Pensieve header: An isolated attempt to compute the multiplication tensor of QU.

The implementation of QU is modified from pensieve://Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Initialization / Utilities

In[∘]:=
```
$k = 1; $U = QU; $E := {$k};
$trim := {ϵ^(k_·) /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
q_ℏ = e^(γ ϵ ℏ);
(* Upper to lower and lower to Upper: *)
U2l = {B_(i_)^(p_·) → e^(-p ℏ γ b_i), B^(p_·) → e^(-p ℏ γ b), T_(i_)^(p_·) → e^(p ℏ t_i), T^(p_·) → e^(p ℏ t), 𝒜_(i_)^(p_·) → e^(p γ α_i), 𝒜^(p_·) → e^(p γ α)};
l2U = {e^(c_. b_(i_) + d_.) :→ B_i^(-c/(ℏ γ)) e^d, e^(c_. b + d_.) :→ B^(-c/(ℏ γ)) e^d,
   e^(c_. t_(i_) + d_.) :→ T_i^(c/ℏ) e^d, e^(c_. t + d_.) :→ T^(c/ℏ) e^d,
   e^(c_. α_(i_) + d_.) :→ 𝒜_i^(c/γ) e^d, e^(c_. α + d_.) :→ 𝒜^(c/γ) e^d,
   e^(ℰ_) :→ e^(Expand@ℰ)};
SS[ℰ_, op_] := Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. U2l, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
Kδ /: Kδ_(i_,j_) := If[i === j, 1, 0];
c_Integer_(k_Integer) := c + O[ϵ]^(k+1);
```

In[∘]:=
```
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@Together[Expand[ℰ] //. e^(x_) e^(y_) :→ e^(x+y) /. e^(x_) :→ e^(CF[x])];
```

In[∘]:=
```
Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs__] := MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

Self-Pair (SP):

In[∘]:=
```
SP_{}[P_] := P; SP_{ξ_→x_, ps___}[P_] := Expand[P // SP_{ps}] /. f_. ξ^(d_·) :→ ∂_{x,d} f
```

DeclareAlgebra

In[∘]:=
```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ⊠ y) ⊠ z;
0 ⊠ _ = _ ⊠ 0 = 0;
(x_Plus) ⊠ y_ := (# ⊠ y) & /@ x; x_ ⊠ (y_Plus) := (x ⊠ #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ⊠ y - y ⊠ x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

In[∘]:=
```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[
  {gp, sr, g, cp, M, pow, k = 0,
   gs = Generators /. {opts},
   cs = Centrals /. {opts} /. Centrals → {}
  },
  U[Generators] = gs;
  (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
  CE[ℰ_] := Collect[ℰ, _U, Expand] /. ⅇ^x_ :> ⅇ^Expand[x] /. $trim;
  U_i_ [ℰ_] := ℰ /. {t : cp :> t_i, u_U :> (#_i &) /@ u};
  U_i_ [NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
  B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
  B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ⊠ (c_. 1_U) := CE[c x]; (c_. 1_U) ⊠ x_ := CE[c x];
  (a_. U[xx___, x_]) ⊠ (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx ⊠ CE@M[a b /. $trim, U@y ⊠ U@x + B[U@x, U@y, $E]] ⊠ U@yy ];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ⊠ U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ⊠ U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List :> l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_s_ :> (l /. x_i_ :> x_s));
    CE[Total[
       CoefficientRules[poly, vs] /. (p_ → c_) :> c U@(us^p)
     ]] /. x_null :> x];
```

```
𝕆U[specs___, 𝔼[L_, Q_, P_]] := 𝕆U[specs, SS@Normal[P e^(L+Q)]];
pow[ℰ_, n_] := pow[ℰ, n - 1] ⊠ ℰ;
𝕊U[ℰ_, ss___Rule] := CE@Total[
   CoefficientRules[ℰ, First /@ {ss}] /.
    (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
σrs___[c_. * u_U] := (c /. (t : cp)j_ ⧴ t j/.{rs}) U[List @@ (u /. v_j_ ⧴ v j/.{rs})];
mj_→k_[c_. * u_U] := CE[((c /. (t : cp)j → tk) DeleteCases[u, _j|k]) ⊠
   U @@ Cases[u, w_j ⧴ wk] ⊠ U @@ Cases[u, _k]];
U /: c_. * u_U * v_U := CE[c u ⊠ v];
Si_[c_. * u_U] := CE[((c /. Si[U, Centrals]) DeleteCases[u, _i]) ⊠
   Ui[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]]];
Δi_→j_,k_[c_. * u_U] := CE[((c /. Δi→j,k[U, Centrals]) DeleteCases[u, _i]) ⊠
   (NCM @@ Cases[u, x_i ⧴ σ1→j,2→k@Δ@U@x] /. NCM[] → U[])]; ]
```

Meta-Operations

In[∘]:=
```
σrs___[ℰ_Plus] := σrs /@ ℰ;
mj_→j_ = Identity; mj_→k_[0] = 0;
mj_→k_[ℰ_Plus] := Simp[mj→k /@ ℰ];
mis___,i_,j_→k_[ℰ_] := mj→k@mis,i→j@ℰ;
Si_[ℰ_Plus] := Simp[Si /@ ℰ];
Δis___[ℰ_Plus] := Simp[Δis /@ ℰ];
```

Implementing CU = $\mathcal{U}(\mathrm{sl}_2^{\gamma\epsilon})$

Verify σ and Δ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\ldots}$.

In[∘]:=
```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[aCU, yCU] = -γ yCU; B[xCU, aCU] = -γ xCU;
B[xCU, yCU] = 2 ϵ aCU - t 1CU;
(S@yCU = -yCU; S@aCU = -aCU; S@xCU = -xCU;)
Si_[CU, Centrals] = {ti → -ti};
Δ@yCU = CU@y1 + CU@y2; Δ@aCU = CU@a1 + CU@a2; Δ@xCU = CU@x1 + CU@x2;
Δi_→j_,k_[CU, Centrals] = {ti → tj + tk};
```

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

```
In[•]:=
ℏ = γ = 1;
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] := SS[q_ℏ - 1] QU@{y, x} + 𝕆_QU[{a}, SS[(1 - T e^(-2 ϵ a ℏ)) / ℏ]];
(S@y_QU := 𝕆_QU[{a, y}, SS[-T^-1 e^(ℏ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU := 𝕆_QU[{a, x}, SS[-e^(ℏ ϵ a) x]];)
S_i_ [QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
Δ@y_QU := 𝕆_QU[{y_1, a_1}_1, {y_2}_2, SS[y_1 + T_1 e^(-ℏ ϵ a_1) y_2]];
Δ@a_QU = QU@a_1 + QU@a_2; Δ@x_QU := 𝕆_QU[{a_1, x_1}_1, {x_2}_2, SS[x_1 + e^(-ℏ ϵ a_1) x_2]];
Δ_(i_→j_,k_) [QU, Centrals] = {t_i → t_j + t_k, T_i → T_j T_k};
```

tSW

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x}\, y\, e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta(e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
In[•]:=
SW_xy[U_, kk_] :=
  SW_xy[U, kk] = Block[{$U = U, $k = kk, $p = kk}, Module[{G, F, fs, f, bs, e, b, es},
     G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
     fs = Flatten@Table[f_(l,i,j,k)[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
     F = fs.(bs = fs /. f_(l_,i_,j_,k_)[η] :→ ϵ^l U@{y^i, a^j, x^k});
     es = Flatten[
       Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ⊠ G - y_U ⊠ F - ∂_η F}}, {b, bs}]];
     F = F /. DSolve[es, fs, η][[1]];
     𝔼[0,
       ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
       F + 0_$k /. {e- → 1, U → Times}
      ] /. (v : η | ξ | t | T | y | a | x) → v_1
    ]];
tSW_(xy_,i_,j_→k_) := SW_xy[$U, $k] /. {ξ_1 → ξ_i, η_1 → η_j, (v : t | T | y | a | x)_1 → v_k};
tSW_(xa,i_,j_→k_) := 𝔼[α_j a_k, e^(-γ α_j) ξ_i x_k, 1];
tSW_(ay,i_,j_→k_) := 𝔼[α_i a_k, e^(-γ α_i) η_j y_k, 1];
```

Exponentials as needed.

Task. Define $\mathrm{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi\,\mathbb{O}(P)}$ to ϵ^k in the algebra U_i, where ξ is a scalar, X is x_i or y_i, and P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form. Should satisfy $U@\mathrm{Exp}_{U_i,k}[\xi, P] == \mathbb{S}_U[e^{\xi\,x},\ x \to \mathbb{O}(P)]$.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi\,\mathbb{O}(P)} = \mathbb{O}(e^{\xi\,P_0}\,F(\xi))$, then $F(\xi = 0) = 1$ and we have:

$\mathbb{O}\left(e^{\xi P_0}(P_0 F(\xi)+\partial_\xi F)\right)=\mathbb{O}\left(\partial_\xi e^{\xi P_0} F(\xi)\right)=\partial_\xi \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)=\partial_\xi e^{\xi\,\mathbb{O}(P)}=e^{\xi\,\mathbb{O}(P)}\,\mathbb{O}(P)=\mathbb{O}\left(e^{\xi P_0} F(\xi)\right)\mathbb{O}(P)$.

This is an ODE for F. Setting inductively $F_k=F_{k-1}+\epsilon^k\varphi$ we find that $F_0=1$ and solve for φ.

In[∘]:=
```
(* Bug: The first line is valid only if 𝕆(e^P0)==e^𝕆(P0). *)
(* Bug: ξ must be a symbol. *)
Exp_{U_i_,0}[ξ_, P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[ξ LQ /. (x | y)_i → 0, ξ LQ /. (t | a)_i → 0, 1]];
Exp_{U_i_,k_}[ξ_, P_] := Block[{$U = U, $k = k},
  Module[{P0, φ, φs, F, j, rhs, at0, atξ},
   P0 = Normal@P /. ϵ → 0;
   φs =
    Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, k}, {j1, 0, 2 k + 1 - j2}, {j3, 0, 2 k + 1 - j2 - j1}];
   F = Normal@Last@Exp_{U_i,k-1}[ξ, P] + ϵ^k φs.(φs /. φ_{js__}[ξ] :→ Times @@ {y_i, a_i, x_i}^{js});
   rhs = Normal@
     Last@m_{i,j→i}[𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + 0_k] m_{i→j}@𝔼[0, 0, P + 0_k]];
   at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, {y_i, a_i, x_i}];
   atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) + P0 F - rhs, {y_i, a_i, x_i}];
   𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + 0_k] /.
    DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]] ]]
```

In[∘]:=
```
$k = 1
```

Out[∘]=
```
1
```

In[∘]:=
```
$Basis = {a_QU, x_QU, y_QU};
Module[{a, b}, Table[{a, b} → CE[B[a, b]], {a, $Basis}, {b, $Basis}]] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \{QU[a], QU[a]\} \to 0 & \{QU[a], QU[x]\} \to \gamma\, QU[x] & \\ \{QU[x], QU[a]\} \to -\gamma\, QU[x] & \{QU[x], QU[x]\} \to 0 & \{QU[x], \\ \{QU[y], QU[a]\} \to \gamma\, QU[y] & \{QU[y], QU[x]\} \to \left(-\frac{1}{\hbar}+\frac{T}{\hbar}\right) QU[\,] - 2\,T\,\epsilon\, QU[a] - \gamma\,\epsilon\,\hbar\, QU[y, x] & \end{pmatrix}$$

In[∘]:=
```
$Basis = {a_QU, x_QU, y_QU};
Module[{a, b},
 DeleteCases[
  Flatten[Table[{a, b, c} → CE[B[a, b] + B[b, a]], {a, $Basis}, {b, $Basis}]], _ → 0]
]
Module[{a, b, c},
 DeleteCases[Flatten[Table[{a, b, c} → CE[B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]]],
    {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
]
$ABasis = {a_QU, x_QU, y_QU, QU[y, y], QU[y, a], QU[y, x], QU[a, a], QU[a, x], QU[x, x]};
Module[{a, b, c},
 DeleteCases[Flatten[Table[{a, b, c} → CE[a ⊠ CE[b ⊠ c] - CE[a ⊠ b] ⊠ c],
    {a, $ABasis}, {b, $ABasis}, {c, $ABasis}]], _ → 0]
]
```

Out[∘]=

{}

Out[∘]=

{}

Out[∘]=

{}

In[∘]:=
```
Module[{a, b, c},
  Table[{a, b, c} → CE[a ⊠ CE[b ⊠ c]], {a, $ABasis}, {b, $ABasis}, {c, $ABasis}]
 ];
```

In[∘]:= `B[x_QU, y_QU] // CE`

Out[∘]=

$\left(\frac{1}{\hbar} - \frac{T}{\hbar}\right)$ QU[] + 2 T ∈ QU[a] + γ ∈ ℏ QU[y, x]

In[∘]:= `B[x_QU, B[x_QU, y_QU]] // CE`

Out[∘]=

(γ ∈ - 3 T γ ∈) QU[x]

In[∘]:= `B[x_QU, B[x_QU, B[x_QU, y_QU]]] // CE`

Out[∘]=

0

In[∘]:= `B[x_QU, B[x_QU, B[x_QU, B[x_QU, y_QU]]]] // CE`

Out[∘]=

0

In[∘]:= `B[x_QU, B[x_QU, B[x_QU, B[x_QU, B[x_QU, y_QU]]]]] // Expand`

Out[∘]=

0

In[∘]:= `B[x_QU, B[x_QU, B[x_QU, B[x_QU, B[x_QU, B[x_QU, y_QU]]]]]] // Expand`

Out[∘]=

0

In[ ]:=
```
adPower[x_, k_, Env_][c_ * y_QU] := CE[c adPower[x, k, Env][y]];
adPower[x_, k_, Env_][y_Plus] := adPower[x, k, Env] /@ y;
adPower[_, 0, Env_][y_] := y;
adPower[x_, k_, Env_][0] = 0;
adPower[x_, k_, Env_][y_QU] :=
  adPower[x, k, Env][y] = CE@B[adPower[x, k - 1, Env][y], x];
adPower[x_, k_][y_] := adPower[x, k, $E][y];
```

In[ ]:= adPower[x_{QU}, 1][y_{QU}]

Out[ ]=

$$\left(-\frac{1}{\hbar}+\frac{T}{\hbar}\right) QU[\,] - 2\,T\,\epsilon\,QU[a] - \gamma\,\epsilon\,\hbar\,QU[y, x]$$

In[ ]:= adPower[x_{QU}, 2][y_{QU}]

Out[ ]=

$$(\gamma\,\epsilon - 3\,T\,\gamma\,\epsilon)\,QU[x]$$

In[ ]:= adPower[x_{QU}, 3][y_{QU}]

Out[ ]=

0

In[ ]:= adPower[x_{QU}, 4][y_{QU}]

Out[ ]=

0

In[ ]:= adPower[x_{QU}, 5][y_{QU}]

Out[ ]=

0

In[ ]:=
```
NilQ[x_QU] = NilQ[y_QU] = True; NilQ[a_QU] = False;
PotQ[c_. * x_QU] := ¬ NilQ[x]; (* Potent Q *)
NilQ[c_ * x_QU] := NilQ[x];
PotQ[x_Plus] := And @@ (PotQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

In[ ]:=
```
Ad[0] = Identity;
Ad[c_. a_QU][z_] := z /. w_QU :> e^(c γ (Count[w,y]-Count[w,x])) w;
Ad[c_. QU[]][z_] := z;
```

In[ ]:=
```
Ad[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
   While[adPower[x, k][y] =!= 0,
    s += adPower[x, k][y] / k!;
    ++k
   ];
   s
  ]
```

In[ ]:= **Ad[x_QU][y_QU]**

Out[ ]=

$$-\frac{\mathrm{QU}[\,]}{\hbar}+\frac{\mathrm{T}\,\mathrm{QU}[\,]}{\hbar}-2\,\mathrm{T}\in\mathrm{QU}[a]+\frac{1}{2}\,\gamma\in\mathrm{QU}[x]-\frac{3}{2}\,\mathrm{T}\,\gamma\in\mathrm{QU}[x]+\mathrm{QU}[y]-\gamma\in\hbar\,\mathrm{QU}[y,x]$$

In[ ]:= **Ad[c a_QU] /@ {QU[], x_QU, y_QU, QU[x, y]}**

Out[ ]=

$$\{\mathrm{QU}[\,],\ e^{-c\,\gamma}\,\mathrm{QU}[x],\ e^{c\,\gamma}\,\mathrm{QU}[y],\ \mathrm{QU}[x,y]\}$$

In[ ]:= **Ad[(α1 + α2 + ϵ fa[λ]) a_QU] /@ {QU[], x_QU, y_QU, QU[x, y]}**

Out[ ]=

$$\left\{\mathrm{QU}[\,],\ e^{-\gamma\,(\alpha 1+\alpha 2+\epsilon\,\mathrm{fa}[\lambda])}\,\mathrm{QU}[x],\ e^{\gamma\,(\alpha 1+\alpha 2+\epsilon\,\mathrm{fa}[\lambda])}\,\mathrm{QU}[y],\ \mathrm{QU}[x,y]\right\}$$

In[ ]:= **Normal@Series[$e^{-\gamma\,(\alpha 1+\alpha 2+\epsilon\,\mathrm{fa}[\lambda])}$, {ϵ, 0, $k}]**

Out[ ]=

$$e^{-((\alpha 1+\alpha 2)\,\gamma)}-e^{-((\alpha 1+\alpha 2)\,\gamma)}\,\gamma\in\mathrm{fa}[\lambda]$$

In[ ]:= **CE[**
Ad[(α1 + α2 + ϵ fa[λ]) a_QU] /@ {QU[], x_QU, y_QU, QU[x, y]} /. $e^{\mathcal{E}\_}$:→ Normal@Series[$e^{\mathcal{E}}$, {ϵ, 0, $k}]]

Out[ ]=

$$\left\{\mathrm{QU}[\,],\ \left(e^{-\alpha 1\,\gamma-\alpha 2\,\gamma}-e^{-\alpha 1\,\gamma-\alpha 2\,\gamma}\,\gamma\in\mathrm{fa}[\lambda]\right)\mathrm{QU}[x],\ \left(e^{\alpha 1\,\gamma+\alpha 2\,\gamma}+e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\in\mathrm{fa}[\lambda]\right)\mathrm{QU}[y],\ \mathrm{QU}[x,y]\right\}$$

In[ ]:= **Ad[a_QU][Ad[x_QU][y_QU]]**

Out[ ]=

$$-\frac{\mathrm{QU}[\,]}{\hbar}+\frac{\mathrm{T}\,\mathrm{QU}[\,]}{\hbar}-2\,\mathrm{T}\in\mathrm{QU}[a]+\frac{1}{2}\,e^{-\gamma}\,\gamma\in\mathrm{QU}[x]-\frac{3}{2}\,e^{-\gamma}\,\mathrm{T}\,\gamma\in\mathrm{QU}[x]+e^{\gamma}\,\mathrm{QU}[y]-\gamma\in\hbar\,\mathrm{QU}[y,x]$$

In[ ]:=
```
λTangent[] = 0;
λTangent[xs___, x_] := CE[Ad[x][λTangent[xs]] + ∂_λ x];
λTangent[xs_List] := λTangent @@ xs
```

In[ ]:= **λTangent[λ y_QU, λ a_QU, λ x_QU]**

Out[ ]=

$$\left(-\frac{e^{\gamma\lambda}\,\lambda}{\hbar}+\frac{e^{\gamma\lambda}\,\mathrm{T}\,\lambda}{\hbar}\right)\mathrm{QU}[\,]+\left(1-2\,e^{\gamma\lambda}\,\mathrm{T}\in\lambda\right)\mathrm{QU}[a]+$$
$$\left(1+\gamma\,\lambda+\frac{1}{2}\,e^{\gamma\lambda}\,\gamma\in\lambda^2-\frac{3}{2}\,e^{\gamma\lambda}\,\mathrm{T}\,\gamma\in\lambda^2\right)\mathrm{QU}[x]+e^{\gamma\lambda}\,\mathrm{QU}[y]-e^{\gamma\lambda}\,\gamma\in\lambda\,\hbar\,\mathrm{QU}[y,x]$$

In[ ]:= **$k = 1**

Out[ ]=

1

In[∘]:= $\mathbf{lhs = \lambda Tangent[\lambda\,\eta1\, y_{QU},\ \alpha1\, a_{QU},\ \lambda\,\xi1\, x_{QU},\ \lambda\,\eta2\, y_{QU},\ \alpha2\, a_{QU},\ \lambda\,\xi2\, x_{QU}]}$

Out[∘]=

$$\left(-\frac{e^{\alpha1\gamma}\eta1\lambda\xi1}{\hbar}+\frac{e^{\alpha1\gamma}T\eta1\lambda\xi1}{\hbar}+\frac{\eta2\lambda\xi1}{\hbar}-\frac{T\eta2\lambda\xi1}{\hbar}+\frac{e^{\alpha1\gamma}\gamma\epsilon\eta1\eta2\lambda^3\xi1^2}{2\hbar}-\right.$$
$$\frac{2e^{\alpha1\gamma}T\gamma\epsilon\eta1\eta2\lambda^3\xi1^2}{\hbar}+\frac{3e^{\alpha1\gamma}T^2\gamma\epsilon\eta1\eta2\lambda^3\xi1^2}{2\hbar}-\frac{e^{\alpha1\gamma+\alpha2\gamma}\eta1\lambda\xi2}{\hbar}+$$
$$\frac{e^{\alpha1\gamma+\alpha2\gamma}T\eta1\lambda\xi2}{\hbar}-\frac{e^{\alpha2\gamma}\eta2\lambda\xi2}{\hbar}+\frac{e^{\alpha2\gamma}T\eta2\lambda\xi2}{\hbar}+\frac{e^{\alpha1\gamma+\alpha2\gamma}\gamma\epsilon\eta1\eta2\lambda^3\xi1\xi2}{\hbar}-$$
$$\frac{4e^{\alpha1\gamma+\alpha2\gamma}T\gamma\epsilon\eta1\eta2\lambda^3\xi1\xi2}{\hbar}+\frac{3e^{\alpha1\gamma+\alpha2\gamma}T^2\gamma\epsilon\eta1\eta2\lambda^3\xi1\xi2}{\hbar}-$$
$$\left.\frac{e^{\alpha2\gamma}\gamma\epsilon\eta2^2\lambda^3\xi1\xi2}{2\hbar}+\frac{2e^{\alpha2\gamma}T\gamma\epsilon\eta2^2\lambda^3\xi1\xi2}{\hbar}-\frac{3e^{\alpha2\gamma}T^2\gamma\epsilon\eta2^2\lambda^3\xi1\xi2}{2\hbar}\right)\mathrm{QU}[\,]+$$
$$\left(-2e^{\alpha1\gamma}T\epsilon\eta1\lambda\xi1+2T\epsilon\eta2\lambda\xi1-2e^{\alpha1\gamma+\alpha2\gamma}T\epsilon\eta1\lambda\xi2-2e^{\alpha2\gamma}T\epsilon\eta2\lambda\xi2\right)\mathrm{QU}[a]+$$
$$\left(e^{-\alpha2\gamma}\xi1+\frac{1}{2}e^{\alpha1\gamma-\alpha2\gamma}\gamma\epsilon\eta1\lambda^2\xi1^2-\frac{3}{2}e^{\alpha1\gamma-\alpha2\gamma}T\gamma\epsilon\eta1\lambda^2\xi1^2+\xi2+e^{\alpha1\gamma}\gamma\epsilon\eta1\lambda^2\xi1\xi2-\right.$$
$$3e^{\alpha1\gamma}T\gamma\epsilon\eta1\lambda^2\xi1\xi2-\gamma\epsilon\eta2\lambda^2\xi1\xi2+3T\gamma\epsilon\eta2\lambda^2\xi1\xi2+\frac{1}{2}e^{\alpha1\gamma+\alpha2\gamma}\gamma\epsilon\eta1\lambda^2\xi2^2-$$
$$\left.\frac{3}{2}e^{\alpha1\gamma+\alpha2\gamma}T\gamma\epsilon\eta1\lambda^2\xi2^2+\frac{1}{2}e^{\alpha2\gamma}\gamma\epsilon\eta2\lambda^2\xi2^2-\frac{3}{2}e^{\alpha2\gamma}T\gamma\epsilon\eta2\lambda^2\xi2^2\right)\mathrm{QU}[x]+$$
$$\left(e^{\alpha1\gamma+\alpha2\gamma}\eta1+e^{\alpha2\gamma}\eta2-e^{\alpha1\gamma+\alpha2\gamma}\gamma\epsilon\eta1\eta2\lambda^2\xi1+3e^{\alpha1\gamma+\alpha2\gamma}T\gamma\epsilon\eta1\eta2\lambda^2\xi1+\right.$$
$$\left.\frac{1}{2}e^{\alpha2\gamma}\gamma\epsilon\eta2^2\lambda^2\xi1-\frac{3}{2}e^{\alpha2\gamma}T\gamma\epsilon\eta2^2\lambda^2\xi1\right)\mathrm{QU}[y]+$$
$$\left(-e^{\alpha1\gamma}\gamma\epsilon\eta1\lambda\xi1\hbar+\gamma\epsilon\eta2\lambda\xi1\hbar-e^{\alpha1\gamma+\alpha2\gamma}\gamma\epsilon\eta1\lambda\xi2\hbar-e^{\alpha2\gamma}\gamma\epsilon\eta2\lambda\xi2\hbar\right)\mathrm{QU}[y,x]$$

In[∘]:= $\mathbf{rhs = CE\Big[\lambda Tangent[(ft0[\lambda]+\epsilon\, ft1[\lambda])\, QU[\,],\ (fy0[\lambda]+\epsilon\, fy1[\lambda])\, y_{QU},}$
$\mathbf{(\alpha1+\alpha2+\epsilon\, fa1[\lambda])\, a_{QU},\ (fx0[\lambda]+\epsilon\, fx1[\lambda])\, x_{QU}]\ /.\ e^{\mathcal{E}\_} :\to Normal@Series\big[e^{\mathcal{E}},\ \{\epsilon, 0, \$k\}\big]\Big]}$

Out[∘]=

$$-e^{\alpha1\gamma+\alpha2\gamma}\gamma\epsilon\hbar\,\mathrm{fx0}[\lambda]\,\mathrm{QU}[y,x]\,\mathrm{fy0}'[\lambda]+$$
$$\mathrm{QU}[a]\left(\epsilon\,\mathrm{fa1}'[\lambda]-2e^{\alpha1\gamma+\alpha2\gamma}T\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]\right)+\mathrm{QU}[x]\left(\gamma\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fa1}'[\lambda]+\mathrm{fx0}'[\lambda]+\right.$$
$$\left.\epsilon\,\mathrm{fx1}'[\lambda]+\frac{1}{2}e^{\alpha1\gamma+\alpha2\gamma}\gamma\epsilon\,\mathrm{fx0}[\lambda]^2\,\mathrm{fy0}'[\lambda]-\frac{3}{2}e^{\alpha1\gamma+\alpha2\gamma}T\gamma\epsilon\,\mathrm{fx0}[\lambda]^2\,\mathrm{fy0}'[\lambda]\right)+$$
$$\mathrm{QU}[y]\left(e^{\alpha1\gamma+\alpha2\gamma}\mathrm{fy0}'[\lambda]+e^{\alpha1\gamma+\alpha2\gamma}\gamma\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fy0}'[\lambda]+e^{\alpha1\gamma+\alpha2\gamma}\epsilon\,\mathrm{fy1}'[\lambda]\right)+$$
$$\mathrm{QU}[\,]\left(\mathrm{ft0}'[\lambda]+\epsilon\,\mathrm{ft1}'[\lambda]-\frac{e^{\alpha1\gamma+\alpha2\gamma}\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\right.$$
$$\frac{e^{\alpha1\gamma+\alpha2\gamma}T\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha1\gamma+\alpha2\gamma}\gamma\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+$$
$$\frac{e^{\alpha1\gamma+\alpha2\gamma}T\gamma\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha1\gamma+\alpha2\gamma}\epsilon\,\mathrm{fx1}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+$$
$$\left.\frac{e^{\alpha1\gamma+\alpha2\gamma}T\epsilon\,\mathrm{fx1}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha1\gamma+\alpha2\gamma}\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy1}'[\lambda]}{\hbar}+\frac{e^{\alpha1\gamma+\alpha2\gamma}T\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy1}'[\lambda]}{\hbar}\right)$$

In[ ]:= **diff = CE[lhs − rhs]**

Out[ ]=

$$
\begin{aligned}
&\mathrm{QU}[a]\Big(-2\,\mathbb{e}^{\alpha1\,\gamma}\,T\,\epsilon\,\eta1\,\lambda\,\xi1+2\,T\,\epsilon\,\eta2\,\lambda\,\xi1-2\,\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\epsilon\,\eta1\,\lambda\,\xi2-\\
&\quad 2\,\mathbb{e}^{\alpha2\,\gamma}\,T\,\epsilon\,\eta2\,\lambda\,\xi2-\epsilon\,\mathrm{fa1}'[\lambda]+2\,\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]\Big)+\\
&\mathrm{QU}[y,x]\Big(-\mathbb{e}^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta1\,\lambda\,\xi1\,\hbar+\gamma\,\epsilon\,\eta2\,\lambda\,\xi1\,\hbar-\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta1\,\lambda\,\xi2\,\hbar-\\
&\quad \mathbb{e}^{\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\lambda\,\xi2\,\hbar+\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\hbar\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]\Big)+\\
&\mathrm{QU}[x]\Big(\mathbb{e}^{-\alpha2\,\gamma}\,\xi1+\frac{1}{2}\,\mathbb{e}^{\alpha1\,\gamma-\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta1\,\lambda^2\,\xi1^2-\frac{3}{2}\,\mathbb{e}^{\alpha1\,\gamma-\alpha2\,\gamma}\,T\,\gamma\,\epsilon\,\eta1\,\lambda^2\,\xi1^2+\xi2+\mathbb{e}^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta1\,\lambda^2\,\xi1\,\xi2-\\
&\quad 3\,\mathbb{e}^{\alpha1\,\gamma}\,T\,\gamma\,\epsilon\,\eta1\,\lambda^2\,\xi1\,\xi2-\gamma\,\epsilon\,\eta2\,\lambda^2\,\xi1\,\xi2+3\,T\,\gamma\,\epsilon\,\eta2\,\lambda^2\,\xi1\,\xi2+\frac{1}{2}\,\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta1\,\lambda^2\,\xi2^2-\\
&\quad \frac{3}{2}\,\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\gamma\,\epsilon\,\eta1\,\lambda^2\,\xi2^2+\frac{1}{2}\,\mathbb{e}^{\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2\,\lambda^2\,\xi2^2-\frac{3}{2}\,\mathbb{e}^{\alpha2\,\gamma}\,T\,\gamma\,\epsilon\,\eta2\,\lambda^2\,\xi2^2-\gamma\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fa1}'[\lambda]-\\
&\quad \mathrm{fx0}'[\lambda]-\epsilon\,\mathrm{fx1}'[\lambda]-\frac{1}{2}\,\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\mathrm{fx0}[\lambda]^2\,\mathrm{fy0}'[\lambda]+\frac{3}{2}\,\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\gamma\,\epsilon\,\mathrm{fx0}[\lambda]^2\,\mathrm{fy0}'[\lambda]\Big)+\mathrm{QU}[y]\\
&\Big(\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\eta1+\mathbb{e}^{\alpha2\,\gamma}\,\eta2-\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta1\,\eta2\,\lambda^2\,\xi1+3\,\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\gamma\,\epsilon\,\eta1\,\eta2\,\lambda^2\,\xi1+\frac{1}{2}\,\mathbb{e}^{\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2^2\,\lambda^2\,\xi1-\\
&\quad \frac{3}{2}\,\mathbb{e}^{\alpha2\,\gamma}\,T\,\gamma\,\epsilon\,\eta2^2\,\lambda^2\,\xi1-\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\mathrm{fy0}'[\lambda]-\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fy0}'[\lambda]-\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\epsilon\,\mathrm{fy1}'[\lambda]\Big)+\\
&\mathrm{QU}[\,]\Big(-\frac{\mathbb{e}^{\alpha1\,\gamma}\,\eta1\,\lambda\,\xi1}{\hbar}+\frac{\mathbb{e}^{\alpha1\,\gamma}\,T\,\eta1\,\lambda\,\xi1}{\hbar}+\frac{\eta2\,\lambda\,\xi1}{\hbar}-\frac{T\,\eta2\,\lambda\,\xi1}{\hbar}+\frac{\mathbb{e}^{\alpha1\,\gamma}\,\gamma\,\epsilon\,\eta1\,\eta2\,\lambda^3\,\xi1^2}{2\,\hbar}-\\
&\quad \frac{2\,\mathbb{e}^{\alpha1\,\gamma}\,T\,\gamma\,\epsilon\,\eta1\,\eta2\,\lambda^3\,\xi1^2}{\hbar}+\frac{3\,\mathbb{e}^{\alpha1\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta1\,\eta2\,\lambda^3\,\xi1^2}{2\,\hbar}-\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\eta1\,\lambda\,\xi2}{\hbar}+\\
&\quad \frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\eta1\,\lambda\,\xi2}{\hbar}-\frac{\mathbb{e}^{\alpha2\,\gamma}\,\eta2\,\lambda\,\xi2}{\hbar}+\frac{\mathbb{e}^{\alpha2\,\gamma}\,T\,\eta2\,\lambda\,\xi2}{\hbar}+\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta1\,\eta2\,\lambda^3\,\xi1\,\xi2}{\hbar}-\\
&\quad \frac{4\,\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\gamma\,\epsilon\,\eta1\,\eta2\,\lambda^3\,\xi1\,\xi2}{\hbar}+\frac{3\,\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta1\,\eta2\,\lambda^3\,\xi1\,\xi2}{\hbar}-\frac{\mathbb{e}^{\alpha2\,\gamma}\,\gamma\,\epsilon\,\eta2^2\,\lambda^3\,\xi1\,\xi2}{2\,\hbar}+\\
&\quad \frac{2\,\mathbb{e}^{\alpha2\,\gamma}\,T\,\gamma\,\epsilon\,\eta2^2\,\lambda^3\,\xi1\,\xi2}{\hbar}-\frac{3\,\mathbb{e}^{\alpha2\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta2^2\,\lambda^3\,\xi1\,\xi2}{2\,\hbar}-\mathrm{ft0}'[\lambda]-\epsilon\,\mathrm{ft1}'[\lambda]+\\
&\quad \frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\\
&\quad \frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\gamma\,\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\epsilon\,\mathrm{fx1}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\\
&\quad \frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\epsilon\,\mathrm{fx1}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy1}'[\lambda]}{\hbar}-\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy1}'[\lambda]}{\hbar}\Big)
\end{aligned}
$$

In[ ]:= **diff /. $\epsilon \to 0$**

Out[ ]=

$$
\begin{aligned}
&\mathrm{QU}[x]\Big(\mathbb{e}^{-\alpha2\,\gamma}\,\xi1+\xi2-\mathrm{fx0}'[\lambda]\Big)+\mathrm{QU}[y]\Big(\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\eta1+\mathbb{e}^{\alpha2\,\gamma}\,\eta2-\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\mathrm{fy0}'[\lambda]\Big)+\\
&\mathrm{QU}[\,]\Big(-\frac{\mathbb{e}^{\alpha1\,\gamma}\,\eta1\,\lambda\,\xi1}{\hbar}+\frac{\mathbb{e}^{\alpha1\,\gamma}\,T\,\eta1\,\lambda\,\xi1}{\hbar}+\frac{\eta2\,\lambda\,\xi1}{\hbar}-\frac{T\,\eta2\,\lambda\,\xi1}{\hbar}-\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\eta1\,\lambda\,\xi2}{\hbar}+\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\eta1\,\lambda\,\xi2}{\hbar}-\\
&\quad \frac{\mathbb{e}^{\alpha2\,\gamma}\,\eta2\,\lambda\,\xi2}{\hbar}+\frac{\mathbb{e}^{\alpha2\,\gamma}\,T\,\eta2\,\lambda\,\xi2}{\hbar}-\mathrm{ft0}'[\lambda]+\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{\mathbb{e}^{\alpha1\,\gamma+\alpha2\,\gamma}\,T\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}\Big)
\end{aligned}
$$

In[∘]:= `{sol1} = DSolve[Coefficient[diff /. ϵ → 0, x_QU] == 0 ∧ fx0[0] == 0, fx0, λ]`

Out[∘]=

$$\left\{\left\{\text{fx0} \to \text{Function}\left[\{\lambda\},\ e^{-\alpha 2\,\gamma}\,\lambda\,\left(\xi 1+e^{\alpha 2\,\gamma}\,\xi 2\right)\right]\right\}\right\}$$

In[∘]:= `CE[diff /. sol1 /. ϵ → 0]`

Out[∘]=

$$\text{QU}[y]\,\left(e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1+e^{\alpha 2\,\gamma}\,\eta 2-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\text{fy0}'[\lambda]\right)+\text{QU}[]\,\left(-\frac{e^{\alpha 1\,\gamma}\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{e^{\alpha 1\,\gamma}\,T\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{\eta 2\,\lambda\,\xi 1}{\hbar}-\frac{T\,\eta 2\,\lambda\,\xi 1}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\eta 1\,\lambda\,\xi 2}{\hbar}-\frac{e^{\alpha 2\,\gamma}\,\eta 2\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 2\,\gamma}\,T\,\eta 2\,\lambda\,\xi 2}{\hbar}-\text{ft0}'[\lambda]+\frac{e^{\alpha 1\,\gamma}\,\lambda\,\xi 1\,\text{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma}\,T\,\lambda\,\xi 1\,\text{fy0}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\lambda\,\xi 2\,\text{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\lambda\,\xi 2\,\text{fy0}'[\lambda]}{\hbar}\right)$$

In[∘]:= `{sol2} = DSolve[Coefficient[CE[diff /. sol1 /. ϵ → 0], y_QU] == 0 ∧ fy0[0] == 0, fy0, λ]`

Out[∘]=

$$\left\{\left\{\text{fy0} \to \text{Function}\left[\{\lambda\},\ e^{-\alpha 1\,\gamma}\,\left(e^{\alpha 1\,\gamma}\,\eta 1+\eta 2\right)\,\lambda\right]\right\}\right\}$$

In[∘]:= `CE[diff /. sol1 /. sol2 /. ϵ → 0]`

Out[∘]=

$$\text{QU}[]\,\left(\frac{2\,\eta 2\,\lambda\,\xi 1}{\hbar}-\frac{2\,T\,\eta 2\,\lambda\,\xi 1}{\hbar}-\text{ft0}'[\lambda]\right)$$

In[∘]:= `{sol3} = DSolve[Coefficient[CE[diff /. sol1 /. sol2 /. ϵ → 0], QU[]] == 0 ∧ ft0[0] == 0, ft0, λ]`

Out[∘]=

$$\left\{\left\{\text{ft0} \to \text{Function}\left[\{\lambda\},\ -\frac{(-1+T)\,\eta 2\,\lambda^2\,\xi 1}{\hbar}\right]\right\}\right\}$$

In[∘]:= `CE[diff /. sol1 /. sol2 /. sol3]`

Out[∘]=

$$2\gamma\epsilon\eta 2\lambda\xi 1\hbar\,\mathrm{QU}[y,x]+\mathrm{QU}[a]\,(4T\epsilon\eta 2\lambda\xi 1-\epsilon\,\mathrm{fa1}'[\lambda])+$$
$$\mathrm{QU}[x]\left(-\frac{1}{2}e^{-\alpha 2\gamma}\gamma\epsilon\eta 2\lambda^2\xi 1^2+\frac{3}{2}e^{-\alpha 2\gamma}T\gamma\epsilon\eta 2\lambda^2\xi 1^2-2\gamma\epsilon\eta 2\lambda^2\xi 1\xi 2+\right.$$
$$\left.6T\gamma\epsilon\eta 2\lambda^2\xi 1\xi 2-e^{-\alpha 2\gamma}\gamma\epsilon\lambda\xi 1\,\mathrm{fa1}'[\lambda]-\gamma\epsilon\lambda\xi 2\,\mathrm{fa1}'[\lambda]-\epsilon\,\mathrm{fx1}'[\lambda]\right)+$$
$$\mathrm{QU}[y]\left(-e^{\alpha 1\gamma+\alpha 2\gamma}\gamma\epsilon\eta 1\eta 2\lambda^2\xi 1+3e^{\alpha 1\gamma+\alpha 2\gamma}T\gamma\epsilon\eta 1\eta 2\lambda^2\xi 1+\frac{1}{2}e^{\alpha 2\gamma}\gamma\epsilon\eta 2^2\lambda^2\xi 1-\right.$$
$$\left.\frac{3}{2}e^{\alpha 2\gamma}T\gamma\epsilon\eta 2^2\lambda^2\xi 1-e^{\alpha 1\gamma+\alpha 2\gamma}\gamma\epsilon\eta 1\,\mathrm{fa1}[\lambda]-e^{\alpha 2\gamma}\gamma\epsilon\eta 2\,\mathrm{fa1}[\lambda]-e^{\alpha 1\gamma+\alpha 2\gamma}\epsilon\,\mathrm{fy1}'[\lambda]\right)+$$
$$\mathrm{QU}[\,]\left(\frac{e^{\alpha 1\gamma}\gamma\epsilon\eta 1\eta 2\lambda^3\xi 1^2}{2\hbar}-\frac{2e^{\alpha 1\gamma}T\gamma\epsilon\eta 1\eta 2\lambda^3\xi 1^2}{\hbar}+\frac{3e^{\alpha 1\gamma}T^2\gamma\epsilon\eta 1\eta 2\lambda^3\xi 1^2}{2\hbar}+\right.$$
$$\frac{e^{\alpha 1\gamma+\alpha 2\gamma}\gamma\epsilon\eta 1\eta 2\lambda^3\xi 1\xi 2}{\hbar}-\frac{4e^{\alpha 1\gamma+\alpha 2\gamma}T\gamma\epsilon\eta 1\eta 2\lambda^3\xi 1\xi 2}{\hbar}+\frac{3e^{\alpha 1\gamma+\alpha 2\gamma}T^2\gamma\epsilon\eta 1\eta 2\lambda^3\xi 1\xi 2}{\hbar}-$$
$$\frac{e^{\alpha 2\gamma}\gamma\epsilon\eta 2^2\lambda^3\xi 1\xi 2}{2\hbar}+\frac{2e^{\alpha 2\gamma}T\gamma\epsilon\eta 2^2\lambda^3\xi 1\xi 2}{\hbar}-\frac{3e^{\alpha 2\gamma}T^2\gamma\epsilon\eta 2^2\lambda^3\xi 1\xi 2}{2\hbar}+$$
$$\frac{e^{\alpha 1\gamma}\gamma\epsilon\eta 1\lambda\xi 1\,\mathrm{fa1}[\lambda]}{\hbar}-\frac{e^{\alpha 1\gamma}T\gamma\epsilon\eta 1\lambda\xi 1\,\mathrm{fa1}[\lambda]}{\hbar}+\frac{\gamma\epsilon\eta 2\lambda\xi 1\,\mathrm{fa1}[\lambda]}{\hbar}-$$
$$\frac{T\gamma\epsilon\eta 2\lambda\xi 1\,\mathrm{fa1}[\lambda]}{\hbar}+\frac{e^{\alpha 1\gamma+\alpha 2\gamma}\gamma\epsilon\eta 1\lambda\xi 2\,\mathrm{fa1}[\lambda]}{\hbar}-\frac{e^{\alpha 1\gamma+\alpha 2\gamma}T\gamma\epsilon\eta 1\lambda\xi 2\,\mathrm{fa1}[\lambda]}{\hbar}+$$
$$\frac{e^{\alpha 2\gamma}\gamma\epsilon\eta 2\lambda\xi 2\,\mathrm{fa1}[\lambda]}{\hbar}-\frac{e^{\alpha 2\gamma}T\gamma\epsilon\eta 2\lambda\xi 2\,\mathrm{fa1}[\lambda]}{\hbar}+\frac{e^{\alpha 1\gamma+\alpha 2\gamma}\epsilon\eta 1\,\mathrm{fx1}[\lambda]}{\hbar}-$$
$$\frac{e^{\alpha 1\gamma+\alpha 2\gamma}T\epsilon\eta 1\,\mathrm{fx1}[\lambda]}{\hbar}+\frac{e^{\alpha 2\gamma}\epsilon\eta 2\,\mathrm{fx1}[\lambda]}{\hbar}-\frac{e^{\alpha 2\gamma}T\epsilon\eta 2\,\mathrm{fx1}[\lambda]}{\hbar}-\epsilon\,\mathrm{ft1}'[\lambda]+$$
$$\left.\frac{e^{\alpha 1\gamma}\epsilon\lambda\xi 1\,\mathrm{fy1}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\gamma}T\epsilon\lambda\xi 1\,\mathrm{fy1}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\gamma+\alpha 2\gamma}\epsilon\lambda\xi 2\,\mathrm{fy1}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\gamma+\alpha 2\gamma}T\epsilon\lambda\xi 2\,\mathrm{fy1}'[\lambda]}{\hbar}\right)$$

In[∘]:= `QU[Generators]`

Out[∘]=

{y, a, x}

In[∘]:=
```
Seepage_U_[{}][_] = 1;
Seepage_U_[ξs_List][P_] := Module[{gs = U[Generators], lx},
  lx = gs〚Length@ξs〛;
  Collect[P, lx, (Ad[Last[ξs] lx_U][𝕆_U[gs, Seepage_U[Most@ξs]@#]] /. U → Times) &]
 ]
```

In[∘]:= `Expand[Seepage_QU[{η, α, ξ}][y a] /. ϵ → 0]`

Out[∘]=

$$a\,e^{\alpha}y-a\,e^{\alpha}\xi+a\,e^{\alpha}T\xi+e^{\alpha}x\,y\,\xi-e^{\alpha}x\,\xi^2+e^{\alpha}T\,x\,\xi^2$$