi_{23_1} + \\
& 4 e^{\frac{1}{2}(2b_{13}-b_{23}-2\alpha_1-2\alpha_{21})} y_{12_3} \eta_{13_2}^2 \xi_{13_1} \xi_{23_1} + e^{\frac{1}{2}(b_{13}+b_{23}-2\alpha_1-2\alpha_{21})} y_{12_3} \eta_{13_2}^2 \xi_{13_1} \xi_{23_1} - \\
& 2 e^{-\alpha_1-\alpha_{21}} y_{12_3} \eta_{13_2}^2 \xi_{13_1} \xi_{23_1} + 2 e^{\frac{1}{2}(-b_{13}+2b_{23}+\alpha_1-2\alpha_{21})} x_{13_3} \eta_{23_2} \xi_{13_1} \xi_{23_1} - \\
& 4 e^{\frac{1}{2}(2b_{13}-b_{23}-3\alpha_{21})} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} + 2 e^{\frac{1}{2}(b_{13}+b_{23}-3\alpha_{21})} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} - \\
& 2 e^{\frac{1}{2}(-b_{13}+2b_{23}-3\alpha_{21})} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} + 2 e^{\frac{3}{2}(b_{23}-\alpha_{21})} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} + \\
& 2 e^{-\frac{3\alpha_{21}}{2}} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_{21})} x_{13_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} \xi_{23_1} + \\
& 2 e^{\frac{1}{2}(3b_{13}-2\alpha_1-2\alpha_{21})} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} - 6 e^{\frac{1}{2}(2b_{13}-b_{23}-2\alpha_1-2\alpha_{21})} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} - \\
& 2 e^{\frac{1}{2}(b_{13}+b_{23}-2\alpha_1-2\alpha_{21})} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} + 2 e^{-\alpha_1-\alpha_{21}} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} + \\
& 4 e^{2b_{13}-b_{23}-\alpha_1-\alpha_{21}} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} + 3 e^{\frac{1}{2}(\alpha_1-2\alpha_{21})} x_{13_3} \eta_{13_2} \xi_{12_2} \xi_{13_1} \xi_{23_1} - \\
& 4 e^{\frac{1}{2}(2b_{13}-b_{23}+\alpha_1-2\alpha_{21})} x_{13_3} \eta_{13_2} \xi_{12_2} \xi_{13_1} \xi_{23_1} + e^{\frac{1}{2}(b_{13}+b_{23}+\alpha_1-2\alpha_{21})} x_{13_3} \eta_{13_2} \xi_{12_2} \xi_{13_1} \xi_{23_1} - \\
& 2 e^{\frac{1}{2}(\alpha_1+\alpha_{21})} x_{13_3} x_{23_3} \xi_{13_2} \xi_{23_1} + x_{13_3} y_{12_3} \eta_{13_2} \xi_{13_2} \xi_{23_1} - e^{\frac{1}{2}(2\alpha_1-\alpha_{21})} x_{13_3} \eta_{23_2} \xi_{13_2} \xi_{23_1} - \\
& e^{\frac{1}{2}(-b_{13}+2b_{23}+2\alpha_1-\alpha_{21})} x_{13_3} \eta_{23_2} \xi_{13_2} \xi_{23_1} - 2 x_{13_3} \eta_{13_2} \xi_{12_1} \xi_{13_2} \xi_{23_1} + \\
& e^{\frac{1}{2}(-\alpha_1-\alpha_{21})} x_{23_3} y_{12_3} \eta_{13_2} \xi_{23_1}^2 - 2 e^{\frac{1}{2}(\alpha_1-2\alpha_{21})} x_{23_3} \eta_{23_2} \xi_{23_1}^2 + 2 e^{\frac{1}{2}(-b_{13}+2b_{23}+\alpha_1-2\alpha_{21})} x_{23_3} \eta_{23_2} \xi_{23_1}^2 + \\
& e^{\frac{1}{2}(-b_{13}+2b_{23}-3\alpha_{21})} y_{12_3} \eta_{13_2} \eta_{23_2} \xi_{23_1}^2 - 2 e^{\frac{1}{2}(-b_{13}+2b_{23}+2\alpha_1-4\alpha_{21})} \eta_{23_2}^2 \xi_{23_1}^2 + \\
& 2 e^{-b_{13}+2b_{23}+\alpha_1-2\alpha_{21}} \eta_{23_2}^2 \xi_{23_1}^2 + 2 e^{\frac{1}{2}(2b_{13}-b_{23}-\alpha_1-\alpha_{21})} x_{23_3} \eta_{13_2} \xi_{12_1} \xi_{23_1}^2 + \\
& 2 e^{\frac{1}{2}(2b_{13}-b_{23}-2\alpha_1-2\alpha_{21})} y_{12_3} \eta_{13_2}^2 \xi_{12_1} \xi_{23_1}^2 - 2 e^{\frac{1}{2}(2b_{13}-b_{23}-2\alpha_1-2\alpha_{21})} \eta_{13_2}^2 \xi_{12_1} \xi_{23_1}^2 + \\
& 2 e^{2b_{13}-b_{23}-\alpha_1-\alpha_{21}} \eta_{13_2}^2 \xi_{12_1}^2 \xi_{23_1}^2 - 3 e^{\frac{1}{2}(-b_{13}+2b_{23}+3\alpha_1-3\alpha_{21})} x_{13_3} \eta_{23_2} \xi_{12_2} \xi_{23_1}^2 + \\
& e^{\frac{3}{2}(\alpha_1-\alpha_{21})} x_{13_3} \eta_{23_2} \xi_{12_2} \xi_{23_1}^2 + e^{\frac{1}{2}(\alpha_1-2\alpha_{21})} x_{13_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} \xi_{23_1}^2 + \\
& e^{\frac{1}{2}(2b_{13}-b_{23}+\alpha_1-2\alpha_{21})} x_{13_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} \xi_{23_1}^2 + e^{-\frac{3}{2}(\alpha_1-\alpha_{21})} x_{23_3} \eta_{12_2} \xi_{12_1} \xi_{23_2} + \\
& e^{\frac{1}{2}(2b_{13}-b_{23}-3\alpha_1+3\alpha_{21})} x_{23_3} \eta_{12_2} \xi_{12_1} \xi_{23_2} - e^{\frac{1}{2}(2b_{13}-b_{23}-2\alpha_1+\alpha_{21})} x_{23_3} y_{23_3} \eta_{13_2} \xi_{12_1} \xi_{23_2} - \\
& e^{\frac{1}{2}(-2\alpha_1+\alpha_{21})} x_{23_3} \eta_{13_2} \xi_{13_1} \xi_{23_2} - e^{\frac{1}{2}(b_{13}+b_{23}-2\alpha_1+\alpha_{21})} x_{23_3} \eta_{13_2} \xi_{13_1} \xi_{23_2} + \\
& 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_{21})} x_{23_3} y_{12_3} \eta_{13_2} \xi_{23_1} \xi_{23_2} - 2 x_{23_3} \eta_{23_2} \xi_{23_1} \xi_{23_2} - 2 e^{\frac{1}{2}(-b_{13}+2b_{23})} x_{23_3} \eta_{23_2} \xi_{23_1} \xi_{23_2} - \\
& e^{\frac{1}{2}(-2\alpha_1+\alpha_{21})} x_{23_3} \eta_{13_2} \xi_{12_1} \xi_{23_1} \xi_{23_2} + 3 e^{\frac{1}{2}(2b_{13}-b_{23}-2\alpha_1+\alpha_{21})} x_{23_3} \eta_{13_2} \xi_{12_1} \xi_{23_1} \xi_{23_2} \Big]
\end{aligned}$$

associativity

$$In[*]:= (dm_{1,2 \rightarrow k} // dm_{k,3 \rightarrow k}) \equiv (dm_{2,3 \rightarrow k} // dm_{1,k \rightarrow k})$$

Time: 2.77 seconds Out[]=

True

co-associativity

$$In[*]:= (d\Delta_{1 \rightarrow 1,2} // d\Delta_{2 \rightarrow 2,3}) \equiv (d\Delta_{1 \rightarrow 2,3} // d\Delta_{2 \rightarrow 1,2})$$

Time: 1.09 seconds Out[]=

True

algebra morphism

$In[] := (d\Delta_{1 \rightarrow 1, 2} d\Delta_{2 \rightarrow 3, 4} // (dm_{1, 3 \rightarrow 1} dm_{2, 4 \rightarrow 2})) \equiv (dm_{1, 2 \rightarrow 3} // d\Delta_{3 \rightarrow 1, 2})$

Time: 11.44 seconds Out[] =

True

antipode

$In[] := \{d\Delta_{i \rightarrow 1, 2} // dS_1 // dm_{1, 2 \rightarrow 1}, d\Delta_{i \rightarrow 1, 2} // dS_2 // dm_{1, 2 \rightarrow 1}, dS_1 // \overline{dS_1}\}$

Time: 36.28 seconds Out[] =

$$\left\{ E_{\{i\} \rightarrow \{1\}} \left[0, 2 e^{-\frac{3}{2} (b_{11} - \alpha_{11})} y_{12_1} y_{13_1} \eta_{12_i} \eta_{13_i} - 2 e^{\frac{3}{2} (-b_{11} + \alpha_{11})} y_{12_1} y_{13_1} \eta_{12_i} \eta_{13_i} - \right. \right. \\ \left. 2 e^{-\frac{3}{2} (b_{21} - \alpha_{21})} y_{13_1} y_{23_1} \eta_{13_i} \eta_{23_i} + 2 e^{\frac{3}{2} (-b_{21} + \alpha_{21})} y_{13_1} y_{23_1} \eta_{13_i} \eta_{23_i} - \right. \\ \left. e^{-\frac{3}{2} (b_{21} - \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i} \eta_{23_i} + e^{\frac{3}{2} (-b_{21} + \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i} \eta_{23_i} - \right. \\ \left. 3 e^{-\frac{3}{2} (b_{21} - \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i}^2 \xi_{12_i} + 3 e^{\frac{3}{2} (-b_{21} + \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i}^2 \xi_{12_i} - \right. \\ \left. 2 e^{-\frac{3}{2} (b_{11} - \alpha_{11})} y_{12_1} y_{13_1} \eta_{13_i}^2 \xi_{23_i} + 2 e^{\frac{3}{2} (-b_{11} + \alpha_{11})} y_{12_1} y_{13_1} \eta_{13_i}^2 \xi_{23_i} \right], \\ E_{\{i\} \rightarrow \{1\}} [0, 0], E_{\{1\} \rightarrow \{1\}} [a_{11} \alpha_{11} + a_{21} \alpha_{21} + b_{11} \beta_{11} + b_{21} \beta_{21} + y_{12_1} \eta_{12_1} + \\ y_{13_1} \eta_{13_1} + y_{23_1} \eta_{23_1} + x_{12_1} \xi_{12_1} + x_{13_1} \xi_{13_1} + x_{23_1} \xi_{23_1}, 0] \left. \right\}$$

$In[] := E_{\{i\} \rightarrow \{1\}} \left[0, 2 e^{-\frac{3}{2} (b_{11} - \alpha_{11})} y_{12_1} y_{13_1} \eta_{12_i} \eta_{13_i} - 2 e^{\frac{3}{2} (-b_{11} + \alpha_{11})} y_{12_1} y_{13_1} \eta_{12_i} \eta_{13_i} - \right. \\ \left. 2 e^{-\frac{3}{2} (b_{21} - \alpha_{21})} y_{13_1} y_{23_1} \eta_{13_i} \eta_{23_i} + 2 e^{\frac{3}{2} (-b_{21} + \alpha_{21})} y_{13_1} y_{23_1} \eta_{13_i} \eta_{23_i} - \right. \\ \left. e^{-\frac{3}{2} (b_{21} - \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i} \eta_{23_i} + e^{\frac{3}{2} (-b_{21} + \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i} \eta_{23_i} - \right. \\ \left. 3 e^{-\frac{3}{2} (b_{21} - \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i}^2 \xi_{12_i} + 3 e^{\frac{3}{2} (-b_{21} + \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i}^2 \xi_{12_i} - \right. \\ \left. 2 e^{-\frac{3}{2} (b_{11} - \alpha_{11})} y_{12_1} y_{13_1} \eta_{13_i}^2 \xi_{23_i} + 2 e^{\frac{3}{2} (-b_{11} + \alpha_{11})} y_{12_1} y_{13_1} \eta_{13_i}^2 \xi_{23_i} \right] // \text{Simplify}$

Time: 0.03 seconds Out[] =

$E_{\{i\} \rightarrow \{1\}} [0, 0]$

quasi-triangular axioms

$In[] := (R_{1,3} // d\Delta_{1 \rightarrow 1, 2}) \equiv (R_{1,3} R_{2,4} // dm_{3,4 \rightarrow 3})$
 $(R_{1,3} // d\Delta_{3 \rightarrow 2, 3}) \equiv (R_{1,3} R_{0,2} // dm_{1,0 \rightarrow 1})$
 $(d\Delta_{i \rightarrow k, j} R_{1,2} // dm_{j,1 \rightarrow 1} // dm_{k,2 \rightarrow 2}) \equiv (R_{1,2} d\Delta_{i \rightarrow j, k} // dm_{1,j \rightarrow 1} // dm_{2,k \rightarrow 2})$

Time: 3.52 seconds Out[] =

True

Out[] =

True

Out[] =

True

$In[] := (R_{1,2} // aSt_2) \equiv (\overline{R}_{1,2})$

Time: 0.06 seconds Out[] =

True

Braid-like Reidemeister moves: R2,R3

```
In[*]:= {R1,2 R3,4 // dm1,3→1 // dm2,4→2, R1,2 R3,4 // dm1,3→1 // dm2,4→2,
  (R1,2 R4,3 R5,6 // dm1,4→1 // dm2,5→5 // dm3,6→3) ≡ (R1,6 R2,3 R4,5 // dm1,4→1 // dm2,5→5 // dm3,6→3)}
```

Time: 4.34 seconds Out[]=

```
{E_{ }→{1,2} [0, 0], E_{ }→{1,2} [0, 0], True}
```

Drinfeld element:

```
In[*]:= $k = 0; $n = 2;
```

```
In[*]:= u_i_ := Module[{i1, i2}, R_i1,i2 // dS_i2 // dm_i2,i1→i]
```

```
In[*]:= u_2
```

Time: 0.11 seconds Out[]=

```
E_{ }→{2} [Log[e^{-b1_2}] - a1_2 b1_2 - e^{-b1_2} x12_2 y12_2, 4 a1_2]
```

Ziping fails to compute the Drinfeld element u . Why?

We still need to find C such that R2c holds:

According to PG.pdf the ribbon element v is a square root of $v^2 = uS(u)$. Then the pivot (a.k.a. distinguished group-like element, spinner) is $C = uv^{-1}$.

```
In[*]:= R3,2 R1,6 C4 // dm1,3→1 // dm2,4→2 // dm2,6→2
C_2
```