

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_h(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

```
In[ ]:= MakePDF []
```

```
In[ ]:= CF = Expand;
```

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```
In[ ]:= x[i_, j_] := x <> i <> j;
a[i_] := a <> i;
y[i_, j_] := y <> i <> j;
b[i_] := b <> i;
```

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```
In[ ]:= Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z_] := (x  $\otimes$  y)  $\otimes$  z;
0  $\otimes$  _ = _  $\otimes$  0 = 0;
(x_Plus)  $\otimes$  y_ := (#  $\otimes$  y) & /@ x; x_  $\otimes$  (y_Plus) := (x  $\otimes$  #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x  $\otimes$  y - y  $\otimes$  x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

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The following ordering agrees with Rosso’s; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xayb.

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```

In[ ]:= $n = 5;
PBWGens = Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
  Table[As[i, 1, $n - 1], a[i]],
  Table[As[i, 1, $n - 1], As[j, i + 1, $n], y[i, j]],
  Table[As[i, 1, $n - 1], b[i]]
}
(#u = U@#) & /@ PBWGens;
SortingRule = Module[{k = 0}, Flatten@Table[{g -> ++k, gi_ -> {i, k}}, {g, PBWGens}]];

```

Time: 0.02 seconds Out[]:=

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```

{x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3,
a4, y12, y13, y14, y15, y23, y24, y25, y34, y35, y45, b1, b2, b3, b4}

```

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```

In[ ]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
ι[s_Symbol] := Rest[Characters[SymbolName[s]]] /.
  {"0" -> 0, "1" -> 1, "2" -> 2, "3" -> 3, "4" -> 4, "5" -> 5, "6" -> 6, "7" -> 7, "8" -> 8, "9" -> 9};

```

```

In[ ]:= {τ[x[3, 4]], ι[x[3, 4]]}

```

Time: 0.00 seconds Out[]:=

```

{x, {3, 4}}

```

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We take the reduction rules RR results from Yamane page

509:

(I) $i = m < j < n$, (II) $i < m < n < j$, (III) $i < m < j = n$,

$\begin{array}{c} i \quad j \\ \hline m \quad n \end{array}$ $\begin{array}{c} i \quad j \\ \hline m \quad n \end{array}$ $\begin{array}{c} i \quad j \\ \hline m \quad n \end{array}$

(IV) $i < m < j < n$, (V) $i < j = m < n$, (VI) $i < j < m < n$.

$\begin{array}{c} i \quad j \\ \hline m \quad n \end{array}$ $\begin{array}{c} i \quad j \\ \hline m \quad n \end{array}$ $\begin{array}{c} i \quad j \\ \hline m \quad n \end{array}$

$$\begin{array}{ll}
 q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 & \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\
 [e_{ij}, e_{mn}] = 0 & \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\
 [e_{ij}, e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} & \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\
 q^2e_{ij}e_{mn} - e_{mn}e_{ij} = qe_{in} & \text{if } ((i, j), (m, n)) \in C_{(V)}.
 \end{array}$$

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```

In[ ]:= RR[xmn_, xij_] /;  $\tau[xij] = \tau[xmn] = "x"$  := Module[{i, j, m, n},
  {i, j} =  $\mathcal{L}[xij]$ ; {m, n} =  $\mathcal{L}[xmn]$ ;
  Which[
    i == m  $\wedge$  j < n,  $q^{-2} U[xij, xmn]$ , (* I *)
    i == m  $\wedge$  j == n, U[xij, xmn],
    i < m < j  $\wedge$  n < j, U[xij, xmn], (* II *)
    i < m < j  $\wedge$  n == j,  $q^{-2} U[xij, xmn]$ , (* III *)
    i < m < j  $\wedge$  n > j, U[xij, xmn] + ( $q^{-2} - q^2$ ) U[x[i, n], x[m, j]], (* IV *)
    j == m,  $q^2 U[xij, xmn] - q U[x[i, n]]$ , (* V *)
    j < m, U[xij, xmn] (* VI *)
  ]
]

```

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```

In[ ]:= 1u = U[];
x_  $\otimes$  (c_. 1u) := CF[c x]; (c_. 1u)  $\otimes$  x_ := CF[c x];
(a_. U[xx_, x_])  $\otimes$  (b_. U[y_, yy_]) := Module[{M},
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[A_, X_] := A X;
  If[OrderedQ[{x, y} /. SortingRule],
    CF@M[a b, U[xx, x, y, yy]],
    U@xx  $\otimes$  CF@M[a b, RR[x, y]]  $\otimes$  U@yy ]
]

```

```

In[ ]:= Echo[U[x34]  $\otimes$  U[x23]]  $\otimes$  U[x12]

```

```

» -q U[x24] + q2 U[x23, x34]

```

Out[]:=

```

q2 U[x14] - q3 U[x12, x24] - q3 U[x13, x34] + q4 U[x12, x23, x34]

```

```

In[ ]:= U[x34]  $\otimes$  Echo[U[x23]  $\otimes$  U[x12]]

```

```

» -q U[x13] + q2 U[x12, x23]

```

Out[]:=

```

q2 U[x14] - q3 U[x12, x24] - q3 U[x13, x34] + q4 U[x12, x23, x34]

```

```

In[ ]:= Module[{Bas = Echo@Select[PBWGens,  $\tau[\#] == "x" \&]$ },
  DeleteCases[_  $\rightarrow$  0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    {X, Y, Z}  $\rightarrow$  ((Xu  $\otimes$  Yu)  $\otimes$  Zu) - (Xu  $\otimes$  (Yu  $\otimes$  Zu))
  ]
]

```

```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45}

```

Out[]:=

```

{}

```

```

In[ ]:= Module[{Bas = Echo@Select[PBWGens,  $\tau[\#] == "x" \&]$ },
  DeleteCases[_  $\rightarrow$  True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    {W, X, Y, Z}  $\rightarrow$   $W_U \otimes ((X_U \otimes Y_U) \otimes Z_U) == W_U \otimes (X_U \otimes (Y_U \otimes Z_U)) ==$ 
       $(W_U \otimes X_U) \otimes (Y_U \otimes Z_U) == ((W_U \otimes X_U) \otimes Y_U) \otimes Z_U == (W_U \otimes (X_U \otimes Y_U)) \otimes Z_U$ 
    ]],
  ]
  >> {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45}

Out[ ]=
  {}

```