Pensieve header: The PBW multiplication tensor for $gl_{n,\epsilon}$ using the nilpotent-only λ-tangent formalism. Some material from pensive://Projects/glneps and from pensieve://Projects/UEA.

```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Conventions

$x_{\alpha,\beta}$ is always with $\alpha \le \beta$ and represents a basis element of the upper triangular matrices. $y_{\alpha,\beta}$ is also always with $\alpha \le \beta$ and represents a basis element of the lower triangular matrices. The adjoint of $x_{\alpha,\beta}$ is $y_{\alpha,\beta}$.

Variable declarations:

```
VarPattern = ((y | x | η | ξ) | (y | x | η | ξ)__)[_];
```

Greek - Latin duality:

```
{y*, x*, η*, ξ*} = {η, ξ, y, x};
(u_αβ__)* := (u*)αβ;
((u : ((y | x | η | ξ) | (y | x | η | ξ)__))[i_])* := u*[i];
(vs_List)* := (v ↦ v*) /@ vs;
```

```
{x1,2, y2,3, η1,2[7]}*
```

```
{ξ1,2, η2,3, y1,2[7]}
```

Weights:

```
Wt[xα_,β_] := {1, α - β}; Wt[yα_,β_] := {0, α - β};
Wt[ξα_,β_] := {0, β - α}; Wt[ηα_,β_] := {1, β - α};   ;
Wt[u_[_]] := Wt[u];
```

The Engine

Canonical Forms:

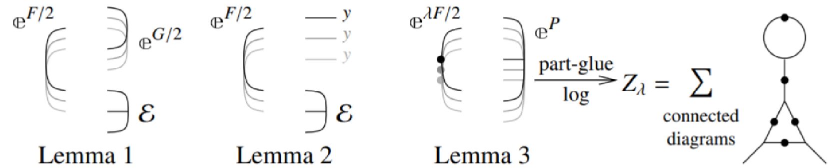
```
LogReduce[ℰ_] := ℰ /. c_ * Log[a_] :→ Log@Factor[a^c] //. Log[a_] + Log[b_] :→ Log@Factor[a b];
CF[ℰ_] := Expand@Collect[ℰ, Cases[ℰ, VarPattern, ∞], CCF] /. CCF → Factor;
CF[ℰ_𝔼] := CF /@ MapAt[LogReduce, ℰ, 1];
CF[ℰ : 𝔼__[___]] := CF /@ ℰ;
```

𝔼 operations:

```
ℰ_𝔼[$] := Length[ℰ] - 1; 𝔼_[ℰs___][$] := 𝔼[ℰs][$];
ℰ_𝔼[k_Integer] := ℰ[[k + 1]]; 𝔼_[ℰs___][k_Integer] := {ℰs}[[k + 1]];
𝔼 /: ℰ1_𝔼 ≡ ℰ2_𝔼 := Inner[CF@#1 == CF@#2 &, ℰ1, ℰ2, And];
𝔼d1_→r1_[ℰ1s___] ≡ 𝔼d2_→r2_[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼 /: ℰ1_𝔼 * ℰ2_𝔼 := 𝔼 @@ Table[CF[ℰ1[kk] + ℰ2[kk]], {kk, 0, Min[ℰ1[$], ℰ2[$]]}];
𝔼d1_→r1_[ℰ1s___] 𝔼d2_→r2_[ℰ2s___] ^:= 𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[ℰ1s] × 𝔼[ℰ2s]);
```

```
𝔼d1_→r1_[ℰ1s___] // 𝔼d2_→r2_[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs, llvs},
  lvs = Cases[{ℰ1s}, (x | y)__[i_] /; MemberQ[is, i], ∞] ⋃
    Cases[{ℰ2s}, (ξ | η)__[i_] /; MemberQ[is, i], ∞]*;
  llvs = Map[$, lvs, {2}];
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ (Zipllvs⋃llvs*[{(llvs)*.(llvs), Times[
      𝔼[ℰ1s] /. Thread[lvs → llvs],
      𝔼[ℰ2s] /. Thread[lvs* → llvs*]
     ]}])
 ]
```

Zipping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.



Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q.

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii}’s and the y_{ii}’s. They have weights {1,0} and {0,0}.

```
Zip_{vs_}[{𝓕_, ℰ_}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0} ∨ Wt[#] == {1, 0}) &]},
   {𝓕, ℰ} // Zip1_{vs} (*//Zip2_{Complement[vs,dvs]}*) // Zip2_{dvs} // Zip3_{Complement[vs,dvs]} // Zip3_{dvs}
  ] // Last
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,
$\left\langle F : \mathcal{E}\,e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1} : \mathcal{E}\right\rangle_B$

```
Zip1_{} = Identity;
Zip1_{vs_}@{𝓕_, 𝔼[Q_, P___]} := Module[{𝓘, F, G, u, v},
  𝓘 = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v}𝓕, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v}Q, 0], {u, vs}, {v, vs}];
  {CF[(vs).(F.Inverse[𝓘 - G.F]).(vs) / 2],
   𝔼[CF[Q - PowerExpand@Log[Det[𝓘 - G.F]] / 2 - vs.G.vs / 2], P]}
 ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E}\,e^{\sum_{i\in B} y_i z_i}\right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j}\left\langle F :\ \mathcal{E}|_{z_B \to z_B + F y_B}\right\rangle_B.$

```
Zip2_{} = Identity;
Zip2_{vs_}@{𝓕_, 𝔼[Q_, P___]} := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[∂_{u,v}𝓕], 0], {u, vs}, {v, vs}];
  Y = Table[∂_v Q, {v, vs}] /. Table[v → 0, {v, vs}];
  CF /@ {𝓕, 𝔼[Q - Y.vs + Y.F.Y / 2, P] /. Thread[vs → vs + F.Y]}
 ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write

```
Zip3vs_@{𝓕_, 𝓔_𝔼} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] = 𝓔⟦kk + 1⟧, {kk, 0, 𝓔@$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0}, ∂u,v 𝓕, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = ∂u Z[m, kk];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = ∂v Z[m, kk, u];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ 𝓔@$, ++kk,
    Z[m + 1, kk] = CF@Sum[
       If[F[u, v] === 0, 0,
        F[u, v]/(2 (m + 1)) (Z[m, kk, u, v] + Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m},
             {jj, 0, kk}])],
       {u, vs}, {v, vs}];
    If[Z[m + 1, kk] =!= 0, $m = m + 1]
   ]];
  CF /@ ({
      𝓕 - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
      𝔼 @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, 𝓔@$}]
     } /. Table[v → 0, {v, vs}])
 ]
```

```
Λ2𝔼d_→r_[Λ_] := Λ2𝔼d→r[$k, Λ];
Λ2𝔼d_→r_[$k_, Λ_] := Module[{k}, 𝔼d→r @@ Table[SeriesCoefficient[Λ, {ϵ, 0, k}], {k, 0, $k}]];
```

Utilities

```
δi_,j_ := If[i == j, 1, 0];
```

```
SSQ[c_. * x_] /; MemberQ[$Basis, x] := ¬ NilQ[x]; (* Semi-Simple Q *)
NilQ[c_ * x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
$k = 1;
```

```
CF[𝓔_] := Expand[𝓔 /. ℯ^L_ :> ℯ^Expand[L]] /. {ℯ^(k_·) /; k > $k → 0};
```

Implementing general universal enveloping algebras

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ¬ OrderedQ[{y, x} /. $PBWRule];
UU_{i_}[1] := U_i[];
UU_{i_}[x_^{p_}] := UU_i @@ Table[x, {p}];
UU_{i_}[ε_] := ε /. {
    U[xs__] ⧴ U_i[xs],
    x_ /; MemberQ[$Basis, x] ⧴ U_i[x]
   };
UU_{i_}[x_, xs__] := UU_{t1}[x] UU_{t2}[xs] // Expand // m_{t1,t2→i};
USimp[ε_] := Collect[ε, Times[U_[__] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
m_{s__}[0] = 0;
m_{s__}[x_Plus] := m_s /@ x;
m_{s__}[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i_→j_}[ε_] := ε /. U_i → U_j;
```

```
m_{i_,j_→k_}[c_. U_{i_}[x___] U_{j_}[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_{i_}[] U_{j_}[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_{i_}[xx___, x_] U_{j_}[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k}) c // USimp
  ];
```

```
UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] :=
  Sort@Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
Supp[ε_] := Union@Cases[{ε}, U_{i_}[___] ⧴ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⊠ y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := USimp[x ⊠ y - y ⊠ x];
```

Predefined Algebras

sl(2)

```
Print["UEA`SetAlgebra knows \"sl2\"."];
```

```
UEA`SetAlgebra knows "sl2".
```

```
SetAlgebra["sl2"] := (
   Print["In sl2: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
   B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
   $Algebra = "sl2";
   $Basis = {e, h, f};
   $PBWRule = {e → 1, h → 2, f → 3};
   NilQ[e] = NilQ[f] = True; NilQ[h] = False;
  );
```

$gl_{n,\epsilon}$

```
Print["UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}."];
```

```
UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}.
```

```
SetAlgebra[gl_{n_,ϵ}] := (
   $Algebra = gl_{n,ϵ};
   $Basis = Flatten@{
      Table[y_{α,β}, {β, 2, n}, {α, 1, β - 1}],
      Table[x_{α,β}, {β, 1, n}, {α, 1, β}]
     };
   (*$Basis=Reverse@SortBy[$Basis,If[#〚1〛===x,{0,#〚3〛-#〚2〛}, {1,#〚2〛-#〚3〛}]&];*)
   NilQ[y__] = True; NilQ[x_{α_,β_}] := (α =!= β); (* Nilpotent Q *)
   $PBWRule = Thread[$Basis → Range@Length@$Basis];
   B[x_{i_,j_}, x_{k_,l_}] := δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j};
   B[y_{i_,j_}, y_{k_,l_}] := CF[-ϵ δ_{j,k} y_{i,l} + ϵ δ_{l,i} y_{k,j}];
   B[x_{i_,j_}, y_{l_,k_}] := CF[δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j} /. x_{α_,β_} ⧴ If[α ≤ β, ϵ x_{α,β}, y_{β,α}]];
   DefRep[0] = Normal@SparseArray[{}, {n, n}];
   DefRep[ℰ_] := ℰ /. {
      x_{α_,β_} ⧴ Normal@SparseArray[{α, β} → 1, {n, n}],
      y_{α_,β_} ⧴ Normal@SparseArray[{β, α} → ϵ, {n, n}]
     }
  );
AdMatrix[x_, Env_] := Module[{A, b, c, P, S, k = 0},
   A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
   M = CF@If[NilQ[x],
      P = S = IdentityMatrix[Length@$Basis];
      While[P =!= 0 P, S += (P = CF[P.A]) / (++k) !]; S,
      Normal[Series[MatrixExp[A], {ϵ, 0, $k}]]
     ];
   If[False ∧ MemberQ[$Basis, x], AdMatrix[x, Env] = M, M]
  ];
AdMatrix[x_] := AdMatrix[x, {$Algebra, $k}];
Ad[x_][y_] := $Basis.AdMatrix[x].Table[Coefficient[y, b], {b, $Basis}];
λTangent[] = 0;
λTangent[xs___, x_] := Ad[x][λTangent[xs]] + ∂_λ x;
λTangent[xs_List] := λTangent @@ xs
TriangularDSolve[eqns_, funs_, iv_] := Monitor[
  Module[{sol = {}, e, fun, der},
   MapThread[
    {e, fun} ↦ sol = Join[sol,
       $Mon =
        MapAt[CF,
         First@DSolve[CF[e /. Derivative[1][f_][v_] ⧴ der[f[v], v] /. sol /. der[f_, v_] ⧴ ∂_v f],
           fun, iv], {1, 2}]
       ],
    {eqns, funs}
   ];
   sol
  ],
  $Mon
 ]
```

$gl_{4,\epsilon}$ Experiments

```
SetAlgebra[gl_{4,ϵ}];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}];
MatrixForm@Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
   {b1, $Basis}, {b2, $Basis}];
DeleteCases[
 Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]], {a, $Basis},
   {b, $Basis}, {c, $Basis}]], _ → 0]
```

1

$\{y_{1,2}\to 1,\ y_{1,3}\to 2,\ y_{2,3}\to 3,\ y_{1,4}\to 4,\ y_{2,4}\to 5,\ y_{3,4}\to 6,\ x_{1,1}\to 7,\ x_{1,2}\to 8,$
$x_{2,2}\to 9,\ x_{1,3}\to 10,\ x_{2,3}\to 11,\ x_{3,3}\to 12,\ x_{1,4}\to 13,\ x_{2,4}\to 14,\ x_{3,4}\to 15,\ x_{4,4}\to 16\}$

$\{\}$

```
SBasis = SortBy[$Basis, If[#[[1]] === x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
PBasis = SBasis;
sol[-1] = Table[If[SSQ[bb], ((bb /. x → ξ1) + (bb /. x → ξ2)), 0], {bb, PBasis}];
Do[
  Block[{$k = k},
   s0 = Thread[PBasis → sol[k - 1]];
   lhs = λTangent@Echo[
      Join[Table[If[NilQ[bb], λ, 1] bb (bb /. {x → ξ1, y → η1}), {bb, PBasis}],
       Table[If[NilQ[bb], λ, 1] bb (bb /. {x → ξ2, y → η2}), {bb, PBasis}]]
     ];
   rhs = λTangent[
     PBasis (PBasis /. {x_{αβ__} :> (x_{αβ} /. s0) + ϵ^k f_{αβ}[λ], y_{αβ__} :> (y_{αβ} /. s0) + ϵ^k g_{αβ}[λ]})
    ];
   Echo@Short[diff = CF[lhs - rhs], 20];
   eqns = Transpose@{
      (Coefficient[diff, #] == 0) & /@ SBasis,
      SBasis /. {x_{αβ__} :> f_{αβ}[0] == 0, y_{αβ__} :> g_{αβ}[0] == 0}
     };
   Echo@Length@eqns;
   unknowns = SBasis /. {x_{αβ__} → f_{αβ}[λ], y_{αβ__} → g_{αβ}[λ]};
   s = TriangularDSolve[eqns, unknowns, λ];
   sol[k] = CF[sol[k - 1] + ϵ^k (PBasis /. {x_{αβ__} → f_{αβ}[λ], y_{αβ__} → g_{αβ}[λ]} /. s)]
  ],
  {k, 0, $k}
 ];
cm[i_, j_ → k_] :=
  Λ2E_{{i,j}→{k}}[sol[$k].PBasis /. {x_{αβ__} :> x_{αβ}[k], y_{αβ__} :> y_{αβ}[k], ξ1_{αβ__} :> ξ_{αβ}[i], η1_{αβ__} :> η_{αβ}[i],
     ξ2_{αβ__} :> ξ_{αβ}[j], η2_{αβ__} :> η_{αβ}[j], λ → 1}];
Short[cm[i, j → k], 20]
```

$\{x_{1,1}\,\xi1_{1,1},\ x_{2,2}\,\xi1_{2,2},\ x_{3,3}\,\xi1_{3,3},\ x_{4,4}\,\xi1_{4,4},\ \lambda\,x_{1,2}\,\xi1_{1,2},\ \lambda\,x_{2,3}\,\xi1_{2,3},\ \lambda\,x_{3,4}\,\xi1_{3,4},\ \lambda\,x_{1,3}\,\xi1_{1,3},$
$\lambda\,x_{2,4}\,\xi1_{2,4},\ \lambda\,x_{1,4}\,\xi1_{1,4},\ \lambda\,y_{1,4}\,\eta1_{1,4},\ \lambda\,y_{1,3}\,\eta1_{1,3},\ \lambda\,y_{2,4}\,\eta1_{2,4},\ \lambda\,y_{1,2}\,\eta1_{1,2},\ \lambda\,y_{2,3}\,\eta1_{2,3},\ \lambda\,y_{3,4}\,\eta1_{3,4},$
$x_{1,1}\,\xi2_{1,1},\ x_{2,2}\,\xi2_{2,2},\ x_{3,3}\,\xi2_{3,3},\ x_{4,4}\,\xi2_{4,4},\ \lambda\,x_{1,2}\,\xi2_{1,2},\ \lambda\,x_{2,3}\,\xi2_{2,3},\ \lambda\,x_{3,4}\,\xi2_{3,4},\ \lambda\,x_{1,3}\,\xi2_{1,3},$
$\lambda\,x_{2,4}\,\xi2_{2,4},\ \lambda\,x_{1,4}\,\xi2_{1,4},\ \lambda\,y_{1,4}\,\eta2_{1,4},\ \lambda\,y_{1,3}\,\eta2_{1,3},\ \lambda\,y_{2,4}\,\eta2_{2,4},\ \lambda\,y_{1,2}\,\eta2_{1,2},\ \lambda\,y_{2,3}\,\eta2_{2,3},\ \lambda\,y_{3,4}\,\eta2_{3,4}\}$

$$e^{\xi 2_{1,1}-\xi 2_{2,2}}\, y_{1,2}\, \eta 1_{1,2} + e^{\xi 2_{1,1}-\xi 2_{3,3}}\, y_{1,3}\, \eta 1_{1,3} + e^{\xi 2_{1,1}-\xi 2_{4,4}}\, y_{1,4}\, \eta 1_{1,4} + e^{\xi 2_{2,2}-\xi 2_{3,3}}\, y_{2,3}\, \eta 1_{2,3} + e^{\xi 2_{2,2}-\xi 2_{4,4}}\, y_{2,4}\, \eta 1_{2,4} +$$
$$e^{\xi 2_{3,3}-\xi 2_{4,4}}\, y_{3,4}\, \eta 1_{3,4} + y_{1,2}\, \eta 2_{1,2} + y_{1,3}\, \eta 2_{1,3} + y_{1,4}\, \eta 2_{1,4} + y_{2,3}\, \eta 2_{2,3} + y_{2,4}\, \eta 2_{2,4} + y_{3,4}\, \eta 2_{3,4} + e^{-\xi 2_{1,1}+\xi 2_{2,2}}\, x_{1,2}\, \xi 1_{1,2} -$$
$$e^{\xi 2_{2,2}-\xi 2_{3,3}}\, \lambda\, y_{2,3}\, \eta 1_{1,3}\, \xi 1_{1,2} - e^{\xi 2_{2,2}-\xi 2_{4,4}}\, \lambda\, y_{2,4}\, \eta 1_{1,4}\, \xi 1_{1,2} - e^{-\xi 2_{1,1}+\xi 2_{2,2}}\, \lambda\, y_{2,3}\, \eta 2_{1,3}\, \xi 1_{1,2} - e^{-\xi 2_{1,1}+\xi 2_{2,2}}\, \lambda\, y_{2,4}\, \eta 2_{1,4}\, \xi 1_{1,2} +$$
$$e^{-\xi 2_{1,1}+\xi 2_{3,3}}\, x_{1,3}\, \xi 1_{1,3} - e^{\xi 2_{3,3}-\xi 2_{4,4}}\, \lambda\, y_{3,4}\, \eta 1_{1,4}\, \xi 1_{1,3} - e^{-\xi 2_{1,1}+\xi 2_{3,3}}\, \lambda\, y_{3,4}\, \eta 2_{1,4}\, \xi 1_{1,3} + e^{-\xi 2_{1,1}+\xi 2_{4,4}}\, x_{1,4}\, \xi 1_{1,4} +$$
$$e^{-\xi 2_{2,2}+\xi 2_{3,3}}\, x_{2,3}\, \xi 1_{2,3} + e^{\xi 2_{1,1}-\xi 2_{2,2}}\, \lambda\, y_{1,2}\, \eta 1_{1,3}\, \xi 1_{2,3} - e^{\xi 2_{3,3}-\xi 2_{4,4}}\, \lambda\, y_{3,4}\, \eta 1_{2,4}\, \xi 1_{2,3} + e^{-\xi 2_{2,2}+\xi 2_{3,3}}\, \lambda\, y_{1,2}\, \eta 2_{1,3}\, \xi 1_{2,3} -$$
$$e^{-\xi 2_{2,2}+\xi 2_{3,3}}\, \lambda\, y_{3,4}\, \eta 2_{2,4}\, \xi 1_{2,3} + e^{-\xi 2_{1,1}+\xi 2_{3,3}}\, \lambda\, x_{1,3}\, \xi 1_{1,2}\, \xi 1_{2,3} - e^{\xi 2_{3,3}-\xi 2_{4,4}}\, \lambda^2\, y_{3,4}\, \eta 1_{1,4}\, \xi 1_{1,2}\, \xi 1_{2,3} -$$
$$e^{-\xi 2_{1,1}+\xi 2_{3,3}}\, \lambda^2\, y_{3,4}\, \eta 2_{1,4}\, \xi 1_{1,2}\, \xi 1_{2,3} + e^{-\xi 2_{2,2}+\xi 2_{4,4}}\, x_{2,4}\, \xi 1_{2,4} + e^{\xi 2_{1,1}-\xi 2_{2,2}}\, \lambda\, y_{1,2}\, \eta 1_{1,4}\, \xi 1_{2,4} + e^{-\xi 2_{2,2}+\xi 2_{4,4}}\, \lambda\, y_{1,2}\, \eta 2_{1,4}\, \xi 1_{2,4} +$$
$$e^{-\xi 2_{1,1}+\xi 2_{4,4}}\, \lambda\, x_{1,4}\, \xi 1_{1,2}\, \xi 1_{2,4} + e^{-\xi 2_{3,3}+\xi 2_{4,4}}\, x_{3,4}\, \xi 1_{3,4} + e^{\xi 2_{1,1}-\xi 2_{3,3}}\, \lambda\, y_{1,3}\, \eta 1_{1,4}\, \xi 1_{3,4} + \ll 162 \gg + x_{1,3}\, f_{1,3}[\lambda]\, f_{3,3}'[\lambda] +$$
$$x_{2,3}\, f_{2,3}[\lambda]\, f_{3,3}'[\lambda] - x_{3,4}\, f_{3,4}[\lambda]\, f_{3,3}'[\lambda] + x_{1,4}\, f_{1,3}[\lambda]\, f_{3,4}[\lambda]\, f_{3,3}'[\lambda] + x_{2,4}\, f_{2,3}[\lambda]\, f_{3,4}[\lambda]\, f_{3,3}'[\lambda] -$$
$$y_{1,3}\, g_{1,3}[\lambda]\, f_{3,3}'[\lambda] + y_{1,2}\, f_{2,3}[\lambda]\, g_{1,3}[\lambda]\, f_{3,3}'[\lambda] - y_{3,4}\, f_{1,3}[\lambda]\, g_{1,4}[\lambda]\, f_{3,3}'[\lambda] - y_{1,3}\, f_{3,4}[\lambda]\, g_{1,4}[\lambda]\, f_{3,3}'[\lambda] +$$
$$y_{1,2}\, f_{2,3}[\lambda]\, f_{3,4}[\lambda]\, g_{1,4}[\lambda]\, f_{3,3}'[\lambda] - y_{2,3}\, g_{2,3}[\lambda]\, f_{3,3}'[\lambda] - y_{3,4}\, f_{2,3}[\lambda]\, g_{2,4}[\lambda]\, f_{3,3}'[\lambda] - y_{2,3}\, f_{3,4}[\lambda]\, g_{2,4}[\lambda]\, f_{3,3}'[\lambda] +$$
$$y_{3,4}\, g_{3,4}[\lambda]\, f_{3,3}'[\lambda] - x_{3,4}\, f_{3,4}'[\lambda] + x_{1,4}\, f_{1,3}[\lambda]\, f_{3,4}'[\lambda] - y_{1,3}\, g_{1,4}[\lambda]\, f_{3,4}'[\lambda] - y_{2,3}\, g_{2,4}[\lambda]\, f_{3,4}'[\lambda] - x_{4,4}\, f_{4,4}'[\lambda] +$$
$$x_{1,4}\, f_{1,4}[\lambda]\, f_{4,4}'[\lambda] + x_{2,4}\, f_{2,4}[\lambda]\, f_{4,4}'[\lambda] + x_{3,4}\, f_{3,4}[\lambda]\, f_{4,4}'[\lambda] - x_{1,4}\, f_{1,3}[\lambda]\, f_{3,4}[\lambda]\, f_{4,4}'[\lambda] - y_{1,4}\, g_{1,4}[\lambda]\, f_{4,4}'[\lambda] +$$
$$y_{1,2}\, f_{2,4}[\lambda]\, g_{1,4}[\lambda]\, f_{4,4}'[\lambda] + y_{1,3}\, f_{3,4}[\lambda]\, g_{1,4}[\lambda]\, f_{4,4}'[\lambda] - y_{2,4}\, g_{2,4}[\lambda]\, f_{4,4}'[\lambda] + y_{2,3}\, f_{3,4}[\lambda]\, g_{2,4}[\lambda]\, f_{4,4}'[\lambda] -$$
$$y_{3,4}\, g_{3,4}[\lambda]\, f_{4,4}'[\lambda] - y_{1,2}\, g_{1,2}'[\lambda] - y_{1,3}\, g_{1,3}'[\lambda] - y_{1,4}\, g_{1,4}'[\lambda] - y_{2,3}\, g_{2,3}'[\lambda] - y_{2,4}\, g_{2,4}'[\lambda] - y_{3,4}\, g_{3,4}'[\lambda]$$

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$$\{x_{1,1}\, \xi 1_{1,1},\ x_{2,2}\, \xi 1_{2,2},\ x_{3,3}\, \xi 1_{3,3},\ x_{4,4}\, \xi 1_{4,4},\ \lambda\, x_{1,2}\, \xi 1_{1,2},\ \lambda\, x_{2,3}\, \xi 1_{2,3},\ \lambda\, x_{3,4}\, \xi 1_{3,4},\ \lambda\, x_{1,3}\, \xi 1_{1,3},$$
$$\lambda\, x_{2,4}\, \xi 1_{2,4},\ \lambda\, x_{1,4}\, \xi 1_{1,4},\ \lambda\, y_{1,4}\, \eta 1_{1,4},\ \lambda\, y_{1,3}\, \eta 1_{1,3},\ \lambda\, y_{2,4}\, \eta 1_{2,4},\ \lambda\, y_{1,2}\, \eta 1_{1,2},\ \lambda\, y_{2,3}\, \eta 1_{2,3},\ \lambda\, y_{3,4}\, \eta 1_{3,4},$$
$$x_{1,1}\, \xi 2_{1,1},\ x_{2,2}\, \xi 2_{2,2},\ x_{3,3}\, \xi 2_{3,3},\ x_{4,4}\, \xi 2_{4,4},\ \lambda\, x_{1,2}\, \xi 2_{1,2},\ \lambda\, x_{2,3}\, \xi 2_{2,3},\ \lambda\, x_{3,4}\, \xi 2_{3,4},\ \lambda\, x_{1,3}\, \xi 2_{1,3},$$
$$\lambda\, x_{2,4}\, \xi 2_{2,4},\ \lambda\, x_{1,4}\, \xi 2_{1,4},\ \lambda\, y_{1,4}\, \eta 2_{1,4},\ \lambda\, y_{1,3}\, \eta 2_{1,3},\ \lambda\, y_{2,4}\, \eta 2_{2,4},\ \lambda\, y_{1,2}\, \eta 2_{1,2},\ \lambda\, y_{2,3}\, \eta 2_{2,3},\ \lambda\, y_{3,4}\, \eta 2_{3,4}\}$$

$$2\, e^{\xi 2_{2,2}-\xi 2_{3,3}} \in \lambda\, y_{1,3}\, \eta 1_{2,3}\, \eta 2_{1,2} + 2\, e^{\xi 2_{3,3}-\xi 2_{4,4}} \in \lambda\, y_{1,4}\, \eta 1_{3,4}\, \eta 2_{1,3} + 2\, e^{\xi 2_{3,3}-\xi 2_{4,4}} \in \lambda\, y_{2,4}\, \eta 1_{3,4}\, \eta 2_{2,3} -$$
$$2\, e^{\xi 2_{1,1}-\xi 2_{2,2}} \in \lambda\, y_{1,4}\, \eta 1_{1,2}\, \eta 2_{2,4} - e^{-\xi 2_{1,1}+2\, \xi 2_{2,2}-\xi 2_{3,3}} \in \lambda^2\, y_{2,3}\, \eta 1_{2,3}\, \eta 2_{1,2}\, \xi 1_{1,2} -$$
$$e^{-\xi 2_{1,1}+\xi 2_{2,2}+\xi 2_{3,3}-\xi 2_{4,4}} \in \lambda^2\, y_{2,4}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{1,2} + \in \lambda^2\, y_{2,4}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{1,2} - e^{-\xi 2_{1,1}+2\, \xi 2_{3,3}-\xi 2_{4,4}} \in \lambda^2\, y_{3,4}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{1,3} +$$
$$e^{-\xi 2_{2,2}+\xi 2_{3,3}} \in \lambda^2\, y_{3,4}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{1,3} + \in \lambda^2\, y_{1,2}\, \eta 1_{2,3}\, \eta 2_{1,2}\, \xi 1_{2,3} - e^{-\xi 2_{2,2}+2\, \xi 2_{3,3}-\xi 2_{4,4}} \in \lambda^2\, y_{3,4}\, \eta 1_{3,4}\, \eta 2_{2,3}\, \xi 1_{2,3} -$$
$$e^{-\xi 2_{1,1}+2\, \xi 2_{3,3}-\xi 2_{4,4}} \in \lambda^3\, y_{3,4}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{1,2}\, \xi 1_{2,3} + e^{-\xi 2_{2,2}+\xi 2_{3,3}} \in \lambda^3\, y_{3,4}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{1,2}\, \xi 1_{2,3} +$$
$$e^{-\xi 2_{2,2}+\xi 2_{3,3}} \in \lambda^2\, y_{1,2}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{2,4} - e^{\xi 2_{1,1}-2\, \xi 2_{2,2}+\xi 2_{4,4}} \in \lambda^2\, y_{1,2}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{2,4} + \in \lambda^2\, y_{1,3}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{3,4} +$$
$$\in \lambda^2\, y_{2,3}\, \eta 1_{3,4}\, \eta 2_{2,3}\, \xi 1_{3,4} - e^{\xi 2_{1,1}-\xi 2_{2,2}-\xi 2_{3,3}+\xi 2_{4,4}} \in \lambda^2\, y_{1,3}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{3,4} + e^{-\xi 2_{2,2}+\xi 2_{3,3}} \in \lambda^3\, y_{1,2}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{2,3}\, \xi 1_{3,4} -$$
$$e^{\xi 2_{1,1}-2\, \xi 2_{2,2}+\xi 2_{4,4}} \in \lambda^3\, y_{1,2}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{2,3}\, \xi 1_{3,4} - 2\, e^{\xi 2_{1,1}-\xi 2_{2,2}} \in \lambda\, x_{1,1}\, \eta 1_{1,2}\, \xi 2_{1,2} + 2\, e^{\xi 2_{1,1}-\xi 2_{2,2}} \in \lambda\, x_{2,2}\, \eta 1_{1,2}\, \xi 2_{1,2} +$$
$$e^{2\, \xi 2_{1,1}-2\, \xi 2_{2,2}} \in \lambda^2\, y_{1,2}\, \eta 1_{1,2}^2\, \xi 2_{1,2} + \ll 1810 \gg + \in \lambda\, x_{1,4}\, \xi 2_{1,4}\, f_{4,4}'[\lambda] - e^{\xi 2_{2,2}-\xi 2_{4,4}} \in \lambda^2\, y_{3,4}\, \eta 1_{2,4}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] -$$
$$e^{-\xi 2_{3,3}+\xi 2_{4,4}} \in \lambda^2\, x_{2,4}\, \xi 1_{3,4}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] - e^{\xi 2_{1,1}-\xi 2_{3,3}} \in \lambda^3\, y_{1,2}\, \eta 1_{1,4}\, \xi 1_{3,4}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] -$$
$$e^{-\xi 2_{3,3}+\xi 2_{4,4}} \in \lambda^3\, y_{1,2}\, \eta 2_{1,4}\, \xi 1_{3,4}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] - e^{\xi 2_{1,1}-\xi 2_{4,4}} \in \lambda^3\, y_{3,4}\, \eta 1_{1,4}\, \xi 2_{1,2}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] + \in \lambda\, x_{2,4}\, \xi 2_{2,4}\, f_{4,4}'[\lambda] +$$
$$e^{\xi 2_{1,1}-\xi 2_{4,4}} \in \lambda^2\, y_{1,2}\, \eta 1_{1,4}\, \xi 2_{2,4}\, f_{4,4}'[\lambda] + \in \lambda^2\, y_{1,2}\, \eta 2_{1,4}\, \xi 2_{2,4}\, f_{4,4}'[\lambda] + \in \lambda\, x_{3,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] +$$
$$e^{\xi 2_{1,1}-\xi 2_{4,4}} \in \lambda^2\, y_{1,3}\, \eta 1_{1,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] + e^{\xi 2_{2,2}-\xi 2_{4,4}} \in \lambda^2\, y_{2,3}\, \eta 1_{2,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] + \in \lambda^2\, y_{1,3}\, \eta 2_{1,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] +$$
$$\in \lambda^2\, y_{2,3}\, \eta 2_{2,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] + e^{\xi 2_{1,1}-\xi 2_{4,4}} \in \lambda^3\, y_{2,3}\, \eta 1_{1,4}\, \xi 2_{1,2}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] - \in \lambda^2\, x_{1,4}\, \xi 2_{1,3}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] -$$
$$\in y_{1,2}\, g_{1,2}'[\lambda] - \in y_{1,3}\, g_{1,3}'[\lambda] - \in y_{1,4}\, g_{1,4}'[\lambda] - \in y_{2,3}\, g_{2,3}'[\lambda] - \in y_{2,4}\, g_{2,4}'[\lambda] - \in y_{3,4}\, g_{3,4}'[\lambda]$$

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$$
\begin{aligned}
&\mathbb{E}_{\{i,j\}\to\{k\}}\Big[y_{1,4}[k]\left(e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]+\eta_{1,4}[j]\right)+\\
&\quad x_{1,1}[k]\,(\xi_{1,1}[i]+\xi_{1,1}[j])+x_{1,2}[k]\left(e^{-\xi_{1,1}[j]+\xi_{2,2}[j]}\,\xi_{1,2}[i]+\xi_{1,2}[j]\right)+\\
&\quad y_{2,4}[k]\left(e^{\xi_{2,2}[j]-\xi_{4,4}[j]}\,\eta_{2,4}[i]+\eta_{2,4}[j]+e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,2}[j]\right)+x_{2,2}[k]\,(\xi_{2,2}[i]+\xi_{2,2}[j])+\\
&\quad x_{1,3}[k]\left(e^{-\xi_{1,1}[j]+\xi_{3,3}[j]}\,\xi_{1,3}[i]+\xi_{1,3}[j]-e^{-\xi_{2,2}[j]+\xi_{3,3}[j]}\,\xi_{1,2}[j]\,\xi_{2,3}[i]\right)+\\
&\quad x_{2,3}[k]\left(e^{-\xi_{2,2}[j]+\xi_{3,3}[j]}\,\xi_{2,3}[i]+\xi_{2,3}[j]\right)+e^{-\xi_{4,4}[j]}\,y_{3,4}[k]\left(e^{\xi_{3,3}[j]}\,\eta_{3,4}[i]+e^{\xi_{4,4}[j]}\,\eta_{3,4}[j]+\right.\\
&\qquad\left.e^{\xi_{1,1}[j]}\,\eta_{1,4}[i]\,\xi_{1,3}[j]+e^{\xi_{2,2}[j]}\,\eta_{2,4}[i]\,\xi_{2,3}[j]+e^{\xi_{1,1}[j]}\,\eta_{1,4}[i]\,\xi_{1,2}[j]\,\xi_{2,3}[j]\right)+\\
&\quad y_{1,2}[k]\left(e^{\xi_{1,1}[j]-\xi_{2,2}[j]}\,\eta_{1,2}[i]+\eta_{1,2}[j]-e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{2,3}[j]-e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{2,4}[j]\right)+\\
&\quad x_{3,3}[k]\,(\xi_{3,3}[i]+\xi_{3,3}[j])+x_{2,4}[k]\left(e^{-\xi_{2,2}[j]+\xi_{4,4}[j]}\,\xi_{2,4}[i]+\xi_{2,4}[j]-e^{-\xi_{3,3}[j]+\xi_{4,4}[j]}\,\xi_{2,3}[j]\,\xi_{3,4}[i]\right)+\\
&\quad x_{3,4}[k]\left(e^{-\xi_{3,3}[j]+\xi_{4,4}[j]}\,\xi_{3,4}[i]+\xi_{3,4}[j]\right)+\\
&\quad y_{1,3}[k]\left(e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]+\eta_{1,3}[j]-e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{3,4}[j]\right)+\\
&\quad y_{2,3}[k]\left(e^{\xi_{2,2}[j]-\xi_{3,3}[j]}\,\eta_{2,3}[i]+\eta_{2,3}[j]+e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{1,2}[j]-\right.\\
&\qquad\left.e^{\xi_{2,2}[j]-\xi_{4,4}[j]}\,\eta_{2,4}[i]\,\xi_{3,4}[j]-e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,2}[j]\,\xi_{3,4}[j]\right)+x_{1,4}[k]\\
&\quad\left(e^{-\xi_{1,1}[j]+\xi_{4,4}[j]}\,\xi_{1,4}[i]+\xi_{1,4}[j]-e^{-\xi_{2,2}[j]+\xi_{4,4}[j]}\,\xi_{1,2}[j]\,\xi_{2,4}[i]-e^{-\xi_{2,2}[j]+\xi_{4,4}[j]}\,\xi_{1,2}[j]\,\xi_{2,3}[i]\,\xi_{3,4}[i]+\right.\\
&\qquad\left.e^{-\xi_{1,1}[j]+\xi_{3,3}[j]}\,\xi_{1,3}[i]\,\xi_{3,4}[j]-e^{-\xi_{2,2}[j]+\xi_{3,3}[j]}\,\xi_{1,2}[j]\,\xi_{2,3}[i]\,\xi_{3,4}[j]\right)+x_{4,4}[k]\,(\xi_{4,4}[i]+\xi_{4,4}[j]),\\
&\ x_{2,2}[k]\left(e^{\xi_{1,1}[j]-\xi_{2,2}[j]}\,\eta_{1,2}[i]\,\xi_{1,2}[j]-e^{\xi_{2,2}[j]-\xi_{3,3}[j]}\,\eta_{2,3}[i]\,\xi_{2,3}[j]-e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{1,2}[j]\,\xi_{2,3}[j]-\right.\\
&\qquad\left.e^{\xi_{2,2}[j]-\xi_{4,4}[j]}\,\eta_{2,4}[i]\,\xi_{2,4}[j]-e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,2}[j]\,\xi_{2,4}[j]\right)+\\
&\quad x_{1,1}[k]\left(-e^{\xi_{1,1}[j]-\xi_{2,2}[j]}\,\eta_{1,2}[i]\,\xi_{1,2}[j]-e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{1,3}[j]-e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,4}[j]+\right.\\
&\qquad\left.e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,3}[j]\,\xi_{3,4}[j]\right)+\ll 14\gg+x_{3,4}[k]\left(e^{\xi_{1,1}[j]-\xi_{\ll 1\gg}[j]}\,\eta_{1,3}[i]\,\xi_{1,4}[j]+\ll 21\gg\right)+\\
&\quad x_{1,4}[k]\left(e^{-\xi_{2,2}[j]+\xi_{4,4}[j]}\,\eta_{1,2}[i]\,\xi_{1,2}[j]\,\xi_{1,4}[i]+e^{-\xi_{3,3}[j]+\xi_{4,4}[j]}\,\eta_{1,3}[i]\,\xi_{1,3}[j]\,\xi_{1,4}[i]+\right.\\
&\qquad\left.\eta_{1,3}[i]\,\xi_{1,3}[i]\,\xi_{1,4}[j]+e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{1,3}[j]\,\xi_{1,4}[j]+\ll 82\gg\right)\Big]
\end{aligned}
$$

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(cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])
```

True