

Pensieve header: The PBW multiplication tensor for $\mathfrak{gl}_n(\epsilon)$ using the λ -tangent formalism. Some material from pensieve://Projects/UEA.

```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

Prolog

```
In[ ]:= BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**", B, m, SetAlgebra, U, UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ },
"."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
UEA` implements / extends {**, B, m, SetAlgebra, U, UB, UProducts, USimp, UU, δ , \$Basis, \$PBWRule}.

Utilities

```
In[ ]:=  $\delta_{i,j}$  := If[i == j, 1, 0];
```

Implementing general universal enveloping algebras

```
In[ ]:= B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
In[*]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_p-] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs_] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
In[*]:= m_s_[0] = 0;
m_s_[x_Plus] := m_s_/@x;
m_s_[sd_SeriesData] := MapAt[m_s_, sd, {3, All}];
m_i->j_[ε_] := ε /. U_i → U_j;
```

```
In[*]:= m_i,j->k_[c_. U_i[x_] U_j[]] := c U_k[x];
m_i,j->k_[c_. U_i[] U_j[y_]] := c U_k[y];
m_i,j->k_[c_. U_i[xx_, x_] U_j[y_, yy_]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i) U_j[yy] // Expand // m_i,j->k)
  c // USimp
];
```

```
In[*]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is_}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[is, d - j]}];
```

```
In[*]:= Supp[ε_] := Union@Cases[ε, U_i[___] => i, ∞];
```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = m_i->σ@i[z], {i, is}];
  z = Expand[y z]; Do[z = m_σ@i,i->i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

Epilog

```
In[*]:= End[]; EndPackage[];
```

Predefined Algebras

$sl(2)$

```
In[*]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
In[*]:= SetAlgebra["sl2"] := (
  Print["In sl2: <e,h,f>/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
);
```

$gl_{n,\epsilon}$

```
In[*]:= Print["UEA`SetAlgebra knows the  $\epsilon$ -nilpotent algebra  $gl_{n,\epsilon}$ ."];
```

UEA`SetAlgebra knows the ϵ -nilpotent algebra $gl_{n,\epsilon}$.

```
In[*]:= SetAlgebra[gl_{n,\epsilon}] := (
  $Basis = Flatten@{
    Table[y_{\alpha,\beta}, {\beta, 2, n}, {\alpha, 1, \beta - 1}],
    Table[x_{\alpha,\beta}, {\beta, 1, n}, {\alpha, 1, \beta}]
  };
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[x_{i,j_}, x_{k,l_}] := \delta_{j,k} x_{i,l} - \delta_{l,i} x_{k,j};
  B[y_{i,j_}, y_{k,l_}] := \epsilon \delta_{j,k} y_{i,l} - \epsilon \delta_{l,i} y_{k,j};
  B[x_{i,j_}, y_{l,k_}] := \delta_{j,k} x_{i,l} - \delta_{l,i} x_{k,j} / . x_{\alpha,\beta_} → If[\alpha ≤ \beta, \epsilon x_{\alpha,\beta}, y_{\alpha,\beta}];
);
```

```
In[*]:= SetAlgebra[gl3,ε];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

```
Out[*]= {y1,2 → 1, y1,3 → 2, y2,3 → 3, x1,1 → 4, x1,2 → 5, x2,2 → 6, x1,3 → 7, x2,3 → 8, x3,3 → 9}
```

```
Out[*]//MatrixForm=
{
  {y1,2, y1,2} → 0, {y1,2, y1,3} → 0, {y1,2, y2,3} → ε y1,3, {y1,2, x1,1} → y1,2,
  {y1,3, y1,2} → 0, {y1,3, y1,3} → 0, {y1,3, y2,3} → 0, {y1,3, x1,1} → y1,3,
  {y2,3, y1,2} → -ε y1,3, {y2,3, y1,3} → 0, {y2,3, y2,3} → 0, {y2,3, x1,1} → y2,3,
  {x1,1, y1,2} → -y2,1, {x1,1, y1,3} → -y3,1, {x1,1, y2,3} → 0, {x1,1, x1,1} → 0,
  {x1,2, y1,2} → ε x1,1 - ε x2,2, {x1,2, y1,3} → -y3,2, {x1,2, y2,3} → 0, {x1,2, x1,1} → -y3,2,
  {x2,2, y1,2} → y2,1, {x2,2, y1,3} → 0, {x2,2, y2,3} → -y3,2, {x2,2, x1,1} → -y3,2,
  {x1,3, y1,2} → -ε x2,3, {x1,3, y1,3} → ε x1,1 - ε x3,3, {x1,3, y2,3} → ε x1,2, {x1,3, x1,1} → -y3,2,
  {x2,3, y1,2} → 0, {x2,3, y1,3} → y2,1, {x2,3, y2,3} → ε x2,2 - ε x3,3, {x2,3, x1,1} → -y3,2,
  {x3,3, y1,2} → 0, {x3,3, y1,3} → y3,1, {x3,3, y2,3} → y3,2, {x3,3, x1,1} → -y3,2
}
```

```
In[*]:= $k = 2;
x0 = ($Basis /. {x → ξ0, y → η0}) . $Basis + 0[ε]$k+1
x1 = ($Basis /. {x → ξ1, y → η1}) . $Basis + 0[ε]$k+1
```

```
Out[*]= (y1,2 η0,2 + y1,3 η0,3 + y2,3 η0,3 + x1,1 ξ0,1,1 + x1,2 ξ0,1,2 + x1,3 ξ0,1,3 + x2,2 ξ0,2,2 + x2,3 ξ0,2,3 + x3,3 ξ0,3,3) + 0[ε]3
```

```
Out[*]= (y1,2 η1,2 + y1,3 η1,3 + y2,3 η1,3 + x1,1 ξ1,1,1 + x1,2 ξ1,1,2 + x1,3 ξ1,1,3 + x2,2 ξ1,2,2 + x2,3 ξ1,2,3 + x3,3 ξ1,3,3) + 0[ε]3
```

```
In[*]:= A = Table[Coefficient[B[b, Normal@x0], c] + 0[ε]$k+1, {c, $Basis}, {b, $Basis}]
```

```
Out[*]= {
  {0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3},
  {η0,2,3 ε + 0[ε]3, 0[ε]3, -η0,1,2 ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3},
  {0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3},
  {-ξ0,1,2 ε + 0[ε]3, -ξ0,1,3 ε + 0[ε]3, 0[ε]3, 0[ε]3, η0,1,2 ε + 0[ε]3, 0[ε]3,
  η0,1,3 ε + 0[ε]3, 0[ε]3, 0[ε]3}, {0[ε]3, 0[ε]3, -ξ0,1,3 ε + 0[ε]3, ξ0,1,2 ε + 0[ε]3,
  (-ξ0,1,1 + ξ0,2,2) + 0[ε]3, -ξ0,1,2 + 0[ε]3, η0,2,3 ε + 0[ε]3, 0[ε]3, 0[ε]3},
  {ξ0,1,2 ε + 0[ε]3, 0[ε]3, -ξ0,2,3 ε + 0[ε]3, 0[ε]3, -η0,1,2 ε + 0[ε]3, 0[ε]3,
  0[ε]3, η0,2,3 ε + 0[ε]3, 0[ε]3}, {0[ε]3, 0[ε]3, 0[ε]3, ξ0,1,3 ε + 0[ε]3,
  ξ0,2,3 ε + 0[ε]3, 0[ε]3, (-ξ0,1,1 + ξ0,3,3) + 0[ε]3, -ξ0,1,2 ε + 0[ε]3, -ξ0,1,3 ε + 0[ε]3},
  {ξ0,1,3 ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, ξ0,2,3 ε + 0[ε]3, -η0,1,2 ε + 0[ε]3,
  (-ξ0,2,2 + ξ0,3,3) + 0[ε]3, -ξ0,2,3 ε + 0[ε]3}, {0[ε]3, ξ0,1,3 ε + 0[ε]3,
  ξ0,2,3 ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, -η0,1,3 ε + 0[ε]3, -η0,2,3 ε + 0[ε]3, 0[ε]3}
}
```

In[*]:= **A // Normal // MatrixForm**

Out[*]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon \eta \theta_{2,3} & 0 & -\epsilon \eta \theta_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\epsilon \xi \theta_{1,2} & -\epsilon \xi \theta_{1,3} & 0 & 0 & \epsilon \eta \theta_{1,2} & 0 & \epsilon \eta \theta_{1,3} & 0 & 0 \\ 0 & 0 & -\epsilon \xi \theta_{1,3} & \xi \theta_{1,2} & -\xi \theta_{1,1} + \xi \theta_{2,2} & -\xi \theta_{1,2} & \epsilon \eta \theta_{2,3} & 0 & 0 \\ \epsilon \xi \theta_{1,2} & 0 & -\epsilon \xi \theta_{2,3} & 0 & -\epsilon \eta \theta_{1,2} & 0 & 0 & \epsilon \eta \theta_{2,3} & 0 \\ 0 & 0 & 0 & \xi \theta_{1,3} & \xi \theta_{2,3} & 0 & -\xi \theta_{1,1} + \xi \theta_{3,3} & -\xi \theta_{1,2} & -\xi \theta_{1,3} \\ \epsilon \xi \theta_{1,3} & 0 & 0 & 0 & 0 & \xi \theta_{2,3} & -\epsilon \eta \theta_{1,2} & -\xi \theta_{2,2} + \xi \theta_{3,3} & -\xi \theta_{2,3} \\ 0 & \epsilon \xi \theta_{1,3} & \epsilon \xi \theta_{2,3} & 0 & 0 & 0 & -\epsilon \eta \theta_{1,3} & -\epsilon \eta \theta_{2,3} & 0 \end{pmatrix}$$

In[*]:= **MatrixExp[A // Normal] /. \epsilon -> 0 // Simplify**

Out[*]=

$$\begin{aligned} & \{ \{1, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0, 0\}, \\ & \{0, 0, 1, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0\}, \\ & \left\{ 0, 0, 0, \frac{(1 - e^{-\xi \theta_{1,1} + \xi \theta_{2,2}}) \xi \theta_{1,2}}{\xi \theta_{1,1} - \xi \theta_{2,2}}, e^{-\xi \theta_{1,1} + \xi \theta_{2,2}}, -\frac{(1 - e^{-\xi \theta_{1,1} + \xi \theta_{2,2}}) \xi \theta_{1,2}}{\xi \theta_{1,1} - \xi \theta_{2,2}}, 0, 0, 0 \right\}, \\ & \{0, 0, 0, 0, 0, 1, 0, 0, 0\}, \{0, 0, 0, \\ & (e^{-\xi \theta_{1,1}} (\xi \theta_{1,1} (-((e^{\xi \theta_{2,2}} - e^{\xi \theta_{3,3}}) \xi \theta_{1,2} \xi \theta_{2,3}) + (e^{\xi \theta_{1,1}} - e^{\xi \theta_{3,3}}) \xi \theta_{1,3} (\xi \theta_{2,2} - \xi \theta_{3,3})) - (e^{\xi \theta_{1,1}} - e^{\xi \theta_{3,3}}) \\ & \xi \theta_{1,3} \xi \theta_{2,2} (\xi \theta_{2,2} - \xi \theta_{3,3}) + \xi \theta_{1,2} \xi \theta_{2,3} ((e^{\xi \theta_{1,1}} - e^{\xi \theta_{3,3}}) \xi \theta_{2,2} - (e^{\xi \theta_{1,1}} - e^{\xi \theta_{2,2}}) \xi \theta_{3,3}))) / \\ & ((\xi \theta_{1,1} - \xi \theta_{2,2}) (\xi \theta_{1,1} - \xi \theta_{3,3}) (\xi \theta_{2,2} - \xi \theta_{3,3})), \frac{e^{-\xi \theta_{1,1}} (e^{\xi \theta_{2,2}} - e^{\xi \theta_{3,3}}) \xi \theta_{2,3}}{\xi \theta_{2,2} - \xi \theta_{3,3}}, \\ & \frac{e^{-\xi \theta_{1,1} - \xi \theta_{2,2}} (-e^{\xi \theta_{1,1}} + e^{\xi \theta_{2,2}}) (e^{\xi \theta_{2,2}} - e^{\xi \theta_{3,3}}) \xi \theta_{1,2} \xi \theta_{2,3}}{(\xi \theta_{1,1} - \xi \theta_{2,2}) (\xi \theta_{2,2} - \xi \theta_{3,3})}, e^{-\xi \theta_{1,1} + \xi \theta_{3,3}}, \\ & \frac{e^{\xi \theta_{3,3}} (e^{-\xi \theta_{1,1}} - e^{-\xi \theta_{2,2}}) \xi \theta_{1,2}}{\xi \theta_{1,1} - \xi \theta_{2,2}}, \frac{1}{(\xi \theta_{1,1} - \xi \theta_{2,2}) (\xi \theta_{1,1} - \xi \theta_{3,3}) (\xi \theta_{2,2} - \xi \theta_{3,3})} \\ & e^{-\xi \theta_{1,1} - \xi \theta_{2,2}} (\xi \theta_{1,1} (e^{\xi \theta_{1,1}} (e^{\xi \theta_{2,2}} - e^{\xi \theta_{3,3}}) \xi \theta_{1,2} \xi \theta_{2,3} - e^{\xi \theta_{2,2}} (e^{\xi \theta_{1,1}} - e^{\xi \theta_{3,3}}) \xi \theta_{1,3} (\xi \theta_{2,2} - \xi \theta_{3,3})) + \\ & e^{\xi \theta_{2,2}} (e^{\xi \theta_{1,1}} - e^{\xi \theta_{3,3}}) \xi \theta_{1,3} \xi \theta_{2,2} (\xi \theta_{2,2} - \xi \theta_{3,3}) - \\ & \xi \theta_{1,2} \xi \theta_{2,3} (e^{\xi \theta_{2,2}} (e^{\xi \theta_{1,1}} - e^{\xi \theta_{3,3}}) \xi \theta_{2,2} - e^{\xi \theta_{3,3}} (e^{\xi \theta_{1,1}} - e^{\xi \theta_{2,2}}) \xi \theta_{3,3})) \Big\}, \\ & \left\{ 0, 0, 0, 0, 0, \frac{(1 - e^{-\xi \theta_{2,2} + \xi \theta_{3,3}}) \xi \theta_{2,3}}{\xi \theta_{2,2} - \xi \theta_{3,3}}, 0, e^{-\xi \theta_{2,2} + \xi \theta_{3,3}}, -\frac{(1 - e^{-\xi \theta_{2,2} + \xi \theta_{3,3}}) \xi \theta_{2,3}}{\xi \theta_{2,2} - \xi \theta_{3,3}} \right\}, \\ & \{0, 0, 0, 0, 0, 0, 0, 0, 1\} \} \end{aligned}$$

In[*]:= **MatrixExp[A // Normal]**

Out[*]=

\$Aborted

In[*]:= **MatrixExp[A]**

Eigenvalues: Unable to find all roots of the characteristic polynomial.

Out[*]=

MatrixExp[{{0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³},
 {η_{02,3}ε + 0[ε]³, 0[ε]³, -η_{01,2}ε + 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³},
 {0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³},
 {-ξ_{01,2}ε + 0[ε]³, -ξ_{01,3}ε + 0[ε]³, 0[ε]³, 0[ε]³, η_{01,2}ε + 0[ε]³, 0[ε]³,
 η_{01,3}ε + 0[ε]³, 0[ε]³, 0[ε]³}, {0[ε]³, 0[ε]³, -ξ_{01,3}ε + 0[ε]³, ξ_{01,2}ε + 0[ε]³,
 (-ξ_{01,1} + ξ_{02,2}) + 0[ε]³, -ξ_{01,2} + 0[ε]³, η_{02,3}ε + 0[ε]³, 0[ε]³, 0[ε]³},
 {ξ_{01,2}ε + 0[ε]³, 0[ε]³, -ξ_{02,3}ε + 0[ε]³, 0[ε]³, -η_{01,2}ε + 0[ε]³, 0[ε]³,
 0[ε]³, η_{02,3}ε + 0[ε]³, 0[ε]³}, {0[ε]³, 0[ε]³, 0[ε]³, ξ_{01,3}ε + 0[ε]³,
 ξ_{02,3}ε + 0[ε]³, 0[ε]³, (-ξ_{01,1} + ξ_{03,3}) + 0[ε]³, -ξ_{01,2} + 0[ε]³, -ξ_{01,3} + 0[ε]³},
 {ξ_{01,3}ε + 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, ξ_{02,3}ε + 0[ε]³, -η_{01,2}ε + 0[ε]³,
 (-ξ_{02,2} + ξ_{03,3}) + 0[ε]³, -ξ_{02,3} + 0[ε]³}, {0[ε]³, ξ_{01,3}ε + 0[ε]³, ξ_{02,3}ε + 0[ε]³,
 0[ε]³, 0[ε]³, 0[ε]³, -η_{01,3}ε + 0[ε]³, -η_{02,3}ε + 0[ε]³, 0[ε]³}}

```
In[*]:= adExp[x_][y_] := Expand@Module[{A, b, c},
  A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
  $Basis.MatrixExp[A].Table[Coefficient[y, b], {b, $Basis}]
]
```

In[*]:= **adExp[x0][x1]**

Out[*]=

$x_{1,2} \eta_{1,2} + x_{1,3} \eta_{1,3} + x_{1,4} \eta_{1,4} + x_{1,5} \eta_{1,5} + x_{2,3} \eta_{2,3} + x_{2,4} \eta_{2,4} + x_{2,5} \eta_{2,5} + x_{3,4} \eta_{3,4} + x_{3,5} \eta_{3,5} + x_{4,5} \eta_{4,5} -$
 $x_{1,3} \eta_{2,3} \xi_{1,2} - x_{1,4} \eta_{2,4} \xi_{1,2} - x_{1,5} \eta_{2,5} \xi_{1,2} - x_{1,4} \eta_{3,4} \xi_{1,3} - x_{1,5} \eta_{3,5} \xi_{1,3} - x_{1,5} \eta_{4,5} \xi_{1,4} +$
 $x_{1,3} \eta_{1,2} \xi_{2,3} - x_{2,4} \eta_{3,4} \xi_{2,3} - x_{2,5} \eta_{3,5} \xi_{2,3} + \frac{1}{2} x_{1,4} \eta_{3,4} \xi_{1,2} \xi_{2,3} + \frac{1}{2} x_{1,5} \eta_{3,5} \xi_{1,2} \xi_{2,3} +$
 $x_{1,4} \eta_{1,2} \xi_{2,4} - x_{2,5} \eta_{4,5} \xi_{2,4} + \frac{1}{2} x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,4} + x_{1,5} \eta_{1,2} \xi_{2,5} + x_{1,4} \eta_{1,3} \xi_{3,4} +$
 $x_{2,4} \eta_{2,3} \xi_{3,4} - x_{3,5} \eta_{4,5} \xi_{3,4} - x_{1,4} \eta_{2,3} \xi_{1,2} \xi_{3,4} + \frac{1}{2} x_{1,5} \eta_{4,5} \xi_{1,3} \xi_{3,4} + \frac{1}{2} x_{1,4} \eta_{1,2} \xi_{2,3} \xi_{3,4} +$
 $\frac{1}{2} x_{2,5} \eta_{4,5} \xi_{2,3} \xi_{3,4} - \frac{1}{6} x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + x_{1,5} \eta_{1,3} \xi_{3,5} + x_{2,5} \eta_{2,3} \xi_{3,5} - x_{1,5} \eta_{2,3} \xi_{1,2} \xi_{3,5} +$
 $\frac{1}{2} x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,5} + x_{1,5} \eta_{1,4} \xi_{4,5} + x_{2,5} \eta_{2,4} \xi_{4,5} + x_{3,5} \eta_{3,4} \xi_{4,5} - x_{1,5} \eta_{2,4} \xi_{1,2} \xi_{4,5} -$
 $x_{1,5} \eta_{3,4} \xi_{1,3} \xi_{4,5} - x_{2,5} \eta_{3,4} \xi_{2,3} \xi_{4,5} + \frac{1}{2} x_{1,5} \eta_{3,4} \xi_{1,2} \xi_{2,3} \xi_{4,5} + \frac{1}{2} x_{1,5} \eta_{1,2} \xi_{2,4} \xi_{4,5} +$
 $\frac{1}{2} x_{1,5} \eta_{1,3} \xi_{3,4} \xi_{4,5} + \frac{1}{2} x_{2,5} \eta_{2,3} \xi_{3,4} \xi_{4,5} - \frac{1}{2} x_{1,5} \eta_{2,3} \xi_{1,2} \xi_{3,4} \xi_{4,5} + \frac{1}{6} x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5}$

In[*]:= **Coefficient**[adExp[x0][y0], x1,5]

Out[*]=

$$\begin{aligned} & \eta_{1,5} - \eta_{2,5} \xi_{1,2} - \eta_{3,5} \xi_{1,3} - \eta_{4,5} \xi_{1,4} + \frac{1}{2} \eta_{3,5} \xi_{1,2} \xi_{2,3} + \frac{1}{2} \eta_{4,5} \xi_{1,2} \xi_{2,4} + \eta_{1,2} \xi_{2,5} + \frac{1}{2} \eta_{4,5} \xi_{1,3} \xi_{3,4} - \\ & \frac{1}{6} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \eta_{1,3} \xi_{3,5} - \eta_{2,3} \xi_{1,2} \xi_{3,5} + \frac{1}{2} \eta_{1,2} \xi_{2,3} \xi_{3,5} + \eta_{1,4} \xi_{4,5} - \eta_{2,4} \xi_{1,2} \xi_{4,5} - \eta_{3,4} \xi_{1,3} \xi_{4,5} + \\ & \frac{1}{2} \eta_{3,4} \xi_{1,2} \xi_{2,3} \xi_{4,5} + \frac{1}{2} \eta_{1,2} \xi_{2,4} \xi_{4,5} + \frac{1}{2} \eta_{1,3} \xi_{3,4} \xi_{4,5} - \frac{1}{2} \eta_{2,3} \xi_{1,2} \xi_{3,4} \xi_{4,5} + \frac{1}{6} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= **λTangent**[] = 0;

λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂_λx;

λTangent[xs_List] := λTangent@@xs

In[*]:= **\$Basis** (\$Basis /. x → ξ)

Out[*]=

$$\{x_{1,2} \xi_{1,2}, x_{1,3} \xi_{1,3}, x_{2,3} \xi_{2,3}, x_{1,4} \xi_{1,4}, x_{2,4} \xi_{2,4}, x_{3,4} \xi_{3,4}, x_{1,5} \xi_{1,5}, x_{2,5} \xi_{2,5}, x_{3,5} \xi_{3,5}, x_{4,5} \xi_{4,5}\}$$

In[*]:= **λTangent@**(λ \$Basis (\$Basis /. x → ξ))

Out[*]=

$$\begin{aligned} & x_{1,2} \xi_{1,2} + x_{1,3} \xi_{1,3} + x_{1,4} \xi_{1,4} + x_{1,5} \xi_{1,5} + x_{2,3} \xi_{2,3} + \lambda x_{1,3} \xi_{1,2} \xi_{2,3} + \\ & x_{2,4} \xi_{2,4} + \lambda x_{1,4} \xi_{1,2} \xi_{2,4} + x_{2,5} \xi_{2,5} + \lambda x_{1,5} \xi_{1,2} \xi_{2,5} + x_{3,4} \xi_{3,4} + \lambda x_{1,4} \xi_{1,3} \xi_{3,4} + \\ & \lambda x_{2,4} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + x_{3,5} \xi_{3,5} + \lambda x_{1,5} \xi_{1,3} \xi_{3,5} + \lambda x_{2,5} \xi_{2,3} \xi_{3,5} + \\ & \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + x_{4,5} \xi_{4,5} + \lambda x_{1,5} \xi_{1,4} \xi_{4,5} + \lambda x_{2,5} \xi_{2,4} \xi_{4,5} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \\ & \lambda x_{3,5} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= **lhs = Join**[λ \$Basis (\$Basis /. x → ξ), λ \$Basis (\$Basis /. x → η)] // λTangent

Out[*]=

$$\begin{aligned} & x_{1,2} \eta_{1,2} + x_{1,3} \eta_{1,3} + x_{1,4} \eta_{1,4} + x_{1,5} \eta_{1,5} + x_{2,3} \eta_{2,3} + \lambda x_{1,3} \eta_{1,2} \eta_{2,3} + x_{2,4} \eta_{2,4} + \lambda x_{1,4} \eta_{1,2} \eta_{2,4} + \\ & x_{2,5} \eta_{2,5} + \lambda x_{1,5} \eta_{1,2} \eta_{2,5} + x_{3,4} \eta_{3,4} + \lambda x_{1,4} \eta_{1,3} \eta_{3,4} + \lambda x_{2,4} \eta_{2,3} \eta_{3,4} + \lambda^2 x_{1,4} \eta_{1,2} \eta_{2,3} \eta_{3,4} + \\ & x_{3,5} \eta_{3,5} + \lambda x_{1,5} \eta_{1,3} \eta_{3,5} + \lambda x_{2,5} \eta_{2,3} \eta_{3,5} + \lambda^2 x_{1,5} \eta_{1,2} \eta_{2,3} \eta_{3,5} + x_{4,5} \eta_{4,5} + \lambda x_{1,5} \eta_{1,4} \eta_{4,5} + \\ & \lambda x_{2,5} \eta_{2,4} \eta_{4,5} + \lambda^2 x_{1,5} \eta_{1,2} \eta_{2,4} \eta_{4,5} + \lambda x_{3,5} \eta_{3,4} \eta_{4,5} + \lambda^2 x_{1,5} \eta_{1,3} \eta_{3,4} \eta_{4,5} + \lambda^2 x_{2,5} \eta_{2,3} \eta_{3,4} \eta_{4,5} + \\ & \lambda^3 x_{1,5} \eta_{1,2} \eta_{2,3} \eta_{3,4} \eta_{4,5} + x_{1,2} \xi_{1,2} + \lambda x_{1,3} \eta_{2,3} \xi_{1,2} + \lambda x_{1,4} \eta_{2,4} \xi_{1,2} + \lambda x_{1,5} \eta_{2,5} \xi_{1,2} + \\ & \lambda^2 x_{1,4} \eta_{2,3} \eta_{3,4} \xi_{1,2} + \lambda^2 x_{1,5} \eta_{2,3} \eta_{3,5} \xi_{1,2} + \lambda^2 x_{1,5} \eta_{2,4} \eta_{4,5} \xi_{1,2} + \lambda^3 x_{1,5} \eta_{2,3} \eta_{3,4} \eta_{4,5} \xi_{1,2} + \\ & x_{1,3} \xi_{1,3} + \lambda x_{1,4} \eta_{3,4} \xi_{1,3} + \lambda x_{1,5} \eta_{3,5} \xi_{1,3} + \lambda^2 x_{1,5} \eta_{3,4} \eta_{4,5} \xi_{1,3} + x_{1,4} \xi_{1,4} + \lambda x_{1,5} \eta_{4,5} \xi_{1,4} + \\ & x_{1,5} \xi_{1,5} + x_{2,3} \xi_{2,3} - \lambda x_{1,3} \eta_{1,2} \xi_{2,3} + \lambda x_{2,4} \eta_{3,4} \xi_{2,3} - \lambda^2 x_{1,4} \eta_{1,2} \eta_{3,4} \xi_{2,3} + \lambda x_{2,5} \eta_{3,5} \xi_{2,3} - \\ & \lambda^2 x_{1,5} \eta_{1,2} \eta_{3,5} \xi_{2,3} + \lambda^2 x_{2,5} \eta_{3,4} \eta_{4,5} \xi_{2,3} - \lambda^3 x_{1,5} \eta_{1,2} \eta_{3,4} \eta_{4,5} \xi_{2,3} + \lambda x_{1,3} \xi_{1,2} \xi_{2,3} + \\ & \lambda^2 x_{1,4} \eta_{3,4} \xi_{1,2} \xi_{2,3} + \lambda^2 x_{1,5} \eta_{3,5} \xi_{1,2} \xi_{2,3} + \lambda^3 x_{1,5} \eta_{3,4} \eta_{4,5} \xi_{1,2} \xi_{2,3} + x_{2,4} \xi_{2,4} - \lambda x_{1,4} \eta_{1,2} \xi_{2,4} + \\ & \lambda x_{2,5} \eta_{4,5} \xi_{2,4} - \lambda^2 x_{1,5} \eta_{1,2} \eta_{4,5} \xi_{2,4} + \lambda x_{1,4} \xi_{1,2} \xi_{2,4} + \lambda^2 x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,4} + x_{2,5} \xi_{2,5} - \\ & \lambda x_{1,5} \eta_{1,2} \xi_{2,5} + \lambda x_{1,5} \xi_{1,2} \xi_{2,5} + x_{3,4} \xi_{3,4} - \lambda x_{1,4} \eta_{1,3} \xi_{3,4} - \lambda x_{2,4} \eta_{2,3} \xi_{3,4} + \lambda x_{3,5} \eta_{4,5} \xi_{3,4} - \\ & \lambda^2 x_{1,5} \eta_{1,3} \eta_{4,5} \xi_{3,4} - \lambda^2 x_{2,5} \eta_{2,3} \eta_{4,5} \xi_{3,4} + \lambda x_{1,4} \xi_{1,3} \xi_{3,4} + \lambda^2 x_{1,5} \eta_{4,5} \xi_{1,3} \xi_{3,4} + \lambda x_{2,4} \xi_{2,3} \xi_{3,4} - \\ & \lambda^2 x_{1,4} \eta_{1,2} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{2,5} \eta_{4,5} \xi_{2,3} \xi_{3,4} - \lambda^3 x_{1,5} \eta_{1,2} \eta_{4,5} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \\ & \lambda^3 x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + x_{3,5} \xi_{3,5} - \lambda x_{1,5} \eta_{1,3} \xi_{3,5} - \lambda x_{2,5} \eta_{2,3} \xi_{3,5} + \lambda x_{1,5} \xi_{1,3} \xi_{3,5} + \\ & \lambda x_{2,5} \xi_{2,3} \xi_{3,5} - \lambda^2 x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + x_{4,5} \xi_{4,5} - \lambda x_{1,5} \eta_{1,4} \xi_{4,5} - \\ & \lambda x_{2,5} \eta_{2,4} \xi_{4,5} - \lambda x_{3,5} \eta_{3,4} \xi_{4,5} + \lambda x_{1,5} \xi_{1,4} \xi_{4,5} + \lambda x_{2,5} \xi_{2,4} \xi_{4,5} - \lambda^2 x_{1,5} \eta_{1,2} \xi_{2,4} \xi_{4,5} + \\ & \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \lambda x_{3,5} \xi_{3,4} \xi_{4,5} - \lambda^2 x_{1,5} \eta_{1,3} \xi_{3,4} \xi_{4,5} - \lambda^2 x_{2,5} \eta_{2,3} \xi_{3,4} \xi_{4,5} + \\ & \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} - \lambda^3 x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= rhs = λTangent@(\$Basis (\$Basis /. x_{αβ} → f_{αβ}[λ]))

Out[*]=

$$\begin{aligned} & x_{1,2} f_{1,2}'[\lambda] + x_{1,3} f_{2,3}[\lambda] f_{1,2}'[\lambda] + x_{1,4} f_{2,4}[\lambda] f_{1,2}'[\lambda] + x_{1,5} f_{2,5}[\lambda] f_{1,2}'[\lambda] + \\ & x_{1,4} f_{2,3}[\lambda] f_{3,4}[\lambda] f_{1,2}'[\lambda] + x_{1,5} f_{2,3}[\lambda] f_{3,5}[\lambda] f_{1,2}'[\lambda] + x_{1,5} f_{2,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] + \\ & x_{1,5} f_{2,3}[\lambda] f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] + x_{1,3} f_{1,3}'[\lambda] + x_{1,4} f_{3,4}[\lambda] f_{1,3}'[\lambda] + x_{1,5} f_{3,5}[\lambda] f_{1,3}'[\lambda] + \\ & x_{1,5} f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,3}'[\lambda] + x_{1,4} f_{1,4}'[\lambda] + x_{1,5} f_{4,5}[\lambda] f_{1,4}'[\lambda] + x_{1,5} f_{1,5}'[\lambda] + x_{2,3} f_{2,3}'[\lambda] + \\ & x_{2,4} f_{3,4}[\lambda] f_{2,3}'[\lambda] + x_{2,5} f_{3,5}[\lambda] f_{2,3}'[\lambda] + x_{2,5} f_{3,4}[\lambda] f_{4,5}[\lambda] f_{2,3}'[\lambda] + x_{2,4} f_{2,4}'[\lambda] + \\ & x_{2,5} f_{4,5}[\lambda] f_{2,4}'[\lambda] + x_{2,5} f_{2,5}'[\lambda] + x_{3,4} f_{3,4}'[\lambda] + x_{3,5} f_{4,5}[\lambda] f_{3,4}'[\lambda] + x_{3,5} f_{3,5}'[\lambda] + x_{4,5} f_{4,5}'[\lambda] \end{aligned}$$

In[*]:= (Coefficient[lhs - rhs, #] == 0) & /@ \$Basis

Out[*]=

$$\begin{aligned} & \{ \eta_{1,2} + \xi_{1,2} - f_{1,2}'[\lambda] = 0, \\ & \eta_{1,3} + \lambda \eta_{1,2} \eta_{2,3} + \lambda \eta_{2,3} \xi_{1,2} + \xi_{1,3} - \lambda \eta_{1,2} \xi_{2,3} + \lambda \xi_{1,2} \xi_{2,3} - f_{2,3}[\lambda] f_{1,2}'[\lambda] - f_{1,3}'[\lambda] = 0, \\ & \eta_{2,3} + \xi_{2,3} - f_{2,3}'[\lambda] = 0, \\ & \eta_{1,4} + \lambda \eta_{1,2} \eta_{2,4} + \lambda \eta_{1,3} \eta_{3,4} + \lambda^2 \eta_{1,2} \eta_{2,3} \eta_{3,4} + \lambda \eta_{2,4} \xi_{1,2} + \lambda^2 \eta_{2,3} \eta_{3,4} \xi_{1,2} + \lambda \eta_{3,4} \xi_{1,3} + \xi_{1,4} - \\ & \lambda^2 \eta_{1,2} \eta_{3,4} \xi_{2,3} + \lambda^2 \eta_{3,4} \xi_{1,2} \xi_{2,3} - \lambda \eta_{1,2} \xi_{2,4} + \lambda \xi_{1,2} \xi_{2,4} - \lambda \eta_{1,3} \xi_{3,4} + \lambda \xi_{1,3} \xi_{3,4} - \lambda^2 \eta_{1,2} \xi_{2,3} \xi_{3,4} + \\ & \lambda^2 \xi_{1,2} \xi_{2,3} \xi_{3,4} - f_{2,4}[\lambda] f_{1,2}'[\lambda] - f_{2,3}[\lambda] f_{3,4}[\lambda] f_{1,2}'[\lambda] - f_{3,4}[\lambda] f_{1,3}'[\lambda] - f_{1,4}'[\lambda] = 0, \\ & \eta_{2,4} + \lambda \eta_{2,3} \eta_{3,4} + \lambda \eta_{3,4} \xi_{2,3} + \xi_{2,4} - \lambda \eta_{2,3} \xi_{3,4} + \lambda \xi_{2,3} \xi_{3,4} - f_{3,4}[\lambda] f_{2,3}'[\lambda] - f_{2,4}'[\lambda] = 0, \\ & \eta_{3,4} + \xi_{3,4} - f_{3,4}'[\lambda] = 0, \\ & \eta_{1,5} + \lambda \eta_{1,2} \eta_{2,5} + \lambda \eta_{1,3} \eta_{3,5} + \lambda^2 \eta_{1,2} \eta_{2,3} \eta_{3,5} + \lambda \eta_{1,4} \eta_{4,5} + \lambda^2 \eta_{1,2} \eta_{2,4} \eta_{4,5} + \lambda^2 \eta_{1,3} \eta_{3,4} \eta_{4,5} + \\ & \lambda^3 \eta_{1,2} \eta_{2,3} \eta_{3,4} \eta_{4,5} + \lambda \eta_{2,5} \xi_{1,2} + \lambda^2 \eta_{2,3} \eta_{3,5} \xi_{1,2} + \lambda^2 \eta_{2,4} \eta_{4,5} \xi_{1,2} + \lambda^3 \eta_{2,3} \eta_{3,4} \eta_{4,5} \xi_{1,2} + \\ & \lambda \eta_{3,5} \xi_{1,3} + \lambda^2 \eta_{3,4} \eta_{4,5} \xi_{1,3} + \lambda \eta_{4,5} \xi_{1,4} + \xi_{1,5} - \lambda^2 \eta_{1,2} \eta_{3,5} \xi_{2,3} - \lambda^3 \eta_{1,2} \eta_{3,4} \eta_{4,5} \xi_{2,3} + \\ & \lambda^2 \eta_{3,5} \xi_{1,2} \xi_{2,3} + \lambda^3 \eta_{3,4} \eta_{4,5} \xi_{1,2} \xi_{2,3} - \lambda^2 \eta_{1,2} \eta_{4,5} \xi_{2,4} + \lambda^2 \eta_{4,5} \xi_{1,2} \xi_{2,4} - \\ & \lambda \eta_{1,2} \xi_{2,5} + \lambda \xi_{1,2} \xi_{2,5} - \lambda^2 \eta_{1,3} \eta_{4,5} \xi_{3,4} + \lambda^2 \eta_{4,5} \xi_{1,3} \xi_{3,4} - \lambda^3 \eta_{1,2} \eta_{4,5} \xi_{2,3} \xi_{3,4} + \\ & \lambda^3 \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} - \lambda \eta_{1,3} \xi_{3,5} + \lambda \xi_{1,3} \xi_{3,5} - \lambda^2 \eta_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 \xi_{1,2} \xi_{2,3} \xi_{3,5} - \\ & \lambda \eta_{1,4} \xi_{4,5} + \lambda \xi_{1,4} \xi_{4,5} - \lambda^2 \eta_{1,2} \xi_{2,4} \xi_{4,5} + \lambda^2 \xi_{1,2} \xi_{2,4} \xi_{4,5} - \lambda^2 \eta_{1,3} \xi_{3,4} \xi_{4,5} + \\ & \lambda^2 \xi_{1,3} \xi_{3,4} \xi_{4,5} - \lambda^3 \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} - f_{2,5}[\lambda] f_{1,2}'[\lambda] - \\ & f_{2,3}[\lambda] f_{3,5}[\lambda] f_{1,2}'[\lambda] - f_{2,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] - f_{2,3}[\lambda] f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] - \\ & f_{3,5}[\lambda] f_{1,3}'[\lambda] - f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,3}'[\lambda] - f_{4,5}[\lambda] f_{1,4}'[\lambda] - f_{1,5}'[\lambda] = 0, \\ & \eta_{2,5} + \lambda \eta_{2,3} \eta_{3,5} + \lambda \eta_{2,4} \eta_{4,5} + \lambda^2 \eta_{2,3} \eta_{3,4} \eta_{4,5} + \lambda \eta_{3,5} \xi_{2,3} + \lambda^2 \eta_{3,4} \eta_{4,5} \xi_{2,3} + \lambda \eta_{4,5} \xi_{2,4} + \xi_{2,5} - \\ & \lambda^2 \eta_{2,3} \eta_{4,5} \xi_{3,4} + \lambda^2 \eta_{4,5} \xi_{2,3} \xi_{3,4} - \lambda \eta_{2,3} \xi_{3,5} + \lambda \xi_{2,3} \xi_{3,5} - \lambda \eta_{2,4} \xi_{4,5} + \lambda \xi_{2,4} \xi_{4,5} - \lambda^2 \eta_{2,3} \xi_{3,4} \xi_{4,5} + \\ & \lambda^2 \xi_{2,3} \xi_{3,4} \xi_{4,5} - f_{3,5}[\lambda] f_{2,3}'[\lambda] - f_{3,4}[\lambda] f_{4,5}[\lambda] f_{2,3}'[\lambda] - f_{4,5}[\lambda] f_{2,4}'[\lambda] - f_{2,5}'[\lambda] = 0, \\ & \eta_{3,5} + \lambda \eta_{3,4} \eta_{4,5} + \lambda \eta_{4,5} \xi_{3,4} + \xi_{3,5} - \lambda \eta_{3,4} \xi_{4,5} + \lambda \xi_{3,4} \xi_{4,5} - f_{4,5}[\lambda] f_{3,4}'[\lambda] - f_{3,5}'[\lambda] = 0, \\ & \eta_{4,5} + \xi_{4,5} - f_{4,5}'[\lambda] = 0 \} \end{aligned}$$

```
In[*]:= {sol} = DSolve[
  Join[
    (Coefficient[lhs - rhs, #] == 0) & /@ $Basis,
    $Basis /. xαβ → fαβ[0] == 0
  ],
  $Basis /. xαβ → fαβ[λ],
  λ
]
```

```
Out[*]= { {f1,2[λ] → λ η1,2 + λ ξ1,2, f1,3[λ] → λ η1,3 + λ ξ1,3 - λ2 η1,2 ξ2,3,
  f2,3[λ] → λ η2,3 + λ ξ2,3, f1,4[λ] → -λ (-η1,4 - ξ1,4 + λ η1,2 ξ2,4 + λ η1,3 ξ3,4),
  f2,4[λ] → -λ (-η2,4 - ξ2,4 + λ η2,3 ξ3,4), f3,4[λ] → λ (η3,4 + ξ3,4), f4,5[λ] → λ η4,5 + λ ξ4,5,
  f3,5[λ] → λ η3,5 + λ ξ3,5 - λ2 η3,4 ξ4,5, f2,5[λ] → λ η2,5 + λ ξ2,5 - λ2 η2,3 ξ3,5 - λ2 η2,4 ξ4,5,
  f1,5[λ] → λ η1,5 + λ ξ1,5 - λ2 η1,2 ξ2,5 - λ2 η1,3 ξ3,5 - λ2 η1,4 ξ4,5 }
```

```
In[*]:= sol /. λ → 1
```

```
Out[*]= {f1,2[1] → η1,2 + ξ1,2, f1,3[1] → η1,3 + ξ1,3 - η1,2 ξ2,3, f2,3[1] → η2,3 + ξ2,3,
  f1,4[1] → η1,4 + ξ1,4 - η1,2 ξ2,4 - η1,3 ξ3,4, f2,4[1] → η2,4 + ξ2,4 - η2,3 ξ3,4,
  f3,4[1] → η3,4 + ξ3,4, f4,5[1] → η4,5 + ξ4,5, f3,5[1] → η3,5 + ξ3,5 - η3,4 ξ4,5,
  f2,5[1] → η2,5 + ξ2,5 - η2,3 ξ3,5 - η2,4 ξ4,5, f1,5[1] → η1,5 + ξ1,5 - η1,2 ξ2,5 - η1,3 ξ3,5 - η1,4 ξ4,5}
```

```
In[*]:= osol = E{1,2}→{2} [x1,1[2] ξ1,1[1] + x1,1[2] ξ1,1[2] + e-ε1,1[2] x1,2[2] ξ1,2[1] + e-ε2,2[1] x1,2[2] ξ1,2[2] +
  e-ε1,1[2] x1,3[2] ξ1,3[1] + e-ε3,3[1] x1,3[2] ξ1,3[2] + e-ε1,1[2] x1,4[2] ξ1,4[1] +
  e-ε4,4[1] x1,4[2] ξ1,4[2] + e-ε1,1[2] x1,5[2] ξ1,5[1] + e-ε5,5[1] x1,5[2] ξ1,5[2] +
  x2,2[2] ξ2,2[1] + x2,2[2] ξ2,2[2] + e-ε2,2[2] x2,3[2] ξ2,3[1] - x1,3[2] ξ1,2[2] ξ2,3[1] +
  e-ε3,3[1] x2,3[2] ξ2,3[2] + e-ε2,2[2] x2,4[2] ξ2,4[1] - x1,4[2] ξ1,2[2] ξ2,4[1] +
  e-ε4,4[1] x2,4[2] ξ2,4[2] + e-ε2,2[2] x2,5[2] ξ2,5[1] - x1,5[2] ξ1,2[2] ξ2,5[1] +
  e-ε5,5[1] x2,5[2] ξ2,5[2] + x3,3[2] ξ3,3[1] + x3,3[2] ξ3,3[2] + e-ε3,3[2] x3,4[2] ξ3,4[1] -
  x1,4[2] ξ1,3[2] ξ3,4[1] - x2,4[2] ξ2,3[2] ξ3,4[1] + e-ε4,4[1] x3,4[2] ξ3,4[2] +
  e-ε3,3[2] x3,5[2] ξ3,5[1] - x1,5[2] ξ1,3[2] ξ3,5[1] - x2,5[2] ξ2,3[2] ξ3,5[1] +
  e-ε5,5[1] x3,5[2] ξ3,5[2] + x4,4[2] ξ4,4[1] + x4,4[2] ξ4,4[2] + e-ε4,4[2] x4,5[2] ξ4,5[1] -
  x1,5[2] ξ1,4[2] ξ4,5[1] - x2,5[2] ξ2,4[2] ξ4,5[1] - x3,5[2] ξ3,4[2] ξ4,5[1] +
  e-ε5,5[1] x4,5[2] ξ4,5[2] + x5,5[2] ξ5,5[1] + x5,5[2] ξ5,5[2], 0, 0] [[1] /. ξi_,i_ → 0
```

```
Out[*]= x1,2[2] ξ1,2[1] + x1,2[2] ξ1,2[2] + x1,3[2] ξ1,3[1] + x1,3[2] ξ1,3[2] +
  x1,4[2] ξ1,4[1] + x1,4[2] ξ1,4[2] + x1,5[2] ξ1,5[1] + x1,5[2] ξ1,5[2] + x2,3[2] ξ2,3[1] -
  x1,3[2] ξ1,2[2] ξ2,3[1] + x2,3[2] ξ2,3[2] + x2,4[2] ξ2,4[1] - x1,4[2] ξ1,2[2] ξ2,4[1] +
  x2,4[2] ξ2,4[2] + x2,5[2] ξ2,5[1] - x1,5[2] ξ1,2[2] ξ2,5[1] + x2,5[2] ξ2,5[2] + x3,4[2] ξ3,4[1] -
  x1,4[2] ξ1,3[2] ξ3,4[1] - x2,4[2] ξ2,3[2] ξ3,4[1] + x3,4[2] ξ3,4[2] + x3,5[2] ξ3,5[1] -
  x1,5[2] ξ1,3[2] ξ3,5[1] - x2,5[2] ξ2,3[2] ξ3,5[1] + x3,5[2] ξ3,5[2] + x4,5[2] ξ4,5[1] -
  x1,5[2] ξ1,4[2] ξ4,5[1] - x2,5[2] ξ2,4[2] ξ4,5[1] - x3,5[2] ξ3,4[2] ξ4,5[1] + x4,5[2] ξ4,5[2]
```

```
In[*]:= Expand@Total[sol /. { $\xi_{\alpha\beta} \Rightarrow \xi_{\alpha\beta}[1]$ ,  $\eta_{\alpha\beta} \Rightarrow \xi_{\alpha\beta}[2]$ , Rule  $\rightarrow$  Times,  $\lambda \rightarrow 1$ ,  $f_{\alpha\beta}[_] \Rightarrow x_{\alpha\beta}[2]$ }] -  
osol  
Out[*]=  
0
```