

Hidden T_EX

A Timing Game

Headers

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Pensieve header: An isolated attempt to compute the multiplication tensor of $\mathcal{U}_q(s-l^+_{3,\epsilon})$. Continues SjabboQU_{SL3}Seepage.nb.

The implementation of QU is modified from pensieve://Projects/SL2Portfolio/SL2PortfolioProgram.nb.

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In[]:=

```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

nb2tex

In[]:=

```
Get["../nb2tex/nb2tex.m"];
nb2tex["Qsl3+eps", PDFWidth -> 7.5, PDFFolder -> "Snips/Qsl3+eps"];
texfile = OpenAppend["Qsl3+eps.tex"];
WriteLine[texfile, "\\end{document}"]; Close[texfile];
Run@
  "\"C:\\Users\\drorb\\AppData\\Local\\Programs\\MiKTeX\\miktex\\bin\\x64\\pdflatex.exe\"
  Qsl3+eps.tex";
```

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Initialization / Utilities

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```
In[*]:=
$k = 1; $U = QU; $E := {$k};
$trim := {ek /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
qh := eγ eh;
(* Upper to lower and lower to Upper: *)
U2l = {Bip → e-p h γ bi, Bp → e-p h γ b, Tip → ep h ti, Tp → ep h t, Aip → ep γ αi, Ap → ep γ α};
l2U = {ec- bi + d- → Bi-c/(h γ) ed, ec- b + d- → B-c/(h γ) ed,
  ec- ti + d- → Tic/h ed, ec- t + d- → Tc/h ed,
  ec- αi + d- → Aic/γ ed, ec- α + d- → Ac/γ ed,
  eδ → eExpand@δ};
SS[ε-, op-] := Collect[Normal@Series[ε, {ε, 0, $k}], ε, op];
SS[ε-] := SS[ε, Together];
SST[ε-, op---] := SS[ε /. U2l, op];
Simp[ε-, op-] := Collect[ε, _CU | _QU, op];
Simp[ε-] := Simp[ε, SS[#, Expand] &];
SimpT[ε-] := Collect[ε, _CU | _QU, SST[#, Expand] &];
Kδ /: Kδi-, j- := If[i === j, 1, 0];
c_Integerk_Integer := c + O[ε]k+1;
```

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DeclareAlgebra

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```
In[*]:=
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x-] := x;
NCM[x-, y-, z---] := (x ⓧ y) ⓧ z;
0 ⓧ _ = _ ⓧ 0 = 0;
(x_Plus) ⓧ y- := (# ⓧ y) & /@ x; x ⓧ (y_Plus) := (x ⓧ #) & /@ y;
B[x-, x-] = 0; B[x-, y-] := x ⓧ y - y ⓧ x;
B[x-, y-, e-] := B[x, y, e] = B[x, y];
```

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```
In[*]:=
DeclareAlgebra[U_Symbol, opts__Rule] := Module[
  {gp, sr, g, cp, M, pow, k = 0,
   gs = Generators /. {opts},
   cs = Centrals /. {opts} /. Centrals → {}},
  ],
  U[Generators] = gs;
  (#U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp; (* gens *)
  sr = Flatten@Table[{g → ++k, gi → {i, k}}, {g, gs}]; (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
```

```

SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
CE[ε_] := Collect[ε, _U, Expand] /. e^x_ := eExpand[x] /. $trim;
U_i[ε_] := ε /. {t : cp => t_i, u_U => (#i &) /@ u};
U_i[NCM[]] = pow[ε_, 0] = U@{} = 1_U = U[];
B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
B[U@(x_)_i_, U@(y_)_j_] /; i != j := 0;
B[U@y_, U@x_] := CE[-B[U@x, U@y]];
x_ ⓧ (c_. 1_U) := CE[c x]; (c_. 1_U) ⓧ x_ := CE[c x];
(a_. U[xx___, x_]) ⓧ (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
  CE@M[a b /. $trim, U[xx, x, y, yy]],
  U@xx ⓧ CE@M[a b /. $trim, U@y ⓧ U@x + B[U@x, U@y, $E]] ⓧ U@yy];
U@{c_. * (l : gp)n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ⓧ U@{r}];
U@{c_. * l : gp, r___} := CE[c U[l] ⓧ U@{r}];
U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
U@{L_Plus, r___} := CE[U@{#, r} & /@ L];
U@{L_, r___} := U@{Expand[L], r};
U[ε_NonCommutativeMultiply] := U /@ ε;
O_U[specs___, poly_] := Module[{sp, null, vs, us},
  sp = Replace[{specs}, L_List => L_null, {1}];
  vs = Join@@(First /@ sp);
  us = Join@@(sp /. L_s_ => (l /. x_i_ => x_s));
  CE[Total[
    CoefficientRules[poly, vs] /. (p_ -> c_) => c U@(usp)
  ] / x_null => x];
O_U[specs___, E[L_, Q_, P_]] := O_U[specs, SS@Normal[P eL+Q]];
pow[ε_, n_] := pow[ε, n - 1] ⓧ ε;
S_U[ε_, ss___Rule] := CE@Total[
  CoefficientRules[ε, First /@ {ss}] /.
    (p_ -> c_) => c NCM@@MapThread[pow, {Last /@ {ss}, p}]];
σ_rs___[c_. * u_U] := (c /. (t : cp)_j_ => t_j /. {rs}) U[List@@(u /. v_j_ => v_j /. {rs})];
m_j->k_[c_. * u_U] := CE[(c /. (t : cp)_j_ => t_k) DeleteCases[u, _j|k] ⓧ
  U@@Cases[u, w_j_ => w_k] ⓧ U@@Cases[u, _k]];
U /: c_. * u_U * v_U := CE[c u ⓧ v];
S_i[c_. * u_U] := CE[(c /. S_i[U, Centrals]) DeleteCases[u, _i] ⓧ
  U_i[NCM@@Reverse@Cases[u, x_i_ => S@U@x]]];
Δ_i->j,k_[c_. * u_U] := CE[(c /. Δ_i->j,k_[U, Centrals]) DeleteCases[u, _i] ⓧ
  (NCM@@Cases[u, x_i_ => σ_1->j, 2->k@Δ@U@x] /. NCM[] -> U[])];

```

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```

In[*]:= Invert_U[ε_] := Module[{A, B, β}, (* ε=A U[] + ε B *)
  A = Coefficient[ε, 1_U] /. e -> 0;
  B = CE[e-1 (ε - A 1_U)];
  S_U[SS[(A + e β)-1], β -> B]
]

```

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Implementing $QU = \mathcal{U}_q(\mathfrak{sl}_{3,\gamma\epsilon}^+)$

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Greek - Latin duality:

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In[*]:=

```

{X*, Y*, Z*, b*, a*, z*, y*, x*, Ξ*, Υ*, Ω*, β*, α*, ζ*, η*, ξ*} =
  {Ξ, Υ, Ω, β, α, ζ, η, ξ, X, Y, Z, b, a, z, y, x};
(u_{αβ})^* := (u^*)_{αβ};
((u : ((y | x | η | ξ) | (y | x | η | ξ) _)) [i_])^* := u^* [i];
(vs_List)^* := (v ↦ v^*) /@ vs;

```

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```

In[*]:=  $\hbar = \gamma = 1;$ 
DeclareAlgebra[QU, Generators  $\rightarrow \{X, Y, Z, b, a, z, y, x\}$ , Centrals  $\rightarrow \{s, t\}$ ];
B[aQU, xQU] = -2 QU@x;
B[bQU, xQU] = QU@x;
B[bQU, yQU] = -2 QU@y;
B[aQU, yQU] = QU@y;
B[aQU, zQU] = -QU@z;
B[bQU, zQU] = -QU@z;
B[yQU, xQU] = -zQU +  $\epsilon \hbar$  QU[y, x];
B[zQU, xQU] = - $\epsilon \hbar$  QU[z, x];
B[bQU, aQU] = 0;
B[zQU, yQU] =  $\epsilon \hbar$  QU[z, y];
B[aQU, xQU] = 2 xQU;
B[bQU, xQU] = -xQU;
B[aQU, yQU] = -yQU;
B[bQU, yQU] = 2 yQU;
B[aQU, zQU] = zQU;
B[bQU, zQU] = zQU;

B[xQU, xQU] = -2  $\epsilon \hbar$  QU@{X, x} +  $\frac{(1-s) \text{QU}[]}{\hbar} + 2 s \in \text{QU}[a]$ ;
B[yQU, yQU] = -2  $\epsilon \hbar$  QU@{Y, y} +  $\frac{(1-t) \text{QU}[]}{\hbar} + 2 t \in \text{QU}[b]$ ;
B[zQU, zQU] = -2  $\epsilon \hbar$  QU@{Z, z} +  $\frac{(1-s t) \text{QU}[]}{\hbar} + 2 s t \in \text{QU}[a] + 2 s t \in \text{QU}[b]$ ;

B[yQU, zQU] = xQU -  $\epsilon \hbar$  QU@{Z, y};
B[xQU, zQU] = - $\epsilon \hbar$  QU@{Z, x} - (s + s  $\epsilon \hbar$ ) QU[Y] + 2 s  $\epsilon \hbar$  QU[Y, a];
B[zQU, xQU] = 2  $\epsilon$  yQU -  $\epsilon \hbar$  QU@{X, z};
B[zQU, yQU] = -2 t  $\epsilon$  QU[x] -  $\epsilon \hbar$  QU[Y, z];
B[yQU, xQU] =  $\epsilon \hbar$  QU[X, y];
B[xQU, yQU] =  $\epsilon \hbar$  QU[Y, x];
B[xQU, yQU] = 2  $\epsilon$  zQU -  $\epsilon \hbar$  QU[X, Y];
B[xQU, zQU] =  $\epsilon \hbar$  QU[X, Z];
B[zQU, yQU] =  $\epsilon \hbar$  QU[Y, Z];

```

```

In[*]:= QU@Generators

```

```

Time: 0.02 seconds Out:=

```

```

{X, Y, Z, b, a, z, y, x}

```

```

In[*]:= $Basis = QU /@ QU@Generators;
Module[{a, b}, Table[{a, b} → CE[B[a, b]], {a, $Basis}, {b, $Basis}]] // MatrixForm
Time: 0.02 seconds Out|=

$$\left\{ \begin{array}{ll} \{QU[X], QU[X]\} \rightarrow \emptyset & \{QU[X], QU[Y]\} \rightarrow 2 \in QU[Z] - \in QU[Y] \\ \{QU[Y], QU[X]\} \rightarrow -2 \in QU[Z] + \in QU[X, Y] & \{QU[Y], QU[Y]\} \rightarrow \emptyset \\ \{QU[Z], QU[X]\} \rightarrow -\in QU[X, Z] & \{QU[Z], QU[Y]\} \rightarrow \in QU[Y, Z] \\ \{QU[b], QU[X]\} \rightarrow -QU[X] & \{QU[b], QU[Y]\} \rightarrow 2 QU[Y] \\ \{QU[a], QU[X]\} \rightarrow 2 QU[X] & \{QU[a], QU[Y]\} \rightarrow -QU[Y] \\ \{QU[z], QU[X]\} \rightarrow 2 \in QU[y] - \in QU[X, z] & \{QU[z], QU[Y]\} \rightarrow -2 t \in QU[x] - \in \\ \{QU[y], QU[X]\} \rightarrow \in QU[X, y] & \{QU[y], QU[Y]\} \rightarrow (1-t) QU[] + 2 t \in QU[t] \\ \{QU[x], QU[X]\} \rightarrow (1-s) QU[] + 2 s \in QU[a] - 2 \in QU[X, x] & \{QU[x], QU[Y]\} \rightarrow \in QU[Y, x] \end{array} \right.$$


In[*]:= Module[{a, b, c},
DeleteCases[Flatten[Table[{a, b, c} → CE[B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]]],
{a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
]
Time: 0.80 seconds Out|=
{ }

```

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The Multiplication Tensor

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```

In[*]:= adPower[x_, k_, Env_] [c_ * y_QU] := CE[c adPower[x, k, Env] [y]];
adPower[x_, k_, Env_] [y_Plus] := adPower[x, k, Env] /@ y;
adPower[_ , 0, Env_] [y_] := y;
adPower[x_, k_, Env_] [0] = 0;
adPower[x_, k_, Env_] [y_QU] :=
adPower[x, k, Env] [y] = CE@B[adPower[x, k-1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, $E] [y];

```

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```

In[*]:= NilQ[z_QU] = NilQ[Z_QU] = NilQ[X_QU] = NilQ[Y_QU] = NilQ[x_QU] = NilQ[y_QU] = True; NilQ[a_QU] = False;
NilQ[b_QU] = False;
PotQ[c_ * x_QU] := ¬ NilQ[x]; (* Potent Q *)
NilQ[c_ * x_QU] := NilQ[x];
PotQ[x_Plus] := And @@ (PotQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));

```

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```

In[*]:= Ad[0] = Identity;
Ad[c_ . a_QU] [z_] := z / . w_QU → ec Y (Sum[ $\frac{B[g_{QU}, a_{QU}]}{g_{QU}}$  Count[w, g], {g, QU@Generators}]) w;
Ad[c_ . b_QU] [z_] := z / . w_QU → ec Y (Sum[ $\frac{B[g_{QU}, b_{QU}]}{g_{QU}}$  Count[w, g], {g, QU@Generators}]) w;
Ad[c_ . QU[]] [z_] := z;

```

```
In[*]:= Table[SeriesCoefficient[Ad[c aQU][gQU], {c, 0, 1}] -
  B[gQU, aQU], {g, Generators // QU}]
Table[SeriesCoefficient[Ad[c bQU][gQU], {c, 0, 1}] -
  B[gQU, bQU], {g, Generators // QU}]
```

Time: 0.06 seconds Out[]=

```
{0, 0, 0, 0, 0, 0, 0, 0}
```

Out[*]=

```
{0, 0, 0, 0, 0, 0, 0, 0}
```

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```
In[*]:= Ad[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k][y] != 0,
    s += adPower[x, k][y] / k!;
    ++k
  ];
  s
]
```

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```
In[*]:= TTλ[] = 0; (* Translated Tangent *)
TTλ[xs___, x_] := CE[Ad[x][TTλ[xs]] + ∂λx];
TTλ[xs_List] := TTλ@@xs
```

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```
In[*]:= SeepageU[{ }, c_] = c;
SeepageU[ξs_List, P_] := Module[{gs = U[Generators], lx},
  lx = gs[[Length@ξs]];
  Collect[P, lx, (Ad[Last[ξs] lxU][0U[gs, SeepageU[Most@ξs, #]]] /. U → Times) &]
]
```

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$$\begin{aligned}
 L_\lambda &= \mathbb{C}\langle \lambda_{\beta_1} x \rangle \mathbb{C}\langle \lambda_{\beta_2} x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle f; x \rangle \mathbb{C}\langle \text{Seepage}(\{f\}, A) \rangle &= \mathbb{C}\langle e^{f; x}; A \rangle \\
 TT(L_\lambda) &= L'_\lambda \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f\}, B). \\
 R_\lambda &= \mathbb{C}\langle f+g \rangle, \quad f=f; x, \quad g=\sum_{w \in W} g_w w & e^f = F e^g = G & \quad R_\lambda = \mathbb{C}\langle FG \rangle \quad \partial_\lambda F = F \partial_\lambda f \\
 TT(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle' \mathbb{C}\langle F(\partial_\lambda FG + \partial_\lambda G) \rangle = S(G)' S(\partial_\lambda FG + \partial_\lambda G).
 \end{aligned}$$

```
In[*]:= $k = 1
```

Time: 0.00 seconds Out[]=

```
1
```

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```
In[ ]:= TTL = CE[TT, @@ (Flatten@Table[
    (If[PotQ[#Qu], 1, λ] #*[k] #Qu) & /@QU@Generators,
    {k, {i, j}}
]]]
```

Time: 12.36 seconds Out[]:=

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```
In[ ]:= mons[k_] := Union@Cases[Coefficient[TTL, ε, k], _QU, ∞];
mons /@ Range[0, $k]
```

Time: 0.01 seconds Out[]:=

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```
{ {QU[], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z] },
  {QU[], QU[a], QU[b], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z], QU[X, x], QU[X, X],
    QU[X, y], QU[X, Y], QU[X, z], QU[X, Z], QU[y, x], QU[Y, a], QU[Y, x], QU[Y, y], QU[Y, Y],
    QU[Y, z], QU[Y, Z], QU[z, x], QU[z, y], QU[z, z], QU[Z, x], QU[Z, y], QU[Z, z] } }
```

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```
In[ ]:= ff = Table[
    Sum[ε^k f_{k,u}[λ], {k, 0, $k}],
    {u, QU@Generators}
] /. f_{0,u}[λ] /; PotQ[uQu] => u*[i] + u*[j]
```

Time: 0.00 seconds Out[]:=

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```
{ f_{0,x}[λ] + ε f_{1,x}[λ], f_{0,y}[λ] + ε f_{1,y}[λ], f_{0,z}[λ] + ε f_{1,z}[λ], β[i] + β[j] + ε f_{1,b}[λ],
  α[i] + α[j] + ε f_{1,a}[λ], f_{0,z}[λ] + ε f_{1,z}[λ], f_{0,y}[λ] + ε f_{1,y}[λ], f_{0,x}[λ] + ε f_{1,x}[λ] }
```

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```
In[ ]:= Short[fs = Flatten[Table[
    f_{k, Times@@m},
    {k, 0, $k}, {m, mons[k]}
], 10]
```

Time: 0.05 seconds Out[]:=

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```
{ f_{0,1}, f_{0,x}, f_{0,x}, f_{0,y}, f_{0,y}, f_{0,z}, f_{0,z}, f_{1,1}, f_{1,a}, f_{1,b}, f_{1,x}, f_{1,x},
  f_{1,y}, f_{1,y}, f_{1,z}, f_{1,z}, f_{1,x,x}, f_{1,x^2}, f_{1,x,y}, f_{1,x,y}, f_{1,x,z}, f_{1,x,z}, f_{1,x,y}, f_{1,a,y},
  f_{1,x,y}, f_{1,y,y}, f_{1,y^2}, f_{1,y,z}, f_{1,y,z}, f_{1,x,z}, f_{1,y,z}, f_{1,z^2}, f_{1,x,z}, f_{1,y,z}, f_{1,z,z} }
```

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```
In[ ]:= gg = Sum[
    ε^k f_{k, Times@@u}[λ] Times@@u,
    {k, 0, $k}, {u, Complement[mons[k], QU /@ QU@Generators]}
]
```

Time: 0.03 seconds Out[]:=

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```
f_{0,1}[λ] + ε f_{1,1}[λ] + x X ε f_{1,x,x}[λ] + X^2 ε f_{1,x^2}[λ] + x y ε f_{1,x,y}[λ] + X y ε f_{1,x,y}[λ] +
  a Y ε f_{1,a,y}[λ] + x Y ε f_{1,x,y}[λ] + X Y ε f_{1,x,y}[λ] + y Y ε f_{1,y,y}[λ] + Y^2 ε f_{1,y^2}[λ] +
  x z ε f_{1,x,z}[λ] + X z ε f_{1,x,z}[λ] + y z ε f_{1,y,z}[λ] + Y z ε f_{1,y,z}[λ] + z^2 ε f_{1,z^2}[λ] +
  x Z ε f_{1,x,z}[λ] + X Z ε f_{1,x,z}[λ] + y Z ε f_{1,y,z}[λ] + Y Z ε f_{1,y,z}[λ] + z Z ε f_{1,z,z}[λ]
```


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In[]:= **G** = **SS** [**e^{gg}**]

Time: 0.11 seconds Out[]:=

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$$e^{f_{0,1}[\lambda]} + e^{f_{0,1}[\lambda]} \in$$

$$\left(f_{1,1}[\lambda] + x X f_{1,xX}[\lambda] + X^2 f_{1,X^2}[\lambda] + x y f_{1,xy}[\lambda] + X y f_{1,xy}[\lambda] + a Y f_{1,aY}[\lambda] + x Y f_{1,xY}[\lambda] + \right.$$

$$X Y f_{1,XY}[\lambda] + y Y f_{1,yY}[\lambda] + Y^2 f_{1,Y^2}[\lambda] + x z f_{1,xz}[\lambda] + X z f_{1,Xz}[\lambda] + y z f_{1,yz}[\lambda] + Y z f_{1,Yz}[\lambda] +$$

$$\left. z^2 f_{1,z^2}[\lambda] + x Z f_{1,xZ}[\lambda] + X Z f_{1,XZ}[\lambda] + y Z f_{1,yZ}[\lambda] + Y Z f_{1,YZ}[\lambda] + z Z f_{1,zZ}[\lambda] \right)$$

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In[]:= **S**[**B_**] := **CE**[**SS@O_{QU}**[**QU@Generators**, **Seepage_{QU}**[**ff**, **SS@CE@B**]]];

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In[]:= **Short**[**SG** = **S**[**G**], **10**]

Time: 23.88 seconds Out[]:=

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$$e^{f_{0,1}[\lambda]} \in \text{QU}[y, x] f_{1,xy}[\lambda] + \ll 25 \gg + \text{QU}[z]$$

$$\left(e^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]+f_{0,1}[\lambda]} \in f_{0,x}[\lambda]^2 f_{1,xy}[\lambda] - e^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]+f_{0,1}[\lambda]} s \in f_{0,x}[\lambda]^2 f_{1,xy}[\lambda] + \right.$$

$$e^{\alpha[i]+\alpha[j]-2\beta[i]-2\beta[j]+f_{0,1}[\lambda]} \in f_{0,x}[\lambda] f_{0,y}[\lambda]^2 f_{1,aY}[\lambda] -$$

$$e^{\alpha[i]+\alpha[j]-2\beta[i]-2\beta[j]+f_{0,1}[\lambda]} t \in f_{0,x}[\lambda] f_{0,y}[\lambda]^2 f_{1,aY}[\lambda] + \ll 18 \gg + e^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]+f_{0,1}[\lambda]}$$

$$\in f_{0,x}[\lambda] f_{0,y}[\lambda] f_{1,zZ}[\lambda] - e^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]+f_{0,1}[\lambda]} s \in f_{0,x}[\lambda] f_{0,y}[\lambda] f_{1,zZ}[\lambda] -$$

$$\left. e^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]+f_{0,1}[\lambda]} \in f_{0,z}[\lambda] f_{1,zZ}[\lambda] + e^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]+f_{0,1}[\lambda]} s t \in f_{0,z}[\lambda] f_{1,zZ}[\lambda] \right)$$

In[]:= **Short**[**Invert_{QU}**[**SG**], **10**]

Time: 0.03 seconds Out[]:=

$$-e^{-f_{0,1}[\lambda]} \in \text{QU}[y, x] f_{1,xy}[\lambda] -$$

$$e^{\alpha[i]+\alpha[j]-2\beta[i]-2\beta[j]-f_{0,1}[\lambda]} \in \text{QU}[Y, a] f_{1,aY}[\lambda] + \ll 28 \gg + \text{QU}[X, z] (\ll 1 \gg) +$$

$$\text{QU}[z] \left(-e^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]-f_{0,1}[\lambda]} \in f_{0,x}[\lambda]^2 f_{1,xy}[\lambda] + e^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]-f_{0,1}[\lambda]} \right.$$

$$s \in f_{0,x}[\lambda]^2 f_{1,xy}[\lambda] - e^{\alpha[i]+\alpha[j]-2\beta[i]-2\beta[j]-f_{0,1}[\lambda]} \in f_{0,x}[\lambda] f_{0,y}[\lambda]^2 f_{1,aY}[\lambda] +$$

$$\ll 21 \gg + e^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]-f_{0,1}[\lambda]} s \in f_{0,x}[\lambda] f_{0,y}[\lambda] f_{1,zZ}[\lambda] +$$

$$\left. e^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]-f_{0,1}[\lambda]} \in f_{0,z}[\lambda] f_{1,zZ}[\lambda] - e^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]-f_{0,1}[\lambda]} s t \in f_{0,z}[\lambda] f_{1,zZ}[\lambda] \right)$$

In[]:= **CE**[**Invert_{QU}**[**SG**] $\boxtimes$ **SG**]

Time: 1.95 seconds Out[]:=

QU[]

$$L_\lambda = \mathbb{C}\langle \lambda_{i_1, i_2}^x \rangle \mathbb{C}\langle \lambda_{i_2, i_1}^x \rangle \quad \text{Seepage is defined by: } \mathbb{C}\langle \lambda_{i_1, i_2}^x \rangle \mathbb{C}\langle \lambda_{i_2, i_1}^x \rangle = \mathbb{C}\langle e^{f_{i_1, i_2}} \rangle$$

$$\text{Tr}(L_\lambda) = L'_\lambda L_\lambda \quad S(B) = \text{Seepage}(\{f_i\}, B).$$

$$R_\lambda = \mathbb{C}\langle f+g \rangle, \quad f = f_{i_1, i_2}, \quad g = \sum_{i_1, i_2} g_{i_1, i_2} w \quad e^f = F \quad e^g = G \quad R_\lambda = \mathbb{C}\langle FG \rangle \quad \partial_\lambda F = F \partial_\lambda f$$

$$\text{Tr}(R_\lambda) = R'_\lambda \partial_\lambda R_\lambda = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle \mathbb{C}\langle F(\partial_\lambda FG + \partial_\lambda G) \rangle = S(G)' S(\partial_\lambda FG + \partial_\lambda G).$$

In[*]:= **Short**[**CE**[(∂_λ (**ff.QU@Generators**) **G** + ∂_λ **G**)], **10**]

Time: 0.03 seconds Out[=

$$\begin{aligned} & e^{f_{0,1}[\lambda]} f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]} \in f_{1,1}[\lambda] f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]} x X \in f_{1,xX}[\lambda] f_{0,1}'[\lambda] + \\ & e^{f_{0,1}[\lambda]} X^2 \in f_{1,X^2}[\lambda] f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]} x y \in f_{1,xy}[\lambda] f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]} X y \in f_{1,xy}[\lambda] f_{0,1}'[\lambda] + \\ & a e^{f_{0,1}[\lambda]} Y \in f_{1,aY}[\lambda] f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]} x Y \in f_{1,xY}[\lambda] f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]} X Y \in f_{1,xY}[\lambda] f_{0,1}'[\lambda] + \\ & \ll 158 \gg + e^{f_{0,1}[\lambda]} Y Z \in f_{1,YZ}[\lambda] + e^{f_{0,1}[\lambda]} Z^2 \in f_{1,Z^2}[\lambda] + e^{f_{0,1}[\lambda]} Z \in f_{1,Z}[\lambda] + e^{f_{0,1}[\lambda]} x Z \in f_{1,xZ}[\lambda] + \\ & e^{f_{0,1}[\lambda]} X Z \in f_{1,XZ}[\lambda] + e^{f_{0,1}[\lambda]} y Z \in f_{1,yZ}[\lambda] + e^{f_{0,1}[\lambda]} Y Z \in f_{1,YZ}[\lambda] + e^{f_{0,1}[\lambda]} z Z \in f_{1,zZ}[\lambda] \end{aligned}$$

In[*]:= **Length@CE**[(∂_λ (**ff.QU@Generators**) **G** + ∂_λ **G**)]

Time: 0.02 seconds Out[=

175

In[*]:= **S**[**CE**[(∂_λ (**ff.QU@Generators**) **G** + ∂_λ **G**)]]

Time: 140.48 seconds Out[=

\$Aborted

pdf

In[*]:= **TTR** = **CE**[**Invert**_{QU}[**SG**] $\otimes$ **S**[(∂_λ (**ff.QU@Generators**) **G** + ∂_λ **G**)]]

Time: 90.61 seconds Out[=

pdf

In[*]:= **Cases**[**TTR**, **x_QU**, **Infinity**] // **Sort**

Cases[**TTL**, **x_QU**, **Infinity**] // **Sort**

Time: 0.01 seconds Out[=

$$\{QU[], QU[a], QU[b], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z], QU[X, x], QU[X, X], QU[X, y], QU[X, Y], QU[X, z], QU[X, Z], QU[y, x], QU[Y, a], QU[Y, x], QU[Y, y], QU[Y, Y], QU[Y, z], QU[Y, Z], QU[z, x], QU[z, y], QU[z, z], QU[Z, x], QU[Z, y], QU[Z, z]\}$$

Out[*]=

$$\{QU[], QU[a], QU[b], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z], QU[X, x], QU[X, X], QU[X, y], QU[X, Y], QU[X, z], QU[X, Z], QU[y, x], QU[Y, a], QU[Y, x], QU[Y, y], QU[Y, Y], QU[Y, z], QU[Y, Z], QU[z, x], QU[z, y], QU[z, z], QU[Z, x], QU[Z, y], QU[Z, z]\}$$

pdf

In[*]:= **Short**[**diff** = **CE**[**TTL** - **TTR**], **10**];

pdf

In[*]:= **Short**[**eqns** = **Flatten**[**Table**[
Coefficient[**Coefficient**[**diff**, **m**], **ε**, **k**] == **0**,
{k, 0, \$k}, **{m, mons[k]}**
]], 10];

pdf

In[*]:= **Short**[**inits** = (**#**[**0**] == **0**) & /@ **fs**, **10**];

pdf

In[*]:= **DSolve**[**eqns** **U** **inits**, **fs**, **λ**]

Time: 101.66 seconds Out[=

pdf

$$\left\{ \begin{aligned} & f_{0,x} \rightarrow \text{Function}[\{\lambda\}, e^{-\beta[j]} \lambda (e^{2\alpha[j]} \xi[i] + e^{\beta[j]} \xi[j])], \\ & f_{0,y} \rightarrow \text{Function}[\{\lambda\}, e^{-\alpha[j]} \lambda (e^{2\beta[j]} \eta[i] + e^{\alpha[j]} \eta[j])], \\ & f_{0,x} \rightarrow \text{Function}[\{\lambda\}, e^{-\beta[i]} \lambda (e^{\beta[i]} \Xi[i] + e^{2\alpha[i]} \Xi[j] + e^{2\alpha[i]} \lambda \eta[i] \Omega[j])], \end{aligned} \right.$$

$f_{0,y} \rightarrow \text{Function}[\{\lambda\}, e^{-\alpha[i]} \lambda (e^{\alpha[i]} \Upsilon[i] + e^{2\beta[i]} \Upsilon[j] - e^{2\beta[i]} s \lambda \xi[i] \Omega[j])],$
 $f_{0,z} \rightarrow \text{Function}[\{\lambda\}, e^{-\beta[j]} \lambda (e^{\alpha[j]+2\beta[j]} \zeta[i] + e^{\beta[j]} \zeta[j] + e^{2\alpha[j]} \lambda \eta[j] \xi[i]),$
 $f_{0,1} \rightarrow \text{Function}[\{\lambda\}, \lambda^2 \xi[i] \Xi[j] - s \lambda^2 \xi[i] \Xi[j] + \lambda^2 \eta[i] \Upsilon[j] - t \lambda^2 \eta[i] \Upsilon[j] +$
 $\lambda^2 \zeta[i] \Omega[j] - s t \lambda^2 \zeta[i] \Omega[j] - s \lambda^3 \eta[i] \xi[i] \Omega[j] + s t \lambda^3 \eta[i] \xi[i] \Omega[j],$
 $f_{0,z} \rightarrow \text{Function}[\{\lambda\}, \lambda (\Omega[i] + e^{\alpha[i]+\beta[i]} \Omega[j]), f_{1,a} \rightarrow$
 $\text{Function}[\{\lambda\}, -2 s \lambda^2 (-\xi[i] \Xi[j] - t \zeta[i] \Omega[j] - \lambda \eta[i] \xi[i] \Omega[j] + t \lambda \eta[i] \xi[i] \Omega[j]),$
 $f_{1,b} \rightarrow \text{Function}[\{\lambda\}, -2 t \lambda^2 (-\eta[i] \Upsilon[j] - s \zeta[i] \Omega[j] + s \lambda \eta[i] \xi[i] \Omega[j]),$
 $f_{1,xy} \rightarrow \text{Function}[\{\lambda\}, -e^{2\alpha[j]-\beta[j]} \lambda^2 \eta[j] \xi[i],$
 $f_{1,xx} \rightarrow \text{Function}[\{\lambda\}, -e^{2\alpha[i]+2\alpha[j]-\beta[i]-\beta[j]} \lambda^2 \xi[i] (2 \Xi[j] + \lambda \eta[i] \Omega[j]),$
 $f_{1,xy} \rightarrow \text{Function}[\{\lambda\}, e^{2\alpha[i]-\alpha[j]-\beta[i]+2\beta[j]} \lambda^2 \eta[i] \Xi[j],$
 $f_{1,x^2} \rightarrow \text{Function}[\{\lambda\}, e^{4\alpha[i]-2\beta[i]} \lambda^3 \eta[i] \Xi[j] \Omega[j],$
 $f_{1,ay} \rightarrow \text{Function}[\{\lambda\}, 2 e^{-\alpha[i]+2\beta[i]} s \lambda^2 \xi[i] \Omega[j],$
 $f_{1,xy} \rightarrow \text{Function}[\{\lambda\}, e^{-\alpha[i]+2\alpha[j]+2\beta[i]-\beta[j]} \lambda^2 \xi[i] (\Upsilon[j] + 2 s \lambda \xi[i] \Omega[j]),$
 $f_{1,yv} \rightarrow \text{Function}[\{\lambda\}, -2 e^{-\alpha[i]-\alpha[j]+2\beta[i]+2\beta[j]} \lambda^2 \eta[i] \Upsilon[j],$
 $f_{1,xv} \rightarrow \text{Function}[\{\lambda\}, e^{\alpha[i]-\beta[i]} \lambda^2 (e^{\alpha[i]} \Xi[j] \Upsilon[i] + 2 e^{2\beta[i]} s \lambda \xi[i] \Xi[j] \Omega[j] +$
 $e^{\alpha[i]} \lambda \eta[i] \Upsilon[i] \Omega[j] - e^{2\beta[i]} \lambda \eta[i] \Upsilon[j] \Omega[j] + 2 e^{2\beta[i]} s \lambda^2 \eta[i] \xi[i] \Omega[j]^2),$
 $f_{1,v^2} \rightarrow \text{Function}[\{\lambda\}, -e^{-2\alpha[i]+4\beta[i]} s \lambda^3 \xi[i] \Upsilon[j] \Omega[j],$
 $f_{1,xz} \rightarrow \text{Function}[\{\lambda\}, e^{2\alpha[j]-\beta[j]} \lambda^2 \zeta[j] \xi[i],$
 $f_{1,x} \rightarrow \text{Function}[\{\lambda\}, -e^{2\alpha[j]-\beta[j]} \lambda^2 (\lambda \xi[i]^2 \Xi[j] - 3 s \lambda \xi[i]^2 \Xi[j] + 2 t \zeta[i] \Upsilon[j] -$
 $\lambda \eta[i] \xi[i] \Upsilon[j] + t \lambda \eta[i] \xi[i] \Upsilon[j] + \lambda \zeta[i] \xi[i] \Omega[j] - 3 s t \lambda \zeta[i] \xi[i] \Omega[j] -$
 $2 s \lambda^2 \eta[i] \xi[i]^2 \Omega[j] + 2 s t \lambda^2 \eta[i] \xi[i]^2 \Omega[j]), f_{1,yz} \rightarrow \text{Function}[\{\lambda\}, -e^{-\alpha[j]-\beta[j]} \lambda^2$
 $(e^{3\beta[j]} \zeta[j] \eta[i] + e^{2\alpha[j]+2\beta[j]} \lambda \eta[i] \eta[j] \xi[i] + e^{3\alpha[j]} \lambda \eta[j]^2 \xi[i]), f_{1,y} \rightarrow \text{Function}[\{\lambda\},$
 $-e^{-\alpha[j]+2\beta[j]} \lambda^2 (-2 \zeta[i] \Xi[j] + 2 s \lambda \eta[i] \xi[i] \Xi[j] + \lambda \eta[i]^2 \Upsilon[j] - 3 t \lambda \eta[i]^2 \Upsilon[j] -$
 $\lambda \zeta[i] \eta[i] \Omega[j] - s t \lambda \zeta[i] \eta[i] \Omega[j] + s \lambda^2 \eta[i]^2 \xi[i] \Omega[j] + s t \lambda^2 \eta[i]^2 \xi[i] \Omega[j]),$
 $f_{1,xz} \rightarrow \text{Function}[\{\lambda\}, -e^{2\alpha[i]+\alpha[j]-\beta[i]-\beta[j]} \lambda^2 (e^{2\beta[j]} \zeta[i] \Xi[j] + 2 e^{\alpha[j]} \lambda \eta[j] \xi[i] \Xi[j] +$
 $e^{2\beta[j]} \lambda \zeta[i] \eta[i] \Omega[j] + e^{\alpha[j]} \lambda^2 \eta[i] \eta[j] \xi[i] \Omega[j]),$
 $f_{1,x} \rightarrow \text{Function}[\{\lambda\}, -e^{2\alpha[i]-\beta[i]} \lambda^3 (\xi[i] \Xi[j]^2 - 3 s \xi[i] \Xi[j]^2 - \eta[i] \Xi[j] \Upsilon[j] +$
 $t \eta[i] \Xi[j] \Upsilon[j] - \zeta[i] \Xi[j] \Omega[j] - s t \zeta[i] \Xi[j] \Omega[j] - 3 s \lambda \eta[i] \xi[i] \Xi[j] \Omega[j] +$
 $s t \lambda \eta[i] \xi[i] \Xi[j] \Omega[j] + \lambda \eta[i]^2 \Upsilon[j] \Omega[j] - t \lambda \eta[i]^2 \Upsilon[j] \Omega[j] -$
 $2 s t \lambda \zeta[i] \eta[i] \Omega[j]^2 - 2 s \lambda^2 \eta[i]^2 \xi[i] \Omega[j]^2 + 2 s t \lambda^2 \eta[i]^2 \xi[i] \Omega[j]^2),$
 $f_{1,yz} \rightarrow \text{Function}[\{\lambda\}, e^{-\alpha[i]+\alpha[j]+2\beta[i]-\beta[j]} \lambda^2 (-e^{2\beta[j]} \zeta[i] \Upsilon[j] + e^{\alpha[j]} \lambda \eta[j] \xi[i] \Upsilon[j] +$
 $3 e^{2\beta[j]} s \lambda \zeta[i] \xi[i] \Omega[j] + 2 e^{\alpha[j]} s \lambda^2 \eta[j] \xi[i]^2 \Omega[j]),$
 $f_{1,y} \rightarrow \text{Function}[\{\lambda\}, e^{-\alpha[i]+2\beta[i]} \lambda^2 (-2 s \lambda \xi[i] \Xi[j] \Upsilon[j] - \lambda \eta[i] \Upsilon[j]^2 + 3 t \lambda \eta[i] \Upsilon[j]^2 -$
 $s \xi[i] \Omega[j] + s \lambda^2 \xi[i]^2 \Xi[j] \Omega[j] - s^2 \lambda^2 \xi[i]^2 \Xi[j] \Omega[j] - \lambda \zeta[i] \Upsilon[j] \Omega[j] +$
 $3 s t \lambda \zeta[i] \Upsilon[j] \Omega[j] - 4 s t \lambda^2 \eta[i] \xi[i] \Upsilon[j] \Omega[j] + 2 s \lambda^2 \zeta[i] \xi[i] \Omega[j]^2 -$
 $4 s^2 t \lambda^2 \zeta[i] \xi[i] \Omega[j]^2 - s^2 \lambda^3 \eta[i] \xi[i]^2 \Omega[j]^2 + 3 s^2 t \lambda^3 \eta[i] \xi[i]^2 \Omega[j]^2),$
 $f_{1,z^2} \rightarrow \text{Function}[\{\lambda\}, e^{2\alpha[j]-\beta[j]} \lambda^3 \zeta[j] \eta[j] \xi[i], f_{1,z} \rightarrow \text{Function}[\{\lambda\}, -e^{\alpha[j]-\beta[j]}$
 $(-2 e^{2\beta[j]} s \lambda^3 \zeta[i] \xi[i] \Xi[j] + e^{\alpha[j]} \lambda^4 \eta[j] \xi[i]^2 \Xi[j] - 3 e^{\alpha[j]} s \lambda^4 \eta[j] \xi[i]^2 \Xi[j] +$
 $2 e^{\alpha[j]} t \lambda^3 \zeta[i] \eta[j] \Upsilon[j] - e^{\alpha[j]} \lambda^4 \eta[i] \eta[j] \xi[i] \Upsilon[j] + e^{\alpha[j]} t \lambda^4 \eta[i] \eta[j] \xi[i] \Upsilon[j] +$
 $e^{2\beta[j]} \lambda^3 \zeta[i]^2 \Omega[j] - 3 e^{2\beta[j]} s t \lambda^3 \zeta[i]^2 \Omega[j] - 2 e^{2\beta[j]} s \lambda^4 \zeta[i] \eta[i] \xi[i] \Omega[j] +$
 $2 e^{2\beta[j]} s t \lambda^4 \zeta[i] \eta[i] \xi[i] \Omega[j] + e^{\alpha[j]} \lambda^4 \zeta[i] \eta[j] \xi[i] \Omega[j] - 3 e^{\alpha[j]} s t \lambda^4 \zeta[i] \eta[j]$
 $\xi[i] \Omega[j] - 2 e^{\alpha[j]} s \lambda^5 \eta[i] \eta[j] \xi[i]^2 \Omega[j] + 2 e^{\alpha[j]} s t \lambda^5 \eta[i] \eta[j] \xi[i]^2 \Omega[j]),$

$f_{1,1} \rightarrow \text{Function}\left[\{\lambda\}, \frac{1}{2} \left(-\lambda^4 \xi[i]^2 \Xi[j]^2 + 4 s \lambda^4 \xi[i]^2 \Xi[j]^2 - 3 s^2 \lambda^4 \xi[i]^2 \Xi[j]^2 + \right. \right.$
 $4 \lambda^3 \zeta[i] \Xi[j] \Upsilon[j] - 4 t \lambda^3 \zeta[i] \Xi[j] \Upsilon[j] - 4 s \lambda^4 \eta[i] \xi[i] \Xi[j] \Upsilon[j] +$
 $4 s t \lambda^4 \eta[i] \xi[i] \Xi[j] \Upsilon[j] - \lambda^4 \eta[i]^2 \Upsilon[j]^2 + 4 t \lambda^4 \eta[i]^2 \Upsilon[j]^2 - 3 t^2 \lambda^4 \eta[i]^2 \Upsilon[j]^2 -$
 $2 s \lambda^3 \eta[i] \xi[i] \Omega[j] + 2 s t \lambda^3 \eta[i] \xi[i] \Omega[j] + 4 s t \lambda^4 \zeta[i] \xi[i] \Xi[j] \Omega[j] -$
 $4 s^2 t \lambda^4 \zeta[i] \xi[i] \Xi[j] \Omega[j] + 2 s \lambda^5 \eta[i] \xi[i]^2 \Xi[j] \Omega[j] - 2 s^2 \lambda^5 \eta[i] \xi[i]^2 \Xi[j] \Omega[j] -$
 $2 s t \lambda^5 \eta[i] \xi[i]^2 \Xi[j] \Omega[j] + 2 s^2 t \lambda^5 \eta[i] \xi[i]^2 \Xi[j] \Omega[j] +$
 $4 s t \lambda^4 \zeta[i] \eta[i] \Upsilon[j] \Omega[j] - 4 s t^2 \lambda^4 \zeta[i] \eta[i] \Upsilon[j] \Omega[j] - 4 s t \lambda^5 \eta[i]^2 \xi[i] \Upsilon[j] \Omega[j] +$
 $4 s t^2 \lambda^5 \eta[i]^2 \xi[i] \Upsilon[j] \Omega[j] - \lambda^4 \zeta[i]^2 \Omega[j]^2 + 4 s t \lambda^4 \zeta[i]^2 \Omega[j]^2 -$
 $3 s^2 t^2 \lambda^4 \zeta[i]^2 \Omega[j]^2 + 2 s \lambda^5 \zeta[i] \eta[i] \xi[i] \Omega[j]^2 - 2 s t \lambda^5 \zeta[i] \eta[i] \xi[i] \Omega[j]^2 -$
 $6 s^2 t \lambda^5 \zeta[i] \eta[i] \xi[i] \Omega[j]^2 + 6 s^2 t^2 \lambda^5 \zeta[i] \eta[i] \xi[i] \Omega[j]^2 -$
 $s^2 \lambda^6 \eta[i]^2 \xi[i]^2 \Omega[j]^2 + 4 s^2 t \lambda^6 \eta[i]^2 \xi[i]^2 \Omega[j]^2 - 3 s^2 t^2 \lambda^6 \eta[i]^2 \xi[i]^2 \Omega[j]^2 \left. \right) \right],$
 $f_{1,xz} \rightarrow \text{Function}\left[\{\lambda\}, -e^{\alpha[i]+2\alpha[j]+\beta[i]-\beta[j]} \lambda^2 \xi[i] \Omega[j] \right],$
 $f_{1,yz} \rightarrow \text{Function}\left[\{\lambda\}, -e^{\alpha[i]-\alpha[j]+\beta[i]+2\beta[j]} \lambda^2 \eta[i] \Omega[j] \right],$
 $f_{1,xz} \rightarrow \text{Function}\left[\{\lambda\}, -e^{2\alpha[i]-\beta[i]} \lambda^2 \left(\Xi[j] \Omega[i] + \lambda \eta[i] \Omega[i] \Omega[j] + e^{\alpha[i]+\beta[i]} \lambda \eta[i] \Omega[j]^2 \right) \right],$
 $f_{1,yz} \rightarrow \text{Function}\left[\{\lambda\}, e^{-\alpha[i]+2\beta[i]} \lambda^2 \left(\Upsilon[j] \Omega[i] - s \lambda \xi[i] \Omega[i] \Omega[j] + e^{\alpha[i]+\beta[i]} s \lambda \xi[i] \Omega[j]^2 \right) \right],$
 $f_{1,zz} \rightarrow \text{Function}\left[\{\lambda\}, -e^{\alpha[i]+\alpha[j]+\beta[i]-\beta[j]} \lambda^2 \left(2 e^{2\beta[j]} \zeta[i] + e^{\alpha[j]} \lambda \eta[j] \xi[i] \right) \Omega[j] \right],$
 $f_{1,z} \rightarrow \text{Function}\left[\{\lambda\}, \right.$
 $-e^{\alpha[i]-\beta[i]} \lambda^2 \left(2 e^{\alpha[i]} \Xi[j] \Upsilon[i] - 2 e^{2\beta[i]} s \lambda \xi[i] \Xi[j] \Omega[j] + 2 e^{\alpha[i]} \lambda \eta[i] \Upsilon[i] \Omega[j] + \right.$
 $2 e^{2\beta[i]} \lambda \eta[i] \Upsilon[j] \Omega[j] - 2 e^{2\beta[i]} t \lambda \eta[i] \Upsilon[j] \Omega[j] + e^{2\beta[i]} \lambda \zeta[i] \Omega[j]^2 -$
 $3 e^{2\beta[i]} s t \lambda \zeta[i] \Omega[j]^2 - 3 e^{2\beta[i]} s \lambda^2 \eta[i] \xi[i] \Omega[j]^2 + 3 e^{2\beta[i]} s t \lambda^2 \eta[i] \xi[i] \Omega[j]^2 \left. \right) \left. \right\}$