

```
In[ ]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW"]
Once[<< KnotTheory`];
```

Time: 0.12 seconds Out[]:=

C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.

Read more at <http://katlas.org/wiki/KnotTheory>.

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_{\hbar}(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.

Read more at <http://katlas.org/wiki/KnotTheory>.

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This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

pdf

This version: April 2020. Original version: July 1994.

```
In[ ]:= MakePDF[]
```

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```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[_e_, op_] := Collect[Normal@Series[_e_, {e, 0, $k}], e, op];
SS[_e_] := SS[_e_, Together];
```

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```

In[ ]:= $N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
ξ[i_, j_] := ξ <> i <> j; α[i_] := α <> i; ℳ[i_] := ℳ <> i;
y[i_, j_] := y <> i <> j; b[i_] := b <> i; B[i_] := B <> i;
η[i_, j_] := η <> i <> j; β[i_] := β <> i;
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_] := τ[s]; ℓ[s_] := ℓ[s];

```

```

In[ ]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm

```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

Greek - Latin duality:

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```

In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]* = ξ[i, j];
ξ[i, j]* = x[i, j]; y[i, j]* = η[i, j];
η[i, j]* = y[i, j]]
Do[As[i, 1, $N - 1], A[i]* = ℳ[i]; ℳ[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i];
b[i]* = β[i]; β[i]* = b[i]]
(vs_List)* := (v ↦ v*) /@ vs;
(u_{αβ})_* := (u^*)_{αβ};
F[u_i] := F[u_i] = ToExpression["F" <> ToString[u]]_i

```

```

In[ ]:= $N = 9

```

Time: 0.05 seconds Out[]:=

9

```

In[ ]:= τ[α1]

```

Time: 0.00 seconds Out[]:=

α

```

Flatten@Table[{ℳ[j]_{i_}^{p_} → e^{p α[j]_i}, B[j]_{i_}^{p_} → e^{-p ℏ b[j]_i}}, {j, 1, $N - 1}];
Join[{e^{χ_} → e^{Expand@12U@χ_}},
Flatten@Table[{e^{c_· b[j]_{i_} + d_} → B[j]_i^{-c/ℏ} e^d, e^{c_· α[j]_{i_} + d_} → ℳ[j]_i^c e^d}, {j, 1, $N - 1}]]];

```

```

In[*]:= ħ = 1;
U21[ε_] := ε /. {A1ip -> ep α1i, B1ip -> e-p ħ b1i, A2ip -> ep α2i, B2ip -> e-p ħ b2i, A3ip -> ep α3i,
  B3ip -> e-p ħ b3i, A4ip -> ep α4i, B4ip -> e-p ħ b4i, A5ip -> ep α5i, B5ip -> e-p ħ b5i, A6ip -> ep α6i,
  B6ip -> e-p ħ b6i, A7ip -> ep α7i, B7ip -> e-p ħ b7i, A8ip -> ep α8i, B8ip -> e-p ħ b8i};

12U[ε_] := ε //. {ed-+c-. b1i -> B1i-c ed, ed-+c-. α1i -> A1ic ed,
  ed-+c-. b2i -> B2i-c ed, ed-+c-. α2i -> A2ic ed, ed-+c-. b3i -> B3i-c ed,
  ed-+c-. α3i -> A3ic ed, ed-+c-. b4i -> B4i-c ed, ed-+c-. α4i -> A4ic ed, ed-+c-. b5i -> B5i-c ed,
  ed-+c-. α5i -> A5ic ed, ed-+c-. b6i -> B6i-c ed, ed-+c-. α6i -> A6ic ed, ed-+c-. b7i -> B7i-c ed,
  ed-+c-. α7i -> A7ic ed, ed-+c-. b8i -> B8i-c ed, ed-+c-. α8i -> A8ic ed, eχ -> Expand[12U[χ]]};

12U[r_Rule] := Module[{U = r[[1]] /. {b -> B, t -> T, α -> A}}, U -> 12U[U21[U] /. r]];
AlsoUpper[rs_List] := rs ∪ (12U /@ rs);

```

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Derivatives in the presence of exponentiated variables:

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```

In[*]:= Db1i[f_] := ∂b1i f - ħ B1i ∂B1i f;
Db2i[f_] := ∂b2i f - ħ B2i ∂B2i f;
Db3i[f_] := ∂b3i f - ħ B3i ∂B3i f;
Db4i[f_] := ∂b4i f - ħ B4i ∂B4i f;
Db5i[f_] := ∂b5i f - ħ B5i ∂B5i f;
Db6i[f_] := ∂b6i f - ħ B6i ∂B6i f;
Db7i[f_] := ∂b7i f - ħ B7i ∂B7i f;
Db8i[f_] := ∂b8i f - ħ B8i ∂B8i f;
Dα1i[f_] := ∂α1i f + A1i ∂A1i f;
Dα2i[f_] := ∂α2i f + A2i ∂A2i f;
Dα3i[f_] := ∂α3i f + A3i ∂A3i f;
Dα4i[f_] := ∂α4i f + A4i ∂A4i f;
Dα5i[f_] := ∂α5i f + A5i ∂A5i f;
Dα6i[f_] := ∂α6i f + A6i ∂A6i f;
Dα7i[f_] := ∂α7i f + A7i ∂A7i f;
Dα8i[f_] := ∂α8i f + A8i ∂A8i f;
Dv-[f_] := ∂v f;

```

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```

In[*]:= LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
  Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars ∪ GreekVars))_;

```

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```

In[ ]:= CF[ $\mathcal{E}$ _] := PPCF@Module[{s1},
  s1 = If[$k ==  $\infty$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor}^p}$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor}^p}$  /.  $q^{n-} \rightarrow \sum_{k=0}^{k} \frac{(\hbar \in \gamma n)^k}{k!}$  /.  $e^{k-}$  /.  $k > $k \rightarrow 0$ 
  ]
  (*Expand@Collect[s1,Cases[s1,VarPattern, $\infty$ ],CCF]/.CCF→Factor*)
]

```

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Weights:

```

In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N],
  Wt[x[i, j]] = {1, i - j}; Wt[y[i, j]] = {0, -(i - j)};
  Wt[ $\xi$ [i, j]] = {0, -(i - j)}; Wt[ $\eta$ [i, j]] = {1, (i - j)};
];
Do[As[i, 1, $N - 1],
  Wt[a[i]] = {1, 0}; Wt[b[i]] = {0, 0};
  Wt[ $\alpha$ [i]] = {0, 0}; Wt[ $\beta$ [i]] = {1, 0};
];
Wt[u_] := Wt[u];

```

```

In[ ]:= $PBWGens

```

Time: 0.03 seconds Out[]:=

```
{x12, x13, x23, a1, a2}
```

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```

In[ ]:= (* q~e $\hbar \gamma$   $\epsilon$ ; Ai[n]~e $\hbar \in a_i$ ; *)

```

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```

In[ ]:= $n = 3; $k =  $\infty$ ; $E := {$n, $k}

```

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```

In[ ]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
  Table[As[i, 1, $n - 1], {a[i], A[i][_]}]
}
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y}] /. $EPBWRule; x_ < y_ := ! OrderedQ[{y, x}] /. $EPBWRule;
$PBWGens := DeleteCases[$EPBWGens, _[_]]

```

```

In[ ]:= $PBWGens*

```

Time: 0.01 seconds Out[]:=

```
{ $\xi$ 12,  $\xi$ 13,  $\xi$ 14,  $\xi$ 15,  $\xi$ 23,  $\xi$ 24,  $\xi$ 25,  $\xi$ 34,  $\xi$ 35,  $\xi$ 45,  $\alpha$ 1,  $\alpha$ 2,  $\alpha$ 3,  $\alpha$ 4}
```

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```

In[*]:= UCF[ $\mathcal{E}_-$ ] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a @ L[AB], k], r]
  }]],
  U[____], F
] /. F → CF]

```

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```

In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j[_] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}_-$ ] := ms[UCF[ $\mathcal{E}$ ]];

```

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```

In[*]:= Supp[ $\mathcal{E}_-$ ] := Union@Cases[{ $\mathcal{E}$ }, Ui[____] ⇒ i, ∞];

```

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```

In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PPNCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /.  $\epsilon^{k-}$  /. k > $k ⇒ 0];
  Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];

```

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```

In[*]:= Δi→j,k[_] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk,k1→k@mj,j1→j[Δj,k[f] Δi→j1,k1[Ui[fs]]]
}];
Δi→j,k,l[_] :=  $\mathcal{E}$  /. Δi→j,k /. Δk→k,l

```

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```
In[*]:= Si[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i→i[Si[f] Si1[Ui1[fs]]]
}];
S̄i[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i→i[S̄i[f] S̄i1[Ui1[fs]]]
}];
```

```
In[*]:= Δ1,2→3[U1[x12]]
```

```
Time: 0.02 seconds Out:=
```

```
Δ1,2→3[U1[x12]]
```

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The $\mathbb{A}$ -algebra

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```
In[*]:= RRk[ai_, aj_] /; τ[ai] == τ[aj] == "a" := Uk[aj, ai];
RRk[ai_, Aj_[n_]] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[Aj[n], ai];
RRk[Aj_[n_], ai_] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[ai, Aj[n]];
RRk[Ai_[m_], Aj_[n_]] /; τ[Ai] == τ[Aj] == "A" := Uk[Aj[n], Ai[m]];
RRk[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@am;
    Uk[xij, am] + γ (δm,i - δm+1,i - δm,j + δm+1,j) Uk[xij] / 2
];
RRk[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@Am; q-n (δn,i - δn+1,i - δn,j + δn+1,j) Uk[xij, Am[n]]
]
```

```
In[*]:= $k
```

```
Time: 0.03 seconds Out:=
```

```
1
```

```
In[*]:= RR0[a[1], x[1, 2]]
```

```
Time: 0.05 seconds Out:=
```

```
γ U0[x12] + U0[x12, a1]
```

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We take the reduction rules RR for xij's from Yamane page

509:

$$\begin{array}{lll}
 \text{(I)} \quad i = m < j < n, & \text{(II)} \quad i < m < n < j, & \text{(III)} \quad i < m < j = n, \\
 \begin{array}{ccc} \underline{i} & \underline{j} & \\ \underline{m} & \underline{n} & \end{array} & \begin{array}{ccc} \underline{i} & \underline{j} & \\ \underline{m} & \underline{n} & \end{array} & \begin{array}{ccc} \underline{i} & \underline{j} & \\ \underline{m} & \underline{n} & \end{array} \\
 \text{(IV)} \quad i < m < j < n, & \text{(V)} \quad i < j = m < n, & \text{(VI)} \quad i < j < m < n. \\
 \begin{array}{ccc} \underline{i} & \underline{j} & \\ \underline{m} & \underline{n} & \end{array} & \begin{array}{ccc} \underline{i} & \underline{j} & \\ \underline{m} & \underline{n} & \end{array} & \begin{array}{ccc} \underline{i} & \underline{j} & \\ \underline{m} & \underline{n} & \end{array}
 \end{array}$$

$$\begin{aligned}
q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} &= 0 && \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\
[e_{ij}, e_{mn}] &= 0 && \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\
[e_{ij}, e_{mn}] &= (q^2 - q^{-2})e_{in}e_{mj} && \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\
q^2e_{ij}e_{mn} - e_{mn}e_{ij} &= qe_{in} && \text{if } ((i, j), (m, n)) \in C_{(V)}.
\end{aligned}$$

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```

In[*]:= RR_k[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 U_k[xij, xmn], (* I∨III *)
    (i < m < j ∧ n < j) ∨ (j < m), U_k[xij, xmn], (* II∨VI *)
    i < m < j ∧ n > j, U_k[xij, xmn] + (q-2 - q2) U_k[x[i, n], x[m, j]], (* IV *)
    j == m, q2 U_k[xij, xmn] - q U_k[x[i, n]] (* V *)
  ]
]

In[*]:= U_1[x34] U_2[x23] U_3[x12] // m1,2→2
Time: 0.06 seconds Out:=
-q U_2[x24] U_3[x12] + q2 U_2[x23, x34] U_3[x12]

In[*]:= U_1[x34] U_2[x23] U_3[x12] // m1,2→2 // m2,3→0
Time: 0.05 seconds Out:=
q2 U_0[x14] - q3 U_0[x12, x24] - q3 U_0[x13, x34] + q4 U_0[x12, x23, x34]

In[*]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U_1[X] U_2[Y] U_3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
Time: 1.05 seconds Out:=
{}

In[*]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U_1[X] U_2[Y] U_3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[*]=
{}

```

```
Module[{Bas = $PBWGens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3 ->2 // m2,4 ->2 // m1,2 ->1) ==
      (P // m3,4 ->3 // m2,3 ->2 // m1,2 ->1) == (P // m1,2 ->1 // m3,4 ->3 // m1,3 ->1) ==
      (P // m1,2 ->1 // m1,3 ->1 // m1,4 ->1) == (P // m2,3 ->2 // m1,2 ->1 // m1,4 ->1)
  ]
]
» {x12, x13, x23, a1, a2}
```

Out[]:=

{ }

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The Co-Product, Co-Associativity, and the Hopf Axiom

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According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

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```
In[ ]:= Do[As[i, 1, $n - 1],
  Δj,k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj,k[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj,k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj,k[x[i, n]] = mk,k1 ->k[mj,j1 ->j[
    q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
  ]
]
```

```
In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui / @ Bas], P -> Δi ->j,k[P]]
]
```

Time: 0.03 seconds Out[]:=

```
Ui[x12] -> Uj[x12] Uk[] - 2 ∈ ħ Uj[x12] Uk[a1] + Uj[] Uk[x12] + 2 ∈ ħ Uj[a1] Uk[x12]
Ui[x13] -> Uj[x13] Uk[] - 2 ∈ ħ Uj[x13] Uk[a1] - 2 ∈ ħ Uj[x13] Uk[a2] +
  Uj[] Uk[x13] + 2 ∈ ħ Uj[a1] Uk[x13] + 2 ∈ ħ Uj[a2] Uk[x13] + 4 ∇ ∈ ħ Uj[x12] Uk[x23]
Ui[x23] -> Uj[x23] Uk[] - 2 ∈ ħ Uj[x23] Uk[a2] + Uj[] Uk[x23] + 2 ∈ ħ Uj[a2] Uk[x23]
Ui[a1] -> Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] -> Uj[a2] Uk[] + Uj[] Uk[a2]
```

```
In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→1,2 // Δ2→2,3) = (P // Δ1→1,3 // Δ1→1,2)
  ]]
]
» {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}
```

Out[]:=

```
{ }
```

```
In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0][Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]]
]
» {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}
```

Out[]:=

```
{ }
```

Is Δ a quantization of the co-bracket?

```
In[ ]:= $k = 1
Time: 0.02 seconds Out[ ]:=
1
```

```
In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1/@Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[_][xs___, A-[n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
```

Time: 0.05 seconds Out[]:=

```
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 ∉ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 ∉ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0
```

pdf

The Antipode

pdf

In[]:=

```

Do[As[i, 1, $n - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj-[x[i, n]] = mj1, j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]

Do[As[i, 1, $n - 1],
  S̄j-[a[i]] = -Uj[a[i]];
  S̄j-[A[i][n_]] = Uj[A[i][-n]];
  S̄j-[x[i, i + 1]] = -q2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  S̄j-[x[i, n]] = mj1, j→j[
    q S̄j[x[i, n - 1]] S̄j1[x[n - 1, n]] - q-1 S̄j[x[n - 1, n]] S̄j1[x[i, n - 1]]
  ]
]

```

In[]:=

```

Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
» {x12, x13, x23, a1, a2, A2[-1], A2[-1], A2[1], A1[2], A1[-2], A1[-1], A1[-2], A1[1], A1[0], A1[-2]}

```

Out[]:=

```

{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A2[-1] → U1[],
 A2[-1] → U1[], A2[1] → U1[], A1[2] → U1[], A1[-2] → U1[],
 A1[-1] → U1[], A1[-2] → U1[], A1[1] → U1[], A1[0] → U1[], A1[-2] → U1[]}

```

```

In[*]:= $k = ∞
Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas],
  P = U1[X];
  X -> UCF[(P // S1 // S1) - P]
]]
]
Time: 0.05 seconds Out:=
∞

» {x12, x13, x23, a1, a2, A2[-1], A1[-2], A2[0], A1[-2], A1[2], A1[0], A1[0], A2[2], A1[0], A2[-1]}
Out[*]=
{}

```

pdf

The Generating Functions for the $\mathbb{A}$ operations

```

In[*]:= BeginProfile[]
Time: 0.03 seconds Out:=
BeginProfile[]

```

```

In[*]:= $k = 1;

```

pdf

Ad, TT, Seepage

pdf

```

In[*]:= PotQ[0] = True;
PotQ[c_. Ui[aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);

```

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```

In[*]:= Ad[x_] [y_] /; PotQ[x] ∧ (Head[y] != Plus) := PPAdPoty [If[y === 0, 0, eUB[y,x]/y y]];
Ad[x_] [y_Plus] /; PotQ[x] := PPAdPotPlus [Ad[x] /@ y];

```

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```

In[*]:= Ad[x_] [y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil @Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]

```

In[*]:= **Ad**[7 $\hbar$ U_j [**x12**] U_k [**a3**]] [U_j [**x23**] U_k [**a4**]]

Time: 0.09 seconds Out[=

$$U_j[x23] U_k[a4] + \left(-7 \hbar - 7 \gamma \in \hbar^2 - \frac{7}{2} \gamma^2 \epsilon^2 \hbar^3 \right) U_j[x13] U_k[a3, a4] + \\ \left(14 \gamma \in \hbar^2 + 14 \gamma^2 \epsilon^2 \hbar^3 \right) U_j[x12, x23] U_k[a3, a4] - 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x12, x13] U_k[a3, a3, a4] + \\ 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x12, x12, x23] U_k[a3, a3, a4] - \frac{686}{3} \gamma^2 \epsilon^2 \hbar^5 U_j[x12, x12, x13] U_k[a3, a3, a3, a4]$$

pdf

In[*]:= **TT** _{λ} [] = 0; (* Translated Tangent *)
TT _{λ} [**xs**____, **x**_] := **PP**_{TT}@**UCF**[**Ad**[**x**][**TT** _{λ} [**xs**]] + ∂_{λ} **x**];
TT _{λ} [**xs****List**] := **TT** _{λ} @@ **xs**

In[*]:= **TT** _{λ} [λ ξ 12 U_0 [**x12**] U_1 [], λ ξ 23 U_0 [**x23**] U_1 []]

Time: 0.01 seconds Out[=

$$\xi 12 U_0[x12] U_1[] + \lambda \xi 12 \xi 23 U_0[x13] U_1[] + \xi 23 U_0[x23] U_1[]$$

pdf

In[*]:= **O** _{i} [**poly**_] := **Total**[
CoefficientRules[**poly**, # _{i} & /@ **\$PBWGens**] /.
(**p**_ → **c**_) => **c** **U** _{i} @@ **Flatten**@**MapThread**[**ConstantArray**, {**\$PBWGens**, **p**}]
];
O _{i} ,**is**__ [**poly**_] := **O** _{i} [**O** _{is} [**poly**]];
P _{i} [**$\mathcal{E}$** _] := **$\mathcal{E}$** /. **U** _{i} [**xs**____] => **Times** @@ (# _{i} & /@ {**xs**});
P _{i} ,**is**__ [**poly**_] := **P** _{i} [**P** _{is} [**poly**]];

In[*]:= **O** _{i} [**x12** _{i} ² **x13** _{i} + 7 **x23** _{i} ¹⁷]

Time: 0.06 seconds Out[=

$$U_i[x12, x12, x13] + \\ 7 U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]$$

In[*]:= **O** _{i,j} [**x12** _{i} ² **x13** _{i} **x24** _{j} + 7 **x23** _{i} ¹⁷ **x25** _{j} ³]

Time: 0.03 seconds Out[=

$$U_i[x12, x12, x13] U_j[x24] + \\ 7 U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23] \\ U_j[x25, x25, x25]$$

In[*]:= **P** _{i,j} @**O** _{i,j} [**x12** _{i} ² **x13** _{i} **x24** _{j} + 7 **x23** _{i} ¹⁷ **x25** _{j} ³]

Time: 0.06 seconds Out[=

$$x12_i^2 x13_i x24_j + 7 x23_i^{17} x25_j^3$$

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```

In[ ]:= Seepage_i_[{ }, c_] = c;
Seepage_i_[ss_List, P_] := Module[{lx},
  lx = $PBWgens[Length@ss];
  CF@Collect[P, lx_i, P_i[Ad[Last[ss] U_i[lx]] [0_i[Seepage_i[Most@ss, #]]]]
    &]
];
Seepage_i_,is_[ss_List, ssss_List, P_] := Seepage_i_[ss, Seepage_is[ssss, P]]

```

pdf

```

In[ ]:= Invert_is_[s_] := Module[{A, B, U1, j}, (* s=A U[] + e B *)
  U1 = Times@@ (U_#[] & /@ {is});
  A = Coefficient[s, U1] /. e -> 0;
  B = UCF[(s - A U1)];
  UCF[
    A^-1 (U1 + Sum[As[j, 1, $k], (-1)^j NonCommutativeMultiply@@ ConstantArray[B/A, j]])]
]

```

```

In[ ]:= $k = 3;
Invert_i,j[7 U_i[] U_j[] + e U_i[x12] U_j[x23]]

```

Time: 0.09 seconds Out[]:=

$$\frac{1}{7} U_i[] U_j[] - \frac{1}{49} e U_i[x12] U_j[x23] + \frac{1}{343} e^2 U_i[x12, x12] U_j[x23, x23] - \frac{e^3 U_i[x12, x12, x12] U_j[x23, x23, x23]}{2401}$$

pdf

```

In[ ]:= TriangularDSolve[eqns_, inits_, fs_, l_, CF_] :=
  PP_TriangularDSolve@Module[{ss = {}, reqns = eqns, rinit = inits, rfs = fs, s, p},
    While[rfs != {},
      {p} = FirstPosition[Count[#, Alternatives@@rfs, ∞, Heads -> True] & /@ reqns, 1];
      {{s}} = DSolve[reqns[[p]] ^ rinit[[p]], rfs[[p]], l];
      AppendTo[ss, s = s /. F_Function -> ReplacePart[F, 2 -> CF[F[[2]]]]];
      reqns = Delete[reqns, p] /. s /. e_ == 0 -> CF[e] == 0;
      rinit = Delete[rinit, p];
      rfs = Delete[rfs, p];
    ];
    ss
  ]

```

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The Generating Functions Engine

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Canonical Forms:

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```
In[*]:= LogReduce[ $\mathcal{E}$ _] :=
   $\mathcal{E}$  /.  $c$ _ * Log[ $a$ _]  $\Rightarrow$  Log@Factor[ $a^c$ ] /. Log[ $a$ _] + Log[ $b$ _]  $\Rightarrow$  Log@Factor[ $a b$ ];
CF[ $\mathcal{E}$ _E] := CF /@ MapAt[LogReduce,  $\mathcal{E}$ , 1];
CF[ $\mathcal{E}$  : E_][_] := CF /@  $\mathcal{E}$ ;
```

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E operations:

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```
In[*]:=  $\mathcal{E}$ _E[$_] := Length[ $\mathcal{E}$ ] - 1; E_[ $\mathcal{E}$ s_] [$_] := E[ $\mathcal{E}$ s] [$];
 $\mathcal{E}$ _E[k_Integer] :=  $\mathcal{E}$ [[k+1]]; E_[ $\mathcal{E}$ s_] [k_Integer] := { $\mathcal{E}$ s}[[k+1]];
E /:  $\mathcal{E}1$ _E  $\equiv$   $\mathcal{E}2$ _E := Inner[CF@#1 == CF@#2 &,  $\mathcal{E}1$ ,  $\mathcal{E}2$ , And];
 $\mathcal{E}_{d1 \rightarrow r1}$ [ $\mathcal{E}1$ s_]  $\equiv$   $\mathcal{E}_{d2 \rightarrow r2}$ [ $\mathcal{E}2$ s_]  $\wedge$  := ( $d1 == d2$ )  $\wedge$  ( $r1 == r2$ )  $\wedge$  (E[ $\mathcal{E}1$ s]  $\equiv$  E[ $\mathcal{E}2$ s]);
E /:  $\mathcal{E}1$ _E *  $\mathcal{E}2$ _E := E @@ Table[CF[ $\mathcal{E}1$ [kk] +  $\mathcal{E}2$ [kk]], {kk, 0, Min[Length[ $\mathcal{E}1$ ], Length[ $\mathcal{E}2$ ]]};
 $\mathcal{E}_{d1 \rightarrow r1}$ [ $\mathcal{E}1$ s_]  $\mathcal{E}_{d2 \rightarrow r2}$ [ $\mathcal{E}2$ s_]  $\wedge$  := E[( $d1 \cup d2$ )  $\rightarrow$  ( $r1 \cup r2$ )] @@ (E[ $\mathcal{E}1$ s] E[ $\mathcal{E}2$ s]);
```

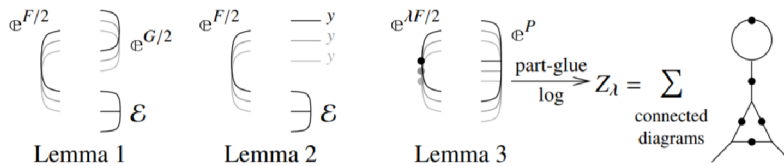
pdf

```
In[*]:=  $\mathcal{E}_{d1 \rightarrow r1}$ [ $\mathcal{E}1$ s_] //  $\mathcal{E}_{d2 \rightarrow r2}$ [ $\mathcal{E}2$ s_] := Module[{is =  $r1 \cap d2$ , lvs, llvs},
  lvs = Cases[{ $\mathcal{E}1$ s}, (Alternatives @@ LatinVars)_ /; MemberQ[is, #],  $\infty$ ]  $\cup$ 
  Cases[{ $\mathcal{E}2$ s}, (Alternatives @@ GreekVars)_ /; MemberQ[is, #],  $\infty$ ]*;
  llvs = lvs /.  $u_{i_}$   $\Rightarrow$   $u_{\#i}$ ;
  E[( $d1 \cup \text{Complement}[d2, is]$ )  $\rightarrow$  ( $r2 \cup \text{Complement}[r1, is]$ )] @@ (Zip[llvs, llvs] [ { (F /@ llvs)* . (F /@ llvs), Times[
    E[ $\mathcal{E}1$ s] /. Thread[lvs  $\rightarrow$  llvs],
    E[ $\mathcal{E}2$ s] /. Thread[lvs*  $\rightarrow$  llvs*]
  ]}])
]
```

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Zippping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii} ’s and the y_{ii} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

pdf

```

In[*]:= Zipvs [{ $\mathcal{F}$ _,  $\mathcal{E}$ _}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0}  $\vee$  Wt[#] == {1, 0}) &]},
  { $\mathcal{F}$ _,  $\mathcal{E}$ _} // Zip1vs (* // Zip2Complement[vs,dvs] *) // Zip2dvs //
  EZip3Complement[vs,dvs] // Zip3dvs
] // Last

```

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Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F: \mathcal{E} \otimes^{\frac{1}{2}} \sum_{i,j \in B} G_{ij} z_i z_j \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1}: \mathcal{E} \right\rangle_B$$

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```

In[*]:= Zip1{} = Identity;
Zip1vs @ { $\mathcal{F}$ _,  $\mathbb{E}[Q_, P\_]$ } := Module[{ $\mathcal{I}$ , F, G, u, v},
  (*Echo[{vs, $\mathcal{F}$ ,Q,P}];*)
   $\mathcal{I}$  = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{\mathcal{F}[u], \mathcal{F}[v]} \mathcal{F}$ , 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v} Q$ , 0], {u, vs}, {v, vs}];
  (*Echo@MatrixForm[ $\mathcal{I} - G.F$ ];*)
  {CF[(F /@ vs) . (F.Inverse[ $\mathcal{I} - G.F$ ]) . (F /@ vs) / 2],
    $\mathbb{E}$ [CF[Q - PowerExpand@Log[Det[ $\mathcal{I} - G.F$ ]] / 2 - vs.G.vs / 2], P]}
]

```

```

In[*]:= Wt /@ {x12$[2], y12$[1],  $\eta$ 12$[1],  $\xi$ 12$[2]}

```

Time: 0.00 seconds Out:=

```
{ {1, -1}, {0, 1}, {1, -1}, {0, 1} }
```

```

In[*]:=  $\mathbb{E}_{\{\} \rightarrow \{1,2\}}$  [x122 y121] //  $\mathbb{E}_{\{1,2\} \rightarrow \{\}}$  [(1 - eb11)  $\eta$ 121  $\xi$ 122]

```

Time: 0.02 seconds Out:=

```
 $\mathbb{E}_{\{\} \rightarrow \{\}}$  [-b11]
```

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Getting rid of linear terms.

Lemma 2. $\left\langle F: \mathcal{E} \otimes^{\frac{1}{2}} \sum_{i,j \in B} F_{ij} y_i y_j \right\rangle_B = \otimes^{\frac{1}{2}} \sum_{i,j \in B} F_{ij} y_i y_j \left\langle F: \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

pdf

```

In[*]:= Zip2{} = Identity;
Zip2vs @ { $\mathcal{F}$ _,  $\mathbb{E}[Q_, P\_]$ } := Module[{F, Y, u, v},
  (*EchoLabel["Zip2"] [{vs, $\mathcal{F}$ ,Q,P}];*)
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[ $\partial_{\mathcal{F}[u], \mathcal{F}[v]} \mathcal{F}$ ], 0], {u, vs}, {v, vs}];
  Y = Table[ $\partial_v Q$ , {v, vs}] /. (*AlsoUpper@*) Table[v  $\rightarrow$  0, {v, vs}];
  CF /@ ({ $\mathcal{F}$ _,  $\mathbb{E}[Q - Y.vs + Y.F.Y / 2$ , P]} /. (*AlsoUpper@*) Thread[vs  $\rightarrow$  vs + F.Y])
]

```

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Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathfrak{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda) (\partial_{z_j} Z_\lambda) \right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

pdf

In[*]:=

```
Zip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  (*Echo[{"Zip3", vs,  $\mathcal{F}$ ,  $\mathcal{E}$ }]*)
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0,  $\mathcal{E}$ @$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0},  $\partial_{F[u], F[v]} \mathcal{F}$ , 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤  $\mathcal{E}$ @$, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2(m+1)}$  (Z[m, kk, u, v] +
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
     $\mathcal{F}$  - Sum[F[u, v] * F[u] * F[v] / 2, {u, vs}, {v, vs}],
    E@@Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0,  $\mathcal{E}$ @$}]
  }) /. (*AlsoUpper@*) Table[v → 0, {v, vs}]
]
```

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In[*]:=

```
EZip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := PPEZip3@Module[
  {n $\mathcal{E}$ , n $\mathcal{F}$ , rc, ps, rr = {(*release rules*)}, fvs},
  rc = 0; n $\mathcal{E}$  = Total[
    CoefficientRules[#, vs] /.
      (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{E}$ [++rc] → c]; c $\mathcal{E}$ [rc] (Times@@vsps))
  ] & /@  $\mathcal{E}$ ;
  rc = 0; fvs = F /@ vs;
  n $\mathcal{F}$  = Total[CoefficientRules[ $\mathcal{F}$ , fvs] /.
    (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{F}$ [++rc] → c]; c $\mathcal{F}$ [rc] (Times@@fvsps))];
  CF[Expand[{n $\mathcal{F}$ , n $\mathcal{E}$ } // Zip3vs] /. rr]
]
```

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“Define” Code

pdf

```

In[ ]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs]);
Define[op_is__ = ε_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[op_nisp, $E_, PPBoot@Block[{i, j, k}, op_isp, $E = ε; op_nis, $E]];
    SD[op_isp, op_{is}, $E]; SD[op_sis__, op_{sis}];
  ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
  } ] ]

```

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$$\begin{aligned}
 L_\lambda &= \mathbb{E}(\lambda_{\tilde{z}_1} x) \mathbb{E}(\lambda_{\tilde{z}_2} x) & \text{Seepage is defined by: } \mathbb{E}_{\text{off}}(f; x_i) \mathbb{D}(\text{Seepage}(f_i, A)) &= \mathbb{D}(e^{f; x_i} A) \\
 \text{TT}(L_\lambda) &= L' \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f_i\}, B). \\
 R_\lambda &= \mathbb{E}(f+g), \quad f = f_i x_i, \quad g = \sum_{|u|=1} g_u w & e^f = F, \quad e^g = G, \quad R_\lambda = \mathbb{D}(FG), \quad \partial_\lambda F = F \partial_\lambda f \\
 \text{TT}(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathbb{D}(FG)' \mathbb{D}(\partial_\lambda FG) = S(G)' \mathbb{D}(F)' \mathbb{D}(F(\partial_\lambda f G + \partial_\lambda G)) = S(G)' S(\partial_\lambda f G + \partial_\lambda G)
 \end{aligned}$$

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In[*]:=

```

Define[amti,j→k = PPamt[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ES[s_][ε_] := ε;
  TTL = ES["TTL"]@PPTTL@TTλ@
    Join[l1 = Table[If[τ[g] == "x", λ, 1] (g*)i Uk[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
  ff = Table[
    Sum[εp fp,uk[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp-,u-[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
  fs = Flatten[Table[
    fp,ℙk[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙk[u] ℙk[u],
    {p, 0, $k}, {u, Complement[mons[p], Uk /@ $PBWGens]}
  ];
  G = Series[εgg, {ε, 0, $k}];
  S[B_] := UCF[SS@Ok[Seepagek[ff, SS@UCF@B]]];
  SG = ES["SG"]@PPSG@S[G];
  TTR = ES["TTR"]@PPTTR@UCF[Invertk[SG] ⊗ S[(∂λ(ff.(#k & /@ $PBWGens)) G + ∂λG)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i,j}→{k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], (α[m]i + α[m]j) a[m]k], 0],
    Total[Cases[fs, fp,-]] /. fp,- => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]

```

pdf

In[*]:=

```

Define[aΔti→j,k = PPaΔt[$k,$n] @Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
  TTL = ES["TTL"] @
    TTλ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p-1] ∪
    Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] => Uj[x] Uk[y], ∞];
  ffj = Table[
    Sum[εp fp,uj[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,uj[λ] /; τ[u] == "a" => If[p == 0, (u*)i, 0];
  ffk = ffj /. j -> k;
  fs = Flatten[Table[
    fp,ℙj,k[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙj,k[u][λ] ℙj,k[u],
    {p, 0, $k}, {u, Complement[mons[p], Uj[] (Uk /@ $PBWGens) ∪ Uk[] (Uj /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oj,k[Seepagej,k[ffj, ffk, SS@UCF@B]]];
  SG = ES["SG"] @ PPSG @ S[G];
  TTR = ES["TTR"] @ PPTTR @ UCF[
    Inverttj,k[SG] ⊗ S[(∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λ G)];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], α[m]i (a[m]j + a[m]k)], 0],
    Total[Cases[fs, fp,-] /. fp,u => u (fp,u /. ss)[λ] /. λ | ħ | γ -> 1]
  ]
]]

```

In[*]:=

```

Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] :=
    mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c_. Ui[x____] => Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,ui[λ] /; τ[u] == "a" => If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp,ℙi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙi[u][λ] ℙi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⓧ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], -α[m]i a[m]i], 0],
    Total[Cases[fs, fp,u]] /. fp,u => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

```

In[*]:= Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] :=
    mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c_. Ui[x---] ⇒ Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,ui[λ] /; τ[u] == "a" ⇒ If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp,ℙi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙi[u][λ] ℙi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[εgg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⊗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], -α[m]i a[m]i], 0],
    Total[Cases[fs, fp,-]] /. fp,- ⇒ u (fp,u /. ss)[λ] /. λ | ℏ | γ → 1
  ]
]]

```

```

In[*]:= $k = 1; $n = 3;

```

```

In[*]:= aSt1 // aSt1

```

Time: 5.94 seconds Out[] =

```

E{1}→{1} [a11 α11 + a21 α21 + x121 ξ121 + x131 ξ131 + x231 ξ231, 0]

```

```
In[*]:= (*Splitter into B tensor A*)
SYi→j, k, l, m := E{i}→{j, k, l, m} [
  Sum[η[r, s]i y[r, s]j + ξ[r, s]i x[r, s]m, {r, 1, $n - 1}, {s, r + 1, $n}] +
  Sum[α[r]i a[r]l + β[r]i b[r]k, {r, 1, $n - 1}], Sequence @@ ConstantArray[0, $k]
]
```

```
In[*]:= SY1→2, 3, 4, 5
```

```
Time: 0.00 seconds Out[ ]=
```

$$E_{\{1\} \rightarrow \{2, 3, 4, 5\}} [a_{14} \alpha_{11} + a_{24} \alpha_{21} + b_{13} \beta_{11} + b_{23} \beta_{21} + y_{112} \eta_{111} + y_{122} \eta_{121} + y_{132} \eta_{131} + y_{222} \eta_{221} + y_{232} \eta_{231} + x_{115} \xi_{111} + x_{125} \xi_{121} + x_{135} \xi_{131} + x_{225} \xi_{221} + x_{235} \xi_{231}, 0]$$

P: A tensor B → k
R in B tensor A

```
In[*]:= Pi, j := E{i, j}→{} @@ PadRight [
  {Sum[η[r, s]j ξ[r, s]i, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[β[r]j α[r]i, {r, 1, $n - 1}]},
  $k + 1]
Ri, j := E{ }→{i, j} @@ PadRight [
  {Sum[y[r, s]i x[r, s]j, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[b[r]i a[r]j, {r, 1, $n - 1}]},
  $k + 1]
```

```
In[*]:= R1, 2 // P2, 3
R1, 2 // P5, 1
```

```
Time: 0.05 seconds Out[ ]=
```

$$E_{\{3\} \rightarrow \{1\}} [b_{11} \beta_{13} + b_{21} \beta_{23} + y_{121} \eta_{123} + y_{131} \eta_{133} + y_{231} \eta_{233}, 0]$$

```
Out[*]=
```

$$E_{\{5\} \rightarrow \{2\}} [a_{12} \alpha_{15} + a_{22} \alpha_{25} + x_{122} \xi_{125} + x_{132} \xi_{135} + x_{232} \xi_{235}, 0]$$

```
In[*]:= Define[bmtj, k→i = Module[{i0, i1, j1, k1}, (Ri0, i1 // aΔti1→j1, k1 // Pj1, j // Pk1, k) /. i0 → i],
  bΔtk→i, j = Module[{i1, j1, k1, k2}, ((Ri, i1 Rj, j1 // amti1, j1→k1) // Pk1, k2) /. k2 → k],
  bSti = Module[{i1, i2}, (Ri, i1 // aSti1 // Pi1, i2) /. i2 → i],
  bSti = Module[{i1, i2}, (Ri, i1 // aSti1 // Pi1, i2) /. i2 → i]]
Define[Ri, j = Ri, j // bSti]
```

```
In[*]:= amt1, 2→3 // l2U
```

```
Time: 0.03 seconds Out[ ]=
```

$$E_{\{1, 2\} \rightarrow \{3\}} \left[a_{13} (\alpha_{11} + \alpha_{12}) + a_{23} (\alpha_{21} + \alpha_{22}) + x_{123} \left(\xi_{121} + e^{\alpha_{11} - \frac{\alpha_{21}}{2}} \xi_{122} \right) + \right. \\
x_{133} \left(\xi_{131} + e^{\frac{\alpha_{11}}{2} + \frac{\alpha_{21}}{2}} \xi_{132} - e^{\alpha_{11} - \frac{\alpha_{21}}{2}} \xi_{122} \xi_{231} \right) + x_{233} \left(\xi_{231} + e^{-\frac{\alpha_{11}}{2} + \alpha_{21}} \xi_{232} \right), \\
-2 e^{\alpha_{11} - \frac{\alpha_{21}}{2}} x_{123} x_{133} \xi_{122} \xi_{131} - e^{\alpha_{11} - \frac{\alpha_{21}}{2}} x_{133} \xi_{122} \xi_{231} + \\
\left. 2 e^{\alpha_{11} - \frac{\alpha_{21}}{2}} x_{123} x_{233} \xi_{122} \xi_{231} - 2 e^{\frac{\alpha_{11}}{2} + \frac{\alpha_{21}}{2}} x_{133} x_{233} \xi_{132} \xi_{231} \right]$$

```
In[*]:= $k = 0; $n = 2;
```

```
In[*]:= bmt1,2→3
» TTL e-Y α1i ξ12i Uj [x12] Uk [] + e-Y α1i ξ12i Uj [] Uk [x12]
» SG Uj [] Uk []
» TTR e-Y α1i Uj [x12] Uk [] fθ,x12j' [λ] + e-Y α1i Uj [] Uk [x12] fθ,x12k' [λ]

Out[*]=
E{1,2}→{3} [b13 β11 + b13 β12 + y123 η121 + y123 η122]
```

Testing **A** and **B** dually paired Hopf algebras

co-associativity

```
In[*]:= {(aΔt1→1,2 // aΔt2→2,3) ≡ (aΔt1→2,3 // aΔt2→1,2), (bΔt1→1,2 // bΔt2→2,3) ≡ (bΔt1→2,3 // bΔt2→1,2) }
» TTL e-Y α1i ξ12i Uj [x12] Uk [] + e-Y α1i ξ12i Uj [] Uk [x12]
» SG Uj [] Uk []
» TTR e-Y α1i Uj [x12] Uk [] fθ,x12j' [λ] + e-Y α1i Uj [] Uk [x12] fθ,«1»»' [λ]

Out[*]=
{True, True}
```

algebra morphism

```
In[*]:= {(aΔti→1,2 aΔtj→3,4 // amt1,3→i // amt2,4→j) ≡ (amti,j→k // aΔtk→i,j),
(bΔti→1,2 bΔtj→3,4 // bmt1,3→i // bmt2,4→j) ≡ (bmti,j→k // bΔtk→i,j) }

Time: 0.08 seconds Out[*]=
{True, True}
```

```
In[*]:= (aΔti→1,2 aΔtj→3,4 // amt1,3→i // amt2,4→j)

Time: 0.03 seconds Out[*]=
E{i,j}→{i,j} [a1i α1i + a1j α1i + a1i α1j + a1j α1j + x12i ξ12i + x12j ξ12i + eα1i x12i ξ12j + eα1i x12j ξ12j]
```

associativity

```
In[*]:= (amt1,2→k // amtk,3→k) ≡
(amt2,3→k // amt1,k→k)

Time: 0.02 seconds Out[*]=
True

In[*]:= {(amt1,2→k // amtk,3→k) ≡ (amt2,3→k // amt1,k→k),
(bmt1,2→k // bmtk,3→k) ≡ (bmt2,3→k // bmt1,k→k) } // Simplify

Time: 0.02 seconds Out[*]=
{True, True}
```

antipode

```
In[*]:= {aΔti→1,2 // aSt1 // amt1,2→1, aΔti→1,2 // aSt2 // amt1,2→1,
bΔti→1,2 // bSt1 // bmt1,2→1, bΔti→1,2 // bSt2 // bmt1,2→1}

Time: 0.08 seconds Out[*]=
{E{i}→{1} [0], E{i}→{1} [0], E{i}→{1} [0], E{i}→{1} [0]}
```

inverse antipode

In[*]:= {aSt_i // aSt_i, bSt_i // bSt_i}

Time: 0.05 seconds Out[]=

{E_{{i}→{i}} [a_{1i} α_{1i} + x_{12i} ξ_{12i}], E_{{i}→{i}} [b_{1i} β_{1i} + y_{12i} η_{12i}] }

Pairing

In[*]:= { (amt_{1,2→3} // P_{3,-3}) ≡ (bΔt_{-3→-1,-2} // P_{1,-1} // P_{2,-2}),
 (bmt_{1,2→3} // P_{-3,3}) ≡ (aΔt_{-3→-1,-2} // P_{-1,1} // P_{-2,2}),
 (aSt₁ // P_{1,-1}) ≡ (bSt₋₁ // P_{1,-1}), (aSt₁ // P_{1,-1}) ≡ (bSt₋₁ // P_{1,-1}) }

Time: 0.05 seconds Out[]=

{True, True, True, True}

In[*]:= P_{1,2}

Time: 0.02 seconds Out[]=

E_{{1,2}→{ }} [α₁₁ β₁₂ + η₁₂₂ ξ₁₂₁]

In[*]:= Define [dm_{i,j→k} = Module [{i1, i2, i3, i4},
 ((sY_{i→i4,i4,i1,i1} // aΔt_{i1→i1,i2} // aΔt_{i2→i2,i3} // aSt_{i3}) (sY_{j→-i1,-i1,-i4,-i4} // bΔt_{-i1→-i1,-i2} //
 bΔt_{-i2→-i2,-i3}) // (P_{i1,-i3} P_{i3,-i1} amt_{i2,-i4→k} bmt_{i4,-i2→k})]]
 Define [dS_i = sY_{i→1,1,2,2} // (bSt₁ aSt₂) // dm_{2,1→i},
 dS_i = sY_{i→1,1,2,2} // (bSt₁ aSt₂) // dm_{2,1→i},
 dΔ_{i→j,k} = (bΔt_{i→1,3} aΔt_{i→4,2}) // (dm_{3,4→k} dm_{1,2→j})]
 (*Notice we take co-opposite on the A-side! to make the double*)

Testing Double

In[*]:= \$k = 0; \$n = 2;

In[*]:= dm_{1,2→3}

» TTL e^{- $\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)$} ξ_{12i} U_j [x₁₂] U_k [] + <<32>> + 2 e^{- $\frac{3\gamma\alpha_2}{2}$} γ ∈ λ ξ_{13i} ξ_{23i} U_j [] U_k [x₁₃, x₂₃]
 » SG U_j [] U_k [] + e^{- $\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)$} ∈ U_j [x₁₂] U_k [a₁] f_{1,a_{1k}x_{12j}} [λ] + <<21>> +
 U_j [] U_k [x₁₃, x₂₃] (e^{- $\frac{3\gamma\alpha_2}{2}$} ∈ f_{0,x_{23k}} [λ] f_{1,x_{12k}x_{23k}} [λ] + e^{- $\frac{3\gamma\alpha_2}{2}$} ∈ f_{1,x_{13k}x_{23k}} [λ])
 » TTR U_j [x₁₂] U_k [x₁₂] (-e^{- $\gamma (2\alpha_1 - \alpha_2)$} γ ∈ f_{1,a_{1j}x_{12k}} [λ] f_{0,x_{12j}} [λ] - e^{- $\gamma (2\alpha_1 - \alpha_2)$} γ ∈ f_{1,a_{1k}x_{12j}} [λ] f_{0,x_{12k}} [λ]) +
 <<1>> + <<1>> + <<25>> + <<1>> + U_j [x₁₃] U_k [x₂₃] (<<1>>) + U_j [] U_k [x₁₃, x₂₃] (<<1>>)

Out[]=

E_{{1,2}→{3}} [a₁₃ α₁₁ + a₁₃ α₁₂ + a₂₃ α₂₁ + a₂₃ α₂₂ + b₁₃ β₁₁ + b₁₃ β₁₂ + b₂₃ β₂₁ + b₂₃ β₂₂ +
 y₁₂₃ η₁₂₁ + e ^{$\frac{1}{2} (-2\alpha_1 + \alpha_2)$} y₁₂₃ η₁₂₂ + y₁₃₃ η₁₃₁ + e ^{$\frac{1}{2} (-\alpha_1 - \alpha_2)$} y₁₃₃ η₁₃₂ + y₂₃₃ η₂₃₁ +
 e ^{$\frac{1}{2} (\alpha_1 - 2\alpha_2)$} y₂₃₃ η₂₃₂ + x₁₂₃ ξ₁₂₁ - e ^{$\frac{1}{2} (-2\alpha_1 + \alpha_2)$} η₁₂₂ ξ₁₂₁ + e ^{$\frac{1}{2} (2b_1 - b_2 - 2\alpha_1 + \alpha_2)$} η₁₂₂ ξ₁₂₁ -
 e ^{$\frac{1}{2} (2b_1 - b_2 - \alpha_1 - \alpha_2)$} y₂₃₃ η₁₃₂ ξ₁₂₁ + e ^{$\frac{1}{2} (2\alpha_1 - \alpha_2)$} x₁₂₃ ξ₁₂₂ + x₁₃₃ ξ₁₃₁ - e ^{$\frac{1}{2} (-\alpha_1 - \alpha_2)$} η₁₃₂ ξ₁₃₁ +
 e ^{$\frac{1}{2} (b_1 + b_2 - \alpha_1 - \alpha_2)$} η₁₃₂ ξ₁₃₁ + e ^{$\frac{1}{2} (\alpha_1 + \alpha_2)$} x₁₃₃ ξ₁₃₂ + x₂₃₃ ξ₂₃₁ + e ^{$\frac{1}{2} (-\alpha_1 - \alpha_2)$} y₁₂₃ η₁₃₂ ξ₂₃₁ -
 e ^{$\frac{1}{2} (\alpha_1 - 2\alpha_2)$} η₂₃₂ ξ₂₃₁ + e ^{$\frac{1}{2} (-b_1 + 2b_2 + \alpha_1 - 2\alpha_2)$} η₂₃₂ ξ₂₃₁ - e ^{$\frac{1}{2} (-\alpha_1 - \alpha_2)$} η₁₃₂ ξ₁₂₁ ξ₂₃₁ +
 e ^{$\frac{1}{2} (2b_1 - b_2 - \alpha_1 - \alpha_2)$} η₁₃₂ ξ₁₂₁ ξ₂₃₁ - e ^{$\frac{1}{2} (2\alpha_1 - \alpha_2)$} x₁₃₃ ξ₁₂₂ ξ₂₃₁ + e ^{$\frac{1}{2} (-\alpha_1 + 2\alpha_2)$} x₂₃₃ ξ₂₃₂,
 -2 y₁₂₃ β₁₂ η₁₂₁ + 2 e ^{$\frac{1}{2} (-2\alpha_1 + \alpha_2)$} y₁₂₃ β₁₁ η₁₂₂ - 2 y₁₃₃ β₁₂ η₁₃₁ - 2 y₁₃₃ β₂₂ η₁₃₁ +

$$\begin{aligned}
& e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} y_{12_3} y_{13_3} \eta_{12_2} \eta_{13_1} + 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} y_{13_3} \beta_{1_1} \eta_{13_2} + 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} y_{13_3} \beta_{2_1} \eta_{13_2} - \\
& e^{\frac{1}{2}(-\alpha_1-\alpha_2)} y_{12_3} y_{13_3} \eta_{12_1} \eta_{13_2} - 2 y_{23_3} \beta_{2_2} \eta_{23_1} - e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} y_{12_3} y_{23_3} \eta_{12_2} \eta_{23_1} + \\
& e^{\frac{1}{2}(-\alpha_1-\alpha_2)} y_{13_3} y_{23_3} \eta_{13_2} \eta_{23_1} + 2 e^{\frac{1}{2}(\alpha_1-2\alpha_2)} y_{23_3} \beta_{2_1} \eta_{23_2} + 4 e^{\frac{1}{2}(\alpha_1-2\alpha_2)} y_{13_3} \eta_{12_1} \eta_{23_2} + \\
& e^{\frac{1}{2}(\alpha_1-2\alpha_2)} y_{12_3} y_{23_3} \eta_{12_1} \eta_{23_2} - e^{\frac{1}{2}(\alpha_1-2\alpha_2)} y_{13_3} y_{23_3} \eta_{13_1} \eta_{23_2} + 4 x_{12_3} \beta_{1_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} \eta_{12_2} \xi_{12_1} + 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} \eta_{12_2} \xi_{12_1} - 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} a_{1_3} \eta_{12_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} a_{1_3} \eta_{12_2} \xi_{12_1} + 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{12_3} y_{12_3} \eta_{12_2} \xi_{12_1} - 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} \beta_{1_2} \eta_{12_2} \xi_{12_1} + \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} \beta_{1_2} \eta_{12_2} \xi_{12_1} - 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} y_{12_3} \eta_{12_1} \eta_{12_2} \xi_{12_1} + \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-4\alpha_1+2\alpha_2)} y_{12_3} \eta_{12_2}^2 \xi_{12_1} - e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} y_{13_3} \eta_{12_2} \eta_{13_1} \xi_{12_1} + \\
& e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{12_3} y_{13_3} \eta_{13_2} \xi_{12_1} - 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \eta_{13_2} \xi_{12_1} + \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} a_{1_3} y_{23_3} \eta_{13_2} \xi_{12_1} - 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \beta_{1_2} \eta_{13_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \beta_{2_1} \eta_{13_2} \xi_{12_1} - 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{13_3} \eta_{12_1} \eta_{13_2} \xi_{12_1} + \\
& e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{12_3} y_{23_3} \eta_{12_1} \eta_{13_2} \xi_{12_1} - e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_1)} y_{13_3} \eta_{12_2} \eta_{13_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_1)} y_{12_3} y_{23_3} \eta_{12_2} \eta_{13_2} \xi_{12_1} + 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{13_3} y_{23_3} \eta_{13_1} \eta_{13_2} \xi_{12_1} - \\
& e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1-2\alpha_2)} y_{13_3} y_{23_3} \eta_{13_2}^2 \xi_{12_1} + e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} y_{23_3} \eta_{12_2} \eta_{23_1} \xi_{12_1} - \\
& e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3}^2 \eta_{13_2} \eta_{23_1} \xi_{12_1} - e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{12_3} y_{23_3} \eta_{23_2} \xi_{12_1} + \\
& e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \eta_{12_2} \eta_{23_2} \xi_{12_1} + e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_2)} y_{23_3}^2 \eta_{13_2} \eta_{23_2} \xi_{12_1} - \\
& 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{12_3} \eta_{12_2} \xi_{12_1}^2 + 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} x_{12_3} \eta_{12_2} \xi_{12_1}^2 + 2 e^{2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2} \eta_{12_2}^2 \xi_{12_1}^2 - \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-4\alpha_1+2\alpha_2)} \eta_{12_2}^2 \xi_{12_1}^2 - e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} x_{12_3} y_{23_3} \eta_{13_2} \xi_{12_1}^2 - \\
& 3 e^{\frac{1}{2}(4b_{1_3}-2b_{2_3}-3\alpha_1)} y_{23_3} \eta_{12_2} \eta_{13_2} \xi_{12_1}^2 + 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_1)} y_{23_3} \eta_{12_2} \eta_{13_2} \xi_{12_1}^2 + \\
& e^{2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2} y_{23_3}^2 \eta_{13_2}^2 \xi_{12_1}^2 - 2 x_{12_3} \eta_{12_2} \xi_{12_1} \xi_{12_2} - 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3})} x_{12_3} \eta_{12_2} \xi_{12_1} \xi_{12_2} + \\
& 2 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}+\alpha_1-2\alpha_2)} x_{12_3} y_{23_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} + 4 x_{13_3} \beta_{1_2} \xi_{13_1} + 4 x_{13_3} \beta_{2_2} \xi_{13_1} + \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1+\alpha_2)} x_{23_3} \eta_{12_2} \xi_{13_1} + e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{13_3} y_{12_3} \eta_{12_2} \xi_{13_1} - 6 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} \eta_{13_2} \xi_{13_1} + \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} \eta_{13_2} \xi_{13_1} + 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} a_{1_3} \eta_{13_2} \xi_{13_1} - \\
& 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} a_{1_3} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} a_{2_3} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} a_{2_3} \eta_{13_2} \xi_{13_1} + \\
& 4 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{12_3} y_{12_3} \eta_{13_2} \xi_{13_1} + 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{13_3} y_{13_3} \eta_{13_2} \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} x_{23_3} y_{23_3} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} \beta_{1_2} \eta_{13_2} \xi_{13_1} + \\
& 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} \beta_{1_2} \eta_{13_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} \beta_{2_2} \eta_{13_2} \xi_{13_1} + 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} \beta_{2_2} \eta_{13_2} \xi_{13_1} - \\
& e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} y_{12_3} \eta_{12_1} \eta_{13_2} \xi_{13_1} + 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_1)} y_{12_3} \eta_{12_2} \eta_{13_2} \xi_{13_1} + \\
& e^{\frac{1}{2}(b_{1_3}+b_{2_3}-3\alpha_1)} y_{12_3} \eta_{12_2} \eta_{13_2} \xi_{13_1} - 2 e^{-\frac{3\alpha_1}{2}} y_{12_3} \eta_{12_2} \eta_{13_2} \xi_{13_1} - \\
& 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} y_{13_3} \eta_{13_1} \eta_{13_2} \xi_{13_1} + 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-2\alpha_1-2\alpha_2)} y_{13_3} \eta_{13_2}^2 \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-2\alpha_1-2\alpha_2)} y_{12_3} y_{23_3} \eta_{13_2}^2 \xi_{13_1} - e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} y_{23_3} \eta_{13_2} \eta_{23_1} \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{12_3} \eta_{23_2} \xi_{13_1} + e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{13_3} y_{23_3} \eta_{23_2} \xi_{13_1} + 4 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} \eta_{12_2} \eta_{23_2} \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} \eta_{12_2} \eta_{23_2} \xi_{13_1} + 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-3\alpha_2)} y_{23_3} \eta_{13_2} \eta_{23_2} \xi_{13_1} - \\
& e^{\frac{1}{2}(b_{1_3}+b_{2_3}-3\alpha_2)} y_{23_3} \eta_{13_2} \eta_{23_2} \xi_{13_1} - 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{13_3} \eta_{12_2} \xi_{12_1} \xi_{13_1} - \\
& 4 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} + 4 e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} + \\
& 2 e^{\frac{1}{2}(b_{1_3}+b_{2_3}-\alpha_1-\alpha_2)} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} - e^{\frac{1}{2}(2b_{1_3}-b_{2_3}-\alpha_1-\alpha_2)} x_{13_3} y_{23_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} +
\end{aligned}$$

$$\begin{aligned}
& 4 e^{\frac{1}{2} (4 b_1 - 2 b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12} \xi_{13} - 6 e^{\frac{1}{2} (2 b_1 - b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12} \xi_{13} - \\
& 2 e^{\frac{1}{2} (b_1 + b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12} \xi_{13} + 2 e^{\frac{3}{2} (b_1 - a_1)} \eta_{12} \eta_{13} \xi_{12} \xi_{13} + 2 e^{-\frac{3 a_1}{2}} \eta_{12} \eta_{13} \xi_{12} \xi_{13} - \\
& 3 e^{\frac{1}{2} (3 b_1 - 2 a_1 - 2 a_2)} y_{23} \eta_{13}^2 \xi_{12} \xi_{13} + 4 e^{\frac{1}{2} (2 b_1 - b_2 - 2 a_1 - 2 a_2)} y_{23} \eta_{13}^2 \xi_{12} \xi_{13} - \\
& 4 e^{2 b_1 - b_2 - a_1 - a_2} y_{23} \eta_{13}^2 \xi_{12} \xi_{13} - 2 e^{\frac{1}{2} (2 a_1 - a_2)} x_{12} x_{13} \xi_{12} \xi_{13} - \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2)} x_{13} \eta_{12} \xi_{12} \xi_{13} - e^{\frac{1}{2} (a_1 - 2 a_2)} x_{12} \eta_{13} \xi_{12} \xi_{13} - \\
& e^{\frac{1}{2} (b_1 + b_2 + a_1 - 2 a_2)} x_{12} \eta_{13} \xi_{12} \xi_{13} + 4 e^{\frac{1}{2} (2 b_1 - b_2 + a_1 - 2 a_2)} x_{13} y_{23} \eta_{13} \xi_{12} \xi_{13} - \\
& 2 e^{\frac{1}{2} (-a_1 - a_2)} x_{13} \eta_{13} \xi_{13}^2 + 2 e^{\frac{1}{2} (b_1 + b_2 - a_1 - a_2)} x_{13} \eta_{13} \xi_{13}^2 - \\
& 2 e^{\frac{1}{2} (b_1 + b_2 - 2 a_1 - 2 a_2)} \eta_{13}^2 \xi_{13}^2 + 2 e^{b_1 + b_2 - a_1 - a_2} \eta_{13}^2 \xi_{13}^2 - e^{\frac{1}{2} (-a_1 + 2 a_2)} x_{13} \eta_{12} \xi_{12} \xi_{13} - \\
& e^{\frac{1}{2} (2 b_1 - b_2 - a_1 + 2 a_2)} x_{13} \eta_{12} \xi_{12} \xi_{13} + e^{\frac{1}{2} (2 b_1 - b_2)} x_{13} y_{23} \eta_{13} \xi_{12} \xi_{13} - \\
& 2 x_{13} \eta_{13} \xi_{13} \xi_{13} - 2 e^{\frac{1}{2} (b_1 + b_2)} x_{13} \eta_{13} \xi_{13} \xi_{13} + 4 x_{23} \beta_2 \xi_{23} - \\
& e^{\frac{1}{2} (-2 a_1 + a_2)} x_{23} y_{12} \eta_{12} \xi_{23} + 3 e^{\frac{1}{2} (-a_1 - a_2)} y_{12} \eta_{13} \xi_{23} + 2 e^{\frac{1}{2} (-a_1 - a_2)} a_2 y_{12} \eta_{13} \xi_{23} + \\
& e^{\frac{1}{2} (-a_1 - a_2)} x_{23} y_{13} \eta_{13} \xi_{23} + 2 e^{\frac{1}{2} (-a_1 - a_2)} y_{12} \beta_1 \eta_{13} \xi_{23} + 2 e^{\frac{1}{2} (-a_1 - a_2)} y_{12} \beta_2 \eta_{13} \xi_{23} + \\
& e^{\frac{1}{2} (-a_1 - a_2)} y_{12} y_{13} \eta_{13} \eta_{13} \xi_{23} - e^{\frac{1}{2} (-a_1 - a_2)} y_{12} y_{23} \eta_{13} \eta_{23} \xi_{23} - \\
& 2 e^{\frac{1}{2} (a_1 - 2 a_2)} \eta_{23} \xi_{23} + 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} \eta_{23} \xi_{23} - 2 e^{\frac{1}{2} (a_1 - 2 a_2)} a_2 \eta_{23} \xi_{23} - \\
& 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} a_2 \eta_{23} \xi_{23} + 2 e^{\frac{1}{2} (a_1 - 2 a_2)} x_{23} y_{23} \eta_{23} \xi_{23} - 2 e^{\frac{1}{2} (a_1 - 2 a_2)} \beta_2 \eta_{23} \xi_{23} + \\
& 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} \beta_2 \eta_{23} \xi_{23} + e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} y_{12} \eta_{12} \eta_{23} \xi_{23} - \\
& 2 e^{\frac{1}{2} (-a_1 - a_2)} y_{12} \eta_{12} \eta_{23} \xi_{23} - e^{\frac{1}{2} (-b_1 + 2 b_2 - a_1 - a_2)} y_{12} \eta_{12} \eta_{23} \xi_{23} - \\
& e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} y_{13} \eta_{13} \eta_{23} \xi_{23} + e^{\frac{1}{2} (-b_1 + 2 b_2 - 3 a_2)} y_{13} \eta_{13} \eta_{23} \xi_{23} + \\
& 2 e^{-\frac{3 a_2}{2}} y_{13} \eta_{13} \eta_{23} \xi_{23} - 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} y_{23} \eta_{23} \eta_{23} \xi_{23} + \\
& 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + 2 a_1 - 4 a_2)} y_{23} \eta_{23}^2 \xi_{23} + 2 e^{\frac{1}{2} (-2 a_1 + a_2)} x_{23} \eta_{12} \xi_{12} \xi_{23} - \\
& 5 e^{\frac{1}{2} (-a_1 - a_2)} \eta_{13} \xi_{12} \xi_{23} + 5 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} a_1 \eta_{13} \xi_{12} \xi_{23} - \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} a_1 \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} a_2 \eta_{13} \xi_{12} \xi_{23} + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} a_2 \eta_{13} \xi_{12} \xi_{23} + 2 e^{\frac{1}{2} (-a_1 - a_2)} x_{12} y_{12} \eta_{13} \xi_{12} \xi_{23} - \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} x_{23} y_{23} \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} \beta_1 \eta_{13} \xi_{12} \xi_{23} + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} \beta_1 \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} \beta_2 \eta_{13} \xi_{12} \xi_{23} + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} \beta_2 \eta_{13} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} y_{12} \eta_{12} \eta_{13} \xi_{12} \xi_{23} + \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2 - 3 a_1)} y_{12} \eta_{12} \eta_{13} \xi_{12} \xi_{23} - e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} y_{13} \eta_{13} \eta_{13} \xi_{12} \xi_{23} - \\
& e^{\frac{1}{2} (2 b_1 - b_2 - 2 a_1 - 2 a_2)} y_{13} \eta_{13}^2 \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (2 b_1 - b_2 - 2 a_1 - 2 a_2)} y_{12} y_{23} \eta_{13}^2 \xi_{12} \xi_{23} + \\
& e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} y_{23} \eta_{13} \eta_{23} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + a_1 - 2 a_2)} x_{12} \eta_{23} \xi_{12} \xi_{23} + \\
& 2 e^{\frac{1}{2} (-a_1 - a_2)} \eta_{12} \eta_{23} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} \eta_{12} \eta_{23} \xi_{12} \xi_{23} - \\
& 2 e^{\frac{1}{2} (b_1 + b_2 - a_1 - a_2)} \eta_{12} \eta_{23} \xi_{12} \xi_{23} + 2 e^{\frac{1}{2} (-b_1 + 2 b_2 - a_1 - a_2)} \eta_{12} \eta_{23} \xi_{12} \xi_{23} - \\
& e^{\frac{1}{2} (2 b_1 - b_2 - 3 a_2)} y_{23} \eta_{13} \eta_{23} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (-a_1 - a_2)} x_{12} \eta_{13} \xi_{12}^2 \xi_{23} + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - a_1 - a_2)} x_{12} \eta_{13} \xi_{12}^2 \xi_{23} + 4 e^{\frac{1}{2} (4 b_1 - 2 b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12}^2 \xi_{23} - \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2 - 3 a_1)} \eta_{12} \eta_{13} \xi_{12}^2 \xi_{23} + 2 e^{\frac{1}{2} (2 b_1 - b_2 - 2 a_1 - 2 a_2)} y_{23} \eta_{13}^2 \xi_{12}^2 \xi_{23} - \\
& 3 e^{2 b_1 - b_2 - a_1 - a_2} y_{23} \eta_{13}^2 \xi_{12}^2 \xi_{23} - e^{\frac{1}{2} (2 a_1 - a_2)} x_{13} \xi_{12} \xi_{23} + 2 e^{\frac{1}{2} (2 a_1 - a_2)} x_{12} x_{23} \xi_{12} \xi_{23} - \\
& 4 e^{\frac{1}{2} (2 a_1 - a_2)} x_{13} \beta_2 \xi_{12} \xi_{23} + x_{13} y_{12} \eta_{12} \xi_{12} \xi_{23} - e^{\frac{1}{2} (a_1 - 2 a_2)} x_{12} y_{12} \eta_{13} \xi_{12} \xi_{23} -
\end{aligned}$$

$$\begin{aligned}
& e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{13_3} y_{13_3} \eta_{13_2} \xi_{12_2} \xi_{23_1} + e^{\frac{1}{2}(-b_1+2b_2+3\alpha_1-3\alpha_2)} x_{12_3} \eta_{23_2} \xi_{12_2} \xi_{23_1} + \\
& e^{\frac{3}{2}(\alpha_1-\alpha_2)} x_{12_3} \eta_{23_2} \xi_{12_2} \xi_{23_1} - 2 e^{\frac{3}{2}(\alpha_1-\alpha_2)} x_{13_3} y_{23_3} \eta_{23_2} \xi_{12_2} \xi_{23_1} + \\
& 2 e^{\frac{1}{2}(2b_1-b_2)} x_{13_3} \eta_{12_2} \xi_{12_1} \xi_{12_2} \xi_{23_1} - e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} \xi_{23_1} - \\
& 3 e^{\frac{1}{2}(2b_1-b_2+\alpha_1-2\alpha_2)} x_{12_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} \xi_{23_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{23_3} \eta_{13_2} \xi_{13_1} \xi_{23_1} + \\
& 4 e^{\frac{1}{2}(2b_1-b_2-\alpha_1-\alpha_2)} x_{23_3} \eta_{13_2} \xi_{13_1} \xi_{23_1} + e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{13_3} y_{12_3} \eta_{13_2} \xi_{13_1} \xi_{23_1} + \\
& 4 e^{\frac{1}{2}(2b_1-b_2-2\alpha_1-2\alpha_2)} y_{12_3} \eta_{13_2}^2 \xi_{13_1} \xi_{23_1} + e^{\frac{1}{2}(b_1+b_2-2\alpha_1-2\alpha_2)} y_{12_3} \eta_{13_2}^2 \xi_{13_1} \xi_{23_1} - \\
& 2 e^{-\alpha_1-\alpha_2} y_{12_3} \eta_{13_2}^2 \xi_{13_1} \xi_{23_1} + 2 e^{\frac{1}{2}(-b_1+2b_2+\alpha_1-2\alpha_2)} x_{13_3} \eta_{23_2} \xi_{13_1} \xi_{23_1} - \\
& 4 e^{\frac{1}{2}(2b_1-b_2-3\alpha_2)} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} + 2 e^{\frac{1}{2}(b_1+b_2-3\alpha_2)} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} - \\
& 2 e^{\frac{1}{2}(-b_1+2b_2-3\alpha_2)} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} + 2 e^{\frac{3}{2}(b_2-\alpha_2)} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} + \\
& 2 e^{-\frac{3\alpha_2}{2}} \eta_{13_2} \eta_{23_2} \xi_{13_1} \xi_{23_1} - 2 e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{13_3} \eta_{13_2} \xi_{12_1} \xi_{13_1} \xi_{23_1} + \\
& 2 e^{\frac{1}{2}(3b_1-2\alpha_1-2\alpha_2)} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} - 6 e^{\frac{1}{2}(2b_1-b_2-2\alpha_1-2\alpha_2)} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} - \\
& 2 e^{\frac{1}{2}(b_1+b_2-2\alpha_1-2\alpha_2)} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} + 2 e^{-\alpha_1-\alpha_2} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} + \\
& 4 e^{2b_1-b_2-\alpha_1-\alpha_2} \eta_{13_2}^2 \xi_{12_1} \xi_{13_1} \xi_{23_1} + 3 e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{13_3} \eta_{13_2} \xi_{12_2} \xi_{13_1} \xi_{23_1} - \\
& 4 e^{\frac{1}{2}(2b_1-b_2+\alpha_1-2\alpha_2)} x_{13_3} \eta_{13_2} \xi_{12_2} \xi_{13_1} \xi_{23_1} + e^{\frac{1}{2}(b_1+b_2+\alpha_1-2\alpha_2)} x_{13_3} \eta_{13_2} \xi_{12_2} \xi_{13_1} \xi_{23_1} - \\
& 2 e^{\frac{1}{2}(\alpha_1+\alpha_2)} x_{13_3} x_{23_3} \xi_{13_2} \xi_{23_1} + x_{13_3} y_{12_3} \eta_{13_2} \xi_{13_2} \xi_{23_1} - e^{\frac{1}{2}(2\alpha_1-\alpha_2)} x_{13_3} \eta_{23_2} \xi_{13_2} \xi_{23_1} - \\
& e^{\frac{1}{2}(-b_1+2b_2+2\alpha_1-\alpha_2)} x_{13_3} \eta_{23_2} \xi_{13_2} \xi_{23_1} - 2 x_{13_3} \eta_{13_2} \xi_{12_1} \xi_{13_2} \xi_{23_1} + \\
& e^{\frac{1}{2}(-\alpha_1-\alpha_2)} x_{23_3} y_{12_3} \eta_{13_2} \xi_{23_1}^2 - 2 e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{23_3} \eta_{23_2} \xi_{23_1}^2 + 2 e^{\frac{1}{2}(-b_1+2b_2+\alpha_1-2\alpha_2)} x_{23_3} \eta_{23_2} \xi_{23_1}^2 + \\
& e^{\frac{1}{2}(-b_1+2b_2-3\alpha_2)} y_{12_3} \eta_{13_2} \eta_{23_2} \xi_{23_1}^2 - 2 e^{\frac{1}{2}(-b_1+2b_2+2\alpha_1-4\alpha_2)} \eta_{23_2}^2 \xi_{23_1}^2 + \\
& 2 e^{-b_1+2b_2+\alpha_1-2\alpha_2} \eta_{23_2}^2 \xi_{23_1}^2 + 2 e^{\frac{1}{2}(2b_1-b_2-\alpha_1-\alpha_2)} x_{23_3} \eta_{13_2} \xi_{12_1} \xi_{23_1}^2 + \\
& 2 e^{\frac{1}{2}(2b_1-b_2-2\alpha_1-2\alpha_2)} y_{12_3} \eta_{13_2}^2 \xi_{12_1} \xi_{23_1}^2 - 2 e^{\frac{1}{2}(2b_1-b_2-2\alpha_1-2\alpha_2)} \eta_{13_2}^2 \xi_{12_1}^2 \xi_{23_1}^2 + \\
& 2 e^{2b_1-b_2-\alpha_1-\alpha_2} \eta_{13_2}^2 \xi_{12_1}^2 \xi_{23_1}^2 - 3 e^{\frac{1}{2}(-b_1+2b_2+3\alpha_1-3\alpha_2)} x_{13_3} \eta_{23_2} \xi_{12_2} \xi_{23_1}^2 + \\
& e^{\frac{3}{2}(\alpha_1-\alpha_2)} x_{13_3} \eta_{23_2} \xi_{12_2} \xi_{23_1}^2 + e^{\frac{1}{2}(\alpha_1-2\alpha_2)} x_{13_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} \xi_{23_1}^2 + \\
& e^{\frac{1}{2}(2b_1-b_2+\alpha_1-2\alpha_2)} x_{13_3} \eta_{13_2} \xi_{12_1} \xi_{12_2} \xi_{23_1}^2 + e^{-\frac{3}{2}(\alpha_1-\alpha_2)} x_{23_3} \eta_{12_2} \xi_{12_1} \xi_{23_2} + \\
& e^{\frac{1}{2}(2b_1-b_2-3\alpha_1+3\alpha_2)} x_{23_3} \eta_{12_2} \xi_{12_1} \xi_{23_2} - e^{\frac{1}{2}(2b_1-b_2-2\alpha_1+\alpha_2)} x_{23_3} y_{23_3} \eta_{13_2} \xi_{12_1} \xi_{23_2} - \\
& e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{23_3} \eta_{13_2} \xi_{13_1} \xi_{23_2} - e^{\frac{1}{2}(b_1+b_2-2\alpha_1+\alpha_2)} x_{23_3} \eta_{13_2} \xi_{13_1} \xi_{23_2} + \\
& 2 e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{23_3} y_{12_3} \eta_{13_2} \xi_{23_1} \xi_{23_2} - 2 x_{23_3} \eta_{23_2} \xi_{23_1} \xi_{23_2} - 2 e^{\frac{1}{2}(-b_1+2b_2)} x_{23_3} \eta_{23_2} \xi_{23_1} \xi_{23_2} - \\
& e^{\frac{1}{2}(-2\alpha_1+\alpha_2)} x_{23_3} \eta_{13_2} \xi_{12_1} \xi_{23_1} \xi_{23_2} + 3 e^{\frac{1}{2}(2b_1-b_2-2\alpha_1+\alpha_2)} x_{23_3} \eta_{13_2} \xi_{12_1} \xi_{23_1} \xi_{23_2} \Big]
\end{aligned}$$

associativity

$$In[*]:= (dm_{1,2 \rightarrow k} // dm_{k,3 \rightarrow k}) \equiv (dm_{2,3 \rightarrow k} // dm_{1,k \rightarrow k})$$

Time: 2.98 seconds Out[]=

True

co-associativity

$$In[*]:= (d\Delta_{1 \rightarrow 1,2} // d\Delta_{2 \rightarrow 2,3}) \equiv (d\Delta_{1 \rightarrow 2,3} // d\Delta_{2 \rightarrow 1,2})$$

Time: 1.19 seconds Out[]=

True

algebra morphism

$\text{In}[*]:= (\mathbf{d}\Delta_{1 \rightarrow 1,2} \mathbf{d}\Delta_{2 \rightarrow 3,4} // (\mathbf{d}\mathbf{m}_{1,3 \rightarrow 1} \mathbf{d}\mathbf{m}_{2,4 \rightarrow 2})) \equiv (\mathbf{d}\mathbf{m}_{1,2 \rightarrow 3} // \mathbf{d}\Delta_{3 \rightarrow 1,2})$

Time: 10.83 seconds Out[]=

True

antipode

$\text{In}[*]:= \{\mathbf{d}\Delta_{i \rightarrow 1,2} // \mathbf{d}\mathbf{S}_1 // \mathbf{d}\mathbf{m}_{1,2 \rightarrow 1}, \mathbf{d}\Delta_{i \rightarrow 1,2} // \mathbf{d}\mathbf{S}_2 // \mathbf{d}\mathbf{m}_{1,2 \rightarrow 1}, \mathbf{d}\mathbf{S}_1 // \overline{\mathbf{d}\mathbf{S}_1}\}$

Time: 40.88 seconds Out[]=

$$\left\{ \mathbb{E}_{\{i\} \rightarrow \{1\}} \left[0, 2 e^{-\frac{3}{2} (b_{11} - \alpha_{11})} y_{12_1} y_{13_1} \eta_{12_i} \eta_{13_i} - 2 e^{\frac{3}{2} (-b_{11} + \alpha_{11})} y_{12_1} y_{13_1} \eta_{12_i} \eta_{13_i} - \right. \right. \\ \left. 2 e^{\frac{3}{2} (b_{21} - \alpha_{21})} y_{13_1} y_{23_1} \eta_{13_i} \eta_{23_i} + 2 e^{\frac{3}{2} (-b_{21} + \alpha_{21})} y_{13_1} y_{23_1} \eta_{13_i} \eta_{23_i} - \right. \\ \left. e^{-\frac{3}{2} (b_{21} - \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i} \eta_{23_i} + e^{\frac{3}{2} (-b_{21} + \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i} \eta_{23_i} - \right. \\ \left. 3 e^{\frac{3}{2} (b_{21} - \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i}^2 \xi_{12_i} + 3 e^{\frac{3}{2} (-b_{21} + \alpha_{21})} y_{12_1} y_{23_1}^2 \eta_{13_i}^2 \xi_{12_i} - \right. \\ \left. 2 e^{\frac{3}{2} (b_{11} - \alpha_{11})} y_{12_1} y_{13_1} \eta_{13_i}^2 \xi_{23_i} + 2 e^{\frac{3}{2} (-b_{11} + \alpha_{11})} y_{12_1} y_{13_1} \eta_{13_i}^2 \xi_{23_i} \right], \\ \mathbb{E}_{\{i\} \rightarrow \{1\}} [0, 0], \mathbb{E}_{\{1\} \rightarrow \{1\}} [a_{11} \alpha_{11} + a_{21} \alpha_{21} + b_{11} \beta_{11} + b_{21} \beta_{21} + y_{12_1} \eta_{12_1} + \\ y_{13_1} \eta_{13_1} + y_{23_1} \eta_{23_1} + x_{12_1} \xi_{12_1} + x_{13_1} \xi_{13_1} + x_{23_1} \xi_{23_1}, 0] \left. \right\}$$

quasi-triangular axioms

$\text{In}[*]:= (\mathbf{R}_{1,3} // \mathbf{d}\Delta_{1 \rightarrow 1,2}) \equiv (\mathbf{R}_{1,3} \mathbf{R}_{2,4} // \mathbf{d}\mathbf{m}_{3,4 \rightarrow 3})$
 $(\mathbf{R}_{1,3} // \mathbf{d}\Delta_{3 \rightarrow 2,3}) \equiv (\mathbf{R}_{1,3} \mathbf{R}_{0,2} // \mathbf{d}\mathbf{m}_{1,0 \rightarrow 1})$
 $(\mathbf{d}\Delta_{i \rightarrow k,j} \mathbf{R}_{1,2} // \mathbf{d}\mathbf{m}_{j,1 \rightarrow 1} // \mathbf{d}\mathbf{m}_{k,2 \rightarrow 2}) \equiv (\mathbf{R}_{1,2} \mathbf{d}\Delta_{i \rightarrow j,k} // \mathbf{d}\mathbf{m}_{1,j \rightarrow 1} // \mathbf{d}\mathbf{m}_{2,k \rightarrow 2})$

Time: 3.17 seconds Out[]=

True

Out[]=

True

Out[]=

True

$\text{In}[*]:= (\mathbf{R}_{1,2} // \mathbf{aSt}_2) \equiv (\overline{\mathbf{R}}_{1,2})$

Time: 0.05 seconds Out[]=

True

Braid-like Reidemeister moves: R2,R3

$\text{In}[*]:= \{\mathbf{R}_{1,2} \overline{\mathbf{R}}_{3,4} // \mathbf{d}\mathbf{m}_{1,3 \rightarrow 1} // \mathbf{d}\mathbf{m}_{2,4 \rightarrow 2}, \overline{\mathbf{R}}_{1,2} \mathbf{R}_{3,4} // \mathbf{d}\mathbf{m}_{1,3 \rightarrow 1} // \mathbf{d}\mathbf{m}_{2,4 \rightarrow 2},$
 $(\mathbf{R}_{1,2} \mathbf{R}_{4,3} \mathbf{R}_{5,6} // \mathbf{d}\mathbf{m}_{1,4 \rightarrow 1} // \mathbf{d}\mathbf{m}_{2,5 \rightarrow 5} // \mathbf{d}\mathbf{m}_{3,6 \rightarrow 3}) \equiv (\mathbf{R}_{1,6} \mathbf{R}_{2,3} \mathbf{R}_{4,5} // \mathbf{d}\mathbf{m}_{1,4 \rightarrow 1} // \mathbf{d}\mathbf{m}_{2,5 \rightarrow 5} // \mathbf{d}\mathbf{m}_{3,6 \rightarrow 3}) \}$

Time: 4.00 seconds Out[]=

$\{\mathbb{E}_{\{i\} \rightarrow \{1,2\}} [0, 0], \mathbb{E}_{\{i\} \rightarrow \{1,2\}} [0, 0], \text{True}\}$

Drinfeld element:

$\text{In}[*]:= \$k = 0; \$n = 2;$

$\text{In}[*]:= \mathbf{u}_{i_} := \text{Module}[\{\mathbf{i1}, \mathbf{i2}\}, \mathbf{R}_{\mathbf{i1}, \mathbf{i2}} // \mathbf{d}\mathbf{S}_{\mathbf{i2}} // \mathbf{d}\mathbf{m}_{\mathbf{i2}, \mathbf{i1} \rightarrow \mathbf{i}}]$

```

In[*]:= u2
Time: 0.00 seconds Out|=

$$E_{\{\} \rightarrow \{2\}} [\mathbf{0}, \mathbf{0}]$$


In[*]:= Ri1,i2 // dSi2
» TTL  $e^{-\gamma \alpha_{1i}} \xi_{12i} U_j[x_{12}] U_k[] + e^{-\gamma \alpha_{1i}} \xi_{12i} U_j[] U_k[x_{12}]$ 
» SG  $U_j[] U_k[]$ 
» TTR  $e^{-\gamma \alpha_{1i}} U_j[x_{12}] U_k[] f_{\theta, x_{12j}}'[\lambda] + e^{-\gamma \alpha_{1i}} U_j[] U_k[x_{12}] f_{\theta, x_{12k}}'[\lambda]$ 
Out[*]=

$$E_{\{\} \rightarrow \{i1, i2\}} [-a_{1i2} b_{1i1} - e^{-b_{1i1}} x_{12i2} y_{12i1}]$$


In[*]:= dSi2
Time: 0.05 seconds Out|=

$$E_{\{i2\} \rightarrow \{i2\}} [-a_{1i2} \alpha_{1i2} - b_{1i2} \beta_{1i2} - e^{-b_{1i2} + \alpha_{1i2}} y_{12i2} \eta_{12i2} - e^{-\alpha_{1i2}} x_{12i2} \xi_{12i2} + \eta_{12i2} \xi_{12i2} - e^{-b_{1i2}} \eta_{12i2} \xi_{12i2}]$$


In[*]:= dmi2,i1-i // Simplify
Time: 0.17 seconds Out|=

$$E_{\{i2, i1\} \rightarrow \{i\}} [a_{1i} (\alpha_{1i1} + \alpha_{1i2}) + b_{1i} (\beta_{1i1} + \beta_{1i2}) + e^{-\alpha_{1i2}} (y_{12i} (\eta_{12i1} + e^{\alpha_{1i2}} \eta_{12i2}) + (-1 + e^{b_{1i}}) \eta_{12i1} \xi_{12i2} + e^{\alpha_{1i2}} x_{12i} (e^{\alpha_{1i2}} \xi_{12i1} + \xi_{12i2}))]$$


In[*]:= E\{\} \rightarrow \{1,2\} [x12 y12] // E\{1,2\} \rightarrow \{\} [(-1 + eb1i)  $\eta_{12i} \xi_{12i}$ ]
Time: 8.47 seconds Out|=
$Aborted

In[*]:= dmi2,i1-i
Time: 0.16 seconds Out|=

$$E_{\{i2, i1\} \rightarrow \{i\}} [a_{1i} \alpha_{1i1} + a_{1i} \alpha_{1i2} + a_{2i} \alpha_{2i1} + a_{2i} \alpha_{2i2} + b_{1i} \beta_{1i1} + b_{1i} \beta_{1i2} + b_{2i} \beta_{2i1} + b_{2i} \beta_{2i2} + e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} y_{12i} \eta_{12i1} + y_{12i} \eta_{12i2} + e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} y_{13i} \eta_{13i1} + y_{13i} \eta_{13i2} + e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} y_{23i} \eta_{23i1} + y_{23i} \eta_{23i2} + e^{\frac{1}{2}(2\alpha_{1i2} - \alpha_{2i2})} x_{12i} \xi_{12i1} + x_{12i} \xi_{12i2} - e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \eta_{12i1} \xi_{12i2} + e^{\frac{1}{2}(2b_{1i} - b_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} \eta_{12i1} \xi_{12i2} - e^{\frac{1}{2}(2b_{1i} - b_{2i} - \alpha_{1i2} - \alpha_{2i2})} y_{23i} \eta_{13i1} \xi_{12i2} + e^{\frac{1}{2}(\alpha_{1i2} + \alpha_{2i2})} x_{13i} \xi_{13i1} + x_{13i} \xi_{13i2} - e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \eta_{13i1} \xi_{13i2} + e^{\frac{1}{2}(b_{1i} + b_{2i} - \alpha_{1i2} - \alpha_{2i2})} \eta_{13i1} \xi_{13i2} + e^{\frac{1}{2}(-\alpha_{1i2} + 2\alpha_{2i2})} x_{23i} \xi_{23i1} + x_{23i} \xi_{23i2} + e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} y_{12i} \eta_{13i1} \xi_{23i2} - e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} \eta_{23i1} \xi_{23i2} + e^{\frac{1}{2}(-b_{1i} + 2b_{2i} + \alpha_{1i2} - 2\alpha_{2i2})} \eta_{23i1} \xi_{23i2} - e^{\frac{1}{2}(2\alpha_{1i2} - \alpha_{2i2})} x_{13i} \xi_{12i1} \xi_{23i2} - e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \eta_{13i1} \xi_{12i2} \xi_{23i2} + e^{\frac{1}{2}(2b_{1i} - b_{2i} - \alpha_{1i2} - \alpha_{2i2})} \eta_{13i1} \xi_{12i2} \xi_{23i2}, 2e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} y_{12i} \beta_{1i2} \eta_{12i1} - 2y_{12i} \beta_{1i1} \eta_{12i2} + 2e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} y_{13i} \beta_{1i2} \eta_{13i1} + 2e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} y_{13i} \beta_{2i2} \eta_{13i1} - e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} y_{12i} y_{13i} \eta_{12i2} \eta_{13i1} - 2y_{13i} \beta_{1i1} \eta_{13i2} - 2y_{13i} \beta_{2i1} \eta_{13i2} + e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} y_{12i} y_{13i} \eta_{12i1} \eta_{13i2} + 2e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} y_{23i} \beta_{2i2} \eta_{23i1} + 4e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} y_{13i} \eta_{12i2} \eta_{23i1} + e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} y_{12i} y_{23i} \eta_{12i2} \eta_{23i1} - e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} y_{13i} y_{23i} \eta_{13i2} \eta_{23i1} - 2y_{23i} \beta_{2i1} \eta_{23i2} - e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} y_{12i} y_{23i} \eta_{12i1} \eta_{23i2} + e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} y_{13i} y_{23i} \eta_{13i1} \eta_{23i2} + 4x_{12i} \beta_{1i1} \xi_{12i2} - 2e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \eta_{12i1} \xi_{12i2} - 2e^{\frac{1}{2}(2b_{1i} - b_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} \eta_{12i1} \xi_{12i2} - 2e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} a_{1i} \eta_{12i1} \xi_{12i2} - 2e^{\frac{1}{2}(2b_{1i} - b_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} a_{1i} \eta_{12i1} \xi_{12i2} + 2e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} x_{12i} y_{12i} \eta_{12i1} \xi_{12i2} -$$


```

$$\begin{aligned}
& 2 e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} \beta_{1i1} \eta_{12i1} \xi_{12i2} + 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} \beta_{1i1} \eta_{12i1} \xi_{12i2} + \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-4\alpha_{1i2}+2\alpha_{2i2})} y_{12i} \eta_{12i1}^2 \xi_{12i2} - 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} y_{12i} \eta_{12i1} \eta_{12i2} \xi_{12i2} + \\
& e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{12i} y_{13i} \eta_{13i1} \xi_{12i2} - 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{23i} \eta_{13i1} \xi_{12i2} + \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} a_{1i} y_{23i} \eta_{13i1} \xi_{12i2} - 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{23i} \beta_{1i1} \eta_{13i1} \xi_{12i2} - \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{23i} \beta_{2i2} \eta_{13i1} \xi_{12i2} - e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2})} y_{13i} \eta_{12i1} \eta_{13i1} \xi_{12i2} - \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2})} y_{12i} y_{23i} \eta_{12i1} \eta_{13i1} \xi_{12i2} - 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{13i} \eta_{12i2} \eta_{13i1} \xi_{12i2} + \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{12i} y_{23i} \eta_{12i2} \eta_{13i1} \xi_{12i2} - e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{13i} y_{23i} \eta_{13i1}^2 \xi_{12i2} - \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} y_{13i} \eta_{12i1} \eta_{13i2} \xi_{12i2} + 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{13i} y_{23i} \eta_{13i1} \eta_{13i2} \xi_{12i2} - \\
& e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{12i} y_{23i} \eta_{23i1} \xi_{12i2} + e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{23i} \eta_{12i1} \eta_{23i1} \xi_{12i2} + \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{2i2})} y_{23i}^2 \eta_{13i1} \eta_{23i1} \xi_{12i2} + e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} y_{23i} \eta_{12i1} \eta_{23i2} \xi_{12i2} - \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{23i}^2 \eta_{13i1} \eta_{23i2} \xi_{12i2} - 2 x_{12i} \eta_{12i1} \xi_{12i1} \xi_{12i2} - \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i})} x_{12i} \eta_{12i1} \xi_{12i1} \xi_{12i2} + 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} x_{12i} y_{23i} \eta_{13i1} \xi_{12i1} \xi_{12i2} - \\
& 2 e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} x_{12i} \eta_{12i1} \xi_{12i2}^2 + 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} x_{12i} \eta_{12i1} \xi_{12i2}^2 + \\
& 2 e^{2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2}} \eta_{12i1}^2 \xi_{12i2}^2 - 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-4\alpha_{1i2}+2\alpha_{2i2})} \eta_{12i1}^2 \xi_{12i2}^2 - \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{12i} y_{23i} \eta_{13i1} \xi_{12i2}^2 - 3 e^{\frac{1}{2}(4b_{1i}-2b_{2i}-3\alpha_{1i2})} y_{23i} \eta_{12i1} \eta_{13i1} \xi_{12i2}^2 + \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2})} y_{23i} \eta_{12i1} \eta_{13i1} \xi_{12i2}^2 + e^{2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2}} y_{23i}^2 \eta_{13i1}^2 \xi_{12i2}^2 - \\
& e^{\frac{1}{2}(-\alpha_{1i2}+2\alpha_{2i2})} x_{13i} \eta_{12i1} \xi_{12i2} \xi_{13i1} - e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}+2\alpha_{2i2})} x_{13i} \eta_{12i1} \xi_{12i2} \xi_{13i1} + \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i})} x_{13i} y_{23i} \eta_{13i1} \xi_{12i2} \xi_{13i1} + 4 x_{13i} \beta_{1i1} \xi_{13i2} + 4 x_{13i} \beta_{2i1} \xi_{13i2} + \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} x_{23i} \eta_{12i1} \xi_{13i2} + e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} x_{13i} y_{12i} \eta_{12i1} \xi_{13i2} - \\
& 6 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \eta_{13i1} \xi_{13i2} + 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{13i1} \xi_{13i2} + 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} a_{1i} \eta_{13i1} \xi_{13i2} - 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} a_{1i} \eta_{13i1} \xi_{13i2} - 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} a_{2i} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} a_{2i} \eta_{13i1} \xi_{13i2} + 4 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{12i} y_{12i} \eta_{13i1} \xi_{13i2} + \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{13i} y_{13i} \eta_{13i1} \xi_{13i2} - 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{23i} y_{23i} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \beta_{1i1} \eta_{13i1} \xi_{13i2} + 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \beta_{1i1} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \beta_{2i1} \eta_{13i1} \xi_{13i2} + 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \beta_{2i1} \eta_{13i1} \xi_{13i2} + \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2})} y_{12i} \eta_{12i1} \eta_{13i1} \xi_{13i2} + e^{\frac{1}{2}(b_{1i}+b_{2i}-3\alpha_{1i2})} y_{12i} \eta_{12i1} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{-\frac{3\alpha_{1i2}}{2}} y_{12i} \eta_{12i1} \eta_{13i1} \xi_{13i2} - e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{12i} \eta_{12i2} \eta_{13i1} \xi_{13i2} + \\
& 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{13i} \eta_{13i1}^2 \xi_{13i2} - 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{12i} y_{23i} \eta_{13i1}^2 \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{13i} \eta_{13i1} \eta_{13i2} \xi_{13i2} - 4 e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{12i} \eta_{23i1} \xi_{13i2} + \\
& e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{13i} y_{23i} \eta_{23i1} \xi_{13i2} + 4 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \eta_{12i1} \eta_{23i1} \xi_{13i2} - \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{12i1} \eta_{23i1} \xi_{13i2} + 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{2i2})} y_{23i} \eta_{13i1} \eta_{23i1} \xi_{13i2} - \\
& e^{\frac{1}{2}(b_{1i}+b_{2i}-3\alpha_{2i2})} y_{23i} \eta_{13i1} \eta_{23i1} \xi_{13i2} - e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{23i} \eta_{13i1} \eta_{23i2} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(2\alpha_{1i2}-\alpha_{2i2})} x_{12i} x_{13i} \xi_{12i1} \xi_{13i2} - 4 e^{\frac{1}{2}(2b_{1i}-b_{2i})} x_{13i} \eta_{12i1} \xi_{12i1} \xi_{13i2} - \\
& e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i1} \xi_{13i2} - e^{\frac{1}{2}(b_{1i}+b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i1} \xi_{13i2} + \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} x_{13i} y_{23i} \eta_{13i1} \xi_{12i1} \xi_{13i2} - 2 e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} x_{13i} \eta_{12i1} \xi_{12i2} \xi_{13i2} - \\
& 4 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i2} \xi_{13i2} + 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i2} \xi_{13i2} +
\end{aligned}$$

$$\begin{aligned}
& 2 e^{\frac{1}{2} (b_1+b_2-\alpha_{12}-\alpha_{22})} x_{12_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{13_{12}} - e^{\frac{1}{2} (2 b_1-b_2-\alpha_{12}-\alpha_{22})} x_{13_i} y_{23_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{13_{12}} + \\
& 4 e^{\frac{1}{2} (4 b_1-2 b_2-3 \alpha_{12})} \eta_{12_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{13_{12}} - 6 e^{\frac{1}{2} (2 b_1-b_2-3 \alpha_{12})} \eta_{12_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{13_{12}} - \\
& 2 e^{\frac{1}{2} (b_1+b_2-3 \alpha_{12})} \eta_{12_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{13_{12}} + 2 e^{\frac{3}{2} (b_1-\alpha_{12})} \eta_{12_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{13_{12}} + \\
& 2 e^{-\frac{3 \alpha_{12}}{2}} \eta_{12_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{13_{12}} - 3 e^{\frac{1}{2} (3 b_1-2 \alpha_{12}-2 \alpha_{22})} y_{23_i} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{13_{12}} + \\
& 4 e^{\frac{1}{2} (2 b_1-b_2-2 \alpha_{12}-2 \alpha_{22})} y_{23_i} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{13_{12}} - 4 e^{2 b_1-b_2-\alpha_{12}-\alpha_{22}} y_{23_i} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{13_{12}} - \\
& 2 x_{13_i} \eta_{13_{11}} \xi_{13_{11}} \xi_{13_{12}} - 2 e^{\frac{1}{2} (b_1+b_2)} x_{13_i} \eta_{13_{11}} \xi_{13_{11}} \xi_{13_{12}} - 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} x_{13_i} \eta_{13_{11}} \xi_{13_{12}}^2 + \\
& 2 e^{\frac{1}{2} (b_1+b_2-\alpha_{12}-\alpha_{22})} x_{13_i} \eta_{13_{11}} \xi_{13_{12}}^2 - 2 e^{\frac{1}{2} (b_1+b_2-2 \alpha_{12}-2 \alpha_{22})} \eta_{13_{11}}^2 \xi_{13_{12}}^2 + \\
& 2 e^{b_1+b_2-\alpha_{12}-\alpha_{22}} \eta_{13_{11}}^2 \xi_{13_{12}}^2 + e^{-\frac{3}{2} (\alpha_{12}-\alpha_{22})} x_{23_i} \eta_{12_{11}} \xi_{12_{12}} \xi_{23_{11}} + \\
& e^{\frac{1}{2} (2 b_1-b_2-3 \alpha_{12}+3 \alpha_{22})} x_{23_i} \eta_{12_{11}} \xi_{12_{12}} \xi_{23_{11}} - e^{\frac{1}{2} (2 b_1-b_2-2 \alpha_{12}+\alpha_{22})} x_{23_i} y_{23_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{11}} - \\
& e^{\frac{1}{2} (-2 \alpha_{12}+\alpha_{22})} x_{23_i} \eta_{13_{11}} \xi_{13_{12}} \xi_{23_{11}} - e^{\frac{1}{2} (b_1+b_2-2 \alpha_{12}+\alpha_{22})} x_{23_i} \eta_{13_{11}} \xi_{13_{12}} \xi_{23_{11}} + \\
& 4 x_{23_i} \beta_{2_{11}} \xi_{23_{12}} - e^{\frac{1}{2} (-2 \alpha_{12}+\alpha_{22})} x_{23_i} y_{12_i} \eta_{12_{11}} \xi_{23_{12}} + 3 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} y_{12_i} \eta_{13_{11}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} a_{2_i} y_{12_i} \eta_{13_{11}} \xi_{23_{12}} + e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} x_{23_i} y_{13_i} \eta_{13_{11}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} y_{12_i} \beta_{1_{12}} \eta_{13_{11}} \xi_{23_{12}} + 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} y_{12_i} \beta_{2_{11}} \eta_{13_{11}} \xi_{23_{12}} + \\
& e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} y_{12_i} y_{13_i} \eta_{13_{11}} \eta_{13_{12}} \xi_{23_{12}} - 2 e^{\frac{1}{2} (\alpha_{12}-2 \alpha_{22})} \eta_{23_{11}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (-b_1+2 b_2+\alpha_{12}-2 \alpha_{22})} \eta_{23_{11}} \xi_{23_{12}} - 2 e^{\frac{1}{2} (\alpha_{12}-2 \alpha_{22})} a_{2_i} \eta_{23_{11}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (-b_1+2 b_2+\alpha_{12}-2 \alpha_{22})} a_{2_i} \eta_{23_{11}} \xi_{23_{12}} + 2 e^{\frac{1}{2} (\alpha_{12}-2 \alpha_{22})} x_{23_i} y_{23_i} \eta_{23_{11}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (\alpha_{12}-2 \alpha_{22})} \beta_{2_{11}} \eta_{23_{11}} \xi_{23_{12}} + 2 e^{\frac{1}{2} (-b_1+2 b_2+\alpha_{12}-2 \alpha_{22})} \beta_{2_{11}} \eta_{23_{11}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} y_{12_i} \eta_{12_{11}} \eta_{23_{11}} \xi_{23_{12}} - e^{\frac{1}{2} (-b_1+2 b_2-\alpha_{12}-\alpha_{22})} y_{12_i} \eta_{12_{11}} \eta_{23_{11}} \xi_{23_{12}} + \\
& e^{\frac{1}{2} (-b_1+2 b_2+\alpha_{12}-2 \alpha_{22})} y_{12_i} \eta_{12_{12}} \eta_{23_{11}} \xi_{23_{12}} + e^{\frac{1}{2} (-b_1+2 b_2-3 \alpha_{22})} y_{13_i} \eta_{13_{11}} \eta_{23_{11}} \xi_{23_{12}} + \\
& 2 e^{-\frac{3 \alpha_{22}}{2}} y_{13_i} \eta_{13_{11}} \eta_{23_{11}} \xi_{23_{12}} - e^{\frac{1}{2} (-b_1+2 b_2+\alpha_{12}-2 \alpha_{22})} y_{13_i} \eta_{13_{12}} \eta_{23_{11}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (-b_1+2 b_2+2 \alpha_{12}-4 \alpha_{22})} y_{23_i} \eta_{23_{11}}^2 \xi_{23_{12}} - e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} y_{12_i} y_{23_i} \eta_{13_{11}} \eta_{23_{12}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (-b_1+2 b_2+\alpha_{12}-2 \alpha_{22})} y_{23_i} \eta_{23_{11}} \eta_{23_{12}} \xi_{23_{12}} - e^{\frac{1}{2} (2 \alpha_{12}-\alpha_{22})} x_{13_i} \xi_{12_{11}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (2 \alpha_{12}-\alpha_{22})} x_{12_i} x_{23_i} \xi_{12_{11}} \xi_{23_{12}} - 4 e^{\frac{1}{2} (2 \alpha_{12}-\alpha_{22})} x_{13_i} \beta_{2_{11}} \xi_{12_{11}} \xi_{23_{12}} + \\
& x_{13_i} y_{12_i} \eta_{12_{11}} \xi_{12_{11}} \xi_{23_{12}} - e^{\frac{1}{2} (\alpha_{12}-2 \alpha_{22})} x_{12_i} y_{12_i} \eta_{13_{11}} \xi_{12_{11}} \xi_{23_{12}} - \\
& e^{\frac{1}{2} (\alpha_{12}-2 \alpha_{22})} x_{13_i} y_{13_i} \eta_{13_{11}} \xi_{12_{11}} \xi_{23_{12}} + e^{\frac{1}{2} (-b_1+2 b_2+3 \alpha_{12}-3 \alpha_{22})} x_{12_i} \eta_{23_{11}} \xi_{12_{11}} \xi_{23_{12}} + \\
& e^{\frac{3}{2} (\alpha_{12}-\alpha_{22})} x_{12_i} \eta_{23_{11}} \xi_{12_{11}} \xi_{23_{12}} - 2 e^{\frac{3}{2} (\alpha_{12}-\alpha_{22})} x_{13_i} y_{23_i} \eta_{23_{11}} \xi_{12_{11}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (-2 \alpha_{12}+\alpha_{22})} x_{23_i} \eta_{12_{11}} \xi_{12_{12}} \xi_{23_{12}} - 5 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} + \\
& 5 e^{\frac{1}{2} (2 b_1-b_2-\alpha_{12}-\alpha_{22})} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} - 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} a_{1_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (2 b_1-b_2-\alpha_{12}-\alpha_{22})} a_{1_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} - 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} a_{2_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (2 b_1-b_2-\alpha_{12}-\alpha_{22})} a_{2_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} + 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} x_{12_i} y_{12_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (2 b_1-b_2-\alpha_{12}-\alpha_{22})} x_{23_i} y_{23_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} - 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} \beta_{1_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (2 b_1-b_2-\alpha_{12}-\alpha_{22})} \beta_{1_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} - 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} \beta_{2_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (2 b_1-b_2-\alpha_{12}-\alpha_{22})} \beta_{2_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} + 4 e^{\frac{1}{2} (2 b_1-b_2-3 \alpha_{12})} y_{12_i} \eta_{12_{11}} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (2 b_1-b_2-\alpha_{12}-\alpha_{22})} y_{12_i} \eta_{12_{12}} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}} - e^{\frac{1}{2} (2 b_1-b_2-2 \alpha_{12}-2 \alpha_{22})} y_{13_i} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (2 b_1-b_2-2 \alpha_{12}-2 \alpha_{22})} y_{12_i} y_{23_i} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{23_{12}} - e^{\frac{1}{2} (2 b_1-b_2-\alpha_{12}-\alpha_{22})} y_{13_i} \eta_{13_{11}} \eta_{13_{12}} \xi_{12_{12}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (-b_1+2 b_2+\alpha_{12}-2 \alpha_{22})} x_{12_i} \eta_{23_{11}} \xi_{12_{12}} \xi_{23_{12}} + 2 e^{\frac{1}{2} (-\alpha_{12}-\alpha_{22})} \eta_{12_{11}} \eta_{23_{11}} \xi_{12_{12}} \xi_{23_{12}} -
\end{aligned}$$

$$\begin{aligned}
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - \alpha_{12} - \alpha_{22})} \eta_{12_{11}} \eta_{23_{11}} \xi_{12_{12}} \xi_{23_{12}} - 2 e^{\frac{1}{2} (b_1 + b_2 - \alpha_{12} - \alpha_{22})} \eta_{12_{11}} \eta_{23_{11}} \xi_{12_{12}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (-b_1 + 2 b_2 - \alpha_{12} - \alpha_{22})} \eta_{12_{11}} \eta_{23_{11}} \xi_{12_{12}} \xi_{23_{12}} - e^{\frac{1}{2} (2 b_1 - b_2 - 3 \alpha_{22})} y_{23_i} \eta_{13_{11}} \eta_{23_{11}} \xi_{12_{12}} \xi_{23_{12}} + \\
& e^{\frac{1}{2} (2 b_1 - b_2 - \alpha_{12} - \alpha_{22})} y_{23_i} \eta_{13_{11}} \eta_{23_{12}} \xi_{12_{12}} \xi_{23_{12}} + 2 e^{\frac{1}{2} (2 b_1 - b_2)} x_{13_i} \eta_{12_{11}} \xi_{12_{11}} \xi_{12_{12}} \xi_{23_{12}} - \\
& e^{\frac{1}{2} (\alpha_{12} - 2 \alpha_{22})} x_{12_i} \eta_{13_{11}} \xi_{12_{11}} \xi_{12_{12}} \xi_{23_{12}} - 3 e^{\frac{1}{2} (2 b_1 - b_2 + \alpha_{12} - 2 \alpha_{22})} x_{12_i} \eta_{13_{11}} \xi_{12_{11}} \xi_{12_{12}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (-\alpha_{12} - \alpha_{22})} x_{12_i} \eta_{13_{11}} \xi_{12_{12}}^2 \xi_{23_{12}} + 2 e^{\frac{1}{2} (2 b_1 - b_2 - \alpha_{12} - \alpha_{22})} x_{12_i} \eta_{13_{11}} \xi_{12_{12}}^2 \xi_{23_{12}} + \\
& 4 e^{\frac{1}{2} (4 b_1 - 2 b_2 - 3 \alpha_{12})} \eta_{12_{11}} \eta_{13_{11}} \xi_{12_{12}}^2 \xi_{23_{12}} - 4 e^{\frac{1}{2} (2 b_1 - b_2 - 3 \alpha_{12})} \eta_{12_{11}} \eta_{13_{11}} \xi_{12_{12}}^2 \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - 2 \alpha_{12} - 2 \alpha_{22})} y_{23_i} \eta_{13_{11}}^2 \xi_{12_{12}}^2 \xi_{23_{12}} - 3 e^{2 b_1 - b_2 - \alpha_{12} - \alpha_{22}} y_{23_i} \eta_{13_{11}}^2 \xi_{12_{12}}^2 \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (\alpha_{12} + \alpha_{22})} x_{13_i} x_{23_i} \xi_{13_{11}} \xi_{23_{12}} + x_{13_i} y_{12_i} \eta_{13_{11}} \xi_{13_{11}} \xi_{23_{12}} - \\
& e^{\frac{1}{2} (2 \alpha_{12} - \alpha_{22})} x_{13_i} \eta_{23_{11}} \xi_{13_{11}} \xi_{23_{12}} - e^{\frac{1}{2} (-b_1 + 2 b_2 + 2 \alpha_{12} - \alpha_{22})} x_{13_i} \eta_{23_{11}} \xi_{13_{11}} \xi_{23_{12}} - \\
& 2 x_{13_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{13_{11}} \xi_{23_{12}} - 2 e^{\frac{1}{2} (-\alpha_{12} - \alpha_{22})} x_{23_i} \eta_{13_{11}} \xi_{13_{12}} \xi_{23_{12}} + \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2 - \alpha_{12} - \alpha_{22})} x_{23_i} \eta_{13_{11}} \xi_{13_{12}} \xi_{23_{12}} + e^{\frac{1}{2} (-\alpha_{12} - \alpha_{22})} x_{13_i} y_{12_i} \eta_{13_{11}} \xi_{13_{12}} \xi_{23_{12}} + \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2 - 2 \alpha_{12} - 2 \alpha_{22})} y_{12_i} \eta_{13_{11}}^2 \xi_{13_{12}} \xi_{23_{12}} + e^{\frac{1}{2} (b_1 + b_2 - 2 \alpha_{12} - 2 \alpha_{22})} y_{12_i} \eta_{13_{11}}^2 \xi_{13_{12}} \xi_{23_{12}} - \\
& 2 e^{-\alpha_{12} - \alpha_{22}} y_{12_i} \eta_{13_{11}}^2 \xi_{13_{12}} \xi_{23_{12}} + 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + \alpha_{12} - 2 \alpha_{22})} x_{13_i} \eta_{23_{11}} \xi_{13_{12}} \xi_{23_{12}} - \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2 - 3 \alpha_{22})} \eta_{13_{11}} \eta_{23_{11}} \xi_{13_{12}} \xi_{23_{12}} + 2 e^{\frac{1}{2} (b_1 + b_2 - 3 \alpha_{22})} \eta_{13_{11}} \eta_{23_{11}} \xi_{13_{12}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (-b_1 + 2 b_2 - 3 \alpha_{22})} \eta_{13_{11}} \eta_{23_{11}} \xi_{13_{12}} \xi_{23_{12}} + 2 e^{\frac{3}{2} (b_2 - \alpha_{22})} \eta_{13_{11}} \eta_{23_{11}} \xi_{13_{12}} \xi_{23_{12}} + \\
& 2 e^{-\frac{3 \alpha_{22}}{2}} \eta_{13_{11}} \eta_{23_{11}} \xi_{13_{12}} \xi_{23_{12}} + 3 e^{\frac{1}{2} (\alpha_{12} - 2 \alpha_{22})} x_{13_i} \eta_{13_{11}} \xi_{12_{11}} \xi_{13_{12}} \xi_{23_{12}} - \\
& 4 e^{\frac{1}{2} (2 b_1 - b_2 + \alpha_{12} - 2 \alpha_{22})} x_{13_i} \eta_{13_{11}} \xi_{12_{11}} \xi_{13_{12}} \xi_{23_{12}} + e^{\frac{1}{2} (b_1 + b_2 + \alpha_{12} - 2 \alpha_{22})} x_{13_i} \eta_{13_{11}} \xi_{12_{11}} \xi_{13_{12}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (-\alpha_{12} - \alpha_{22})} x_{13_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{13_{12}} \xi_{23_{12}} + 2 e^{\frac{1}{2} (3 b_1 - 2 \alpha_{12} - 2 \alpha_{22})} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{13_{12}} \xi_{23_{12}} - \\
& 6 e^{\frac{1}{2} (2 b_1 - b_2 - 2 \alpha_{12} - 2 \alpha_{22})} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{13_{12}} \xi_{23_{12}} - 2 e^{\frac{1}{2} (b_1 + b_2 - 2 \alpha_{12} - 2 \alpha_{22})} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{13_{12}} \xi_{23_{12}} + \\
& 2 e^{-\alpha_{12} - \alpha_{22}} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{13_{12}} \xi_{23_{12}} + 4 e^{2 b_1 - b_2 - \alpha_{12} - \alpha_{22}} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{13_{12}} \xi_{23_{12}} + \\
& 2 e^{\frac{1}{2} (-2 \alpha_{12} + \alpha_{22})} x_{23_i} y_{12_i} \eta_{13_{11}} \xi_{23_{11}} \xi_{23_{12}} - 2 x_{23_i} \eta_{23_{11}} \xi_{23_{11}} \xi_{23_{12}} - \\
& 2 e^{\frac{1}{2} (-b_1 + 2 b_2)} x_{23_i} \eta_{23_{11}} \xi_{23_{11}} \xi_{23_{12}} - e^{\frac{1}{2} (-2 \alpha_{12} + \alpha_{22})} x_{23_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{11}} \xi_{23_{12}} + \\
& 3 e^{\frac{1}{2} (2 b_1 - b_2 - 2 \alpha_{12} + \alpha_{22})} x_{23_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{11}} \xi_{23_{12}} + e^{\frac{1}{2} (-\alpha_{12} - \alpha_{22})} x_{23_i} y_{12_i} \eta_{13_{11}} \xi_{23_{12}}^2 - \\
& 2 e^{\frac{1}{2} (\alpha_{12} - 2 \alpha_{22})} x_{23_i} \eta_{23_{11}} \xi_{23_{12}}^2 + 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + \alpha_{12} - 2 \alpha_{22})} x_{23_i} \eta_{23_{11}} \xi_{23_{12}}^2 + \\
& e^{\frac{1}{2} (-b_1 + 2 b_2 - 3 \alpha_{22})} y_{12_i} \eta_{13_{11}} \eta_{23_{11}} \xi_{23_{12}}^2 - 2 e^{\frac{1}{2} (-b_1 + 2 b_2 + 2 \alpha_{12} - 4 \alpha_{22})} \eta_{23_{11}}^2 \xi_{23_{12}}^2 + \\
& 2 e^{-b_1 + 2 b_2 + \alpha_{12} - 2 \alpha_{22}} \eta_{23_{11}}^2 \xi_{23_{12}}^2 - 3 e^{\frac{1}{2} (-b_1 + 2 b_2 + 3 \alpha_{12} - 3 \alpha_{22})} x_{13_i} \eta_{23_{11}} \xi_{12_{11}} \xi_{23_{12}}^2 + \\
& e^{\frac{3}{2} (\alpha_{12} - \alpha_{22})} x_{13_i} \eta_{23_{11}} \xi_{12_{11}} \xi_{23_{12}}^2 + 2 e^{\frac{1}{2} (2 b_1 - b_2 - \alpha_{12} - \alpha_{22})} x_{23_i} \eta_{13_{11}} \xi_{12_{12}} \xi_{23_{12}}^2 + \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - 2 \alpha_{12} - 2 \alpha_{22})} y_{12_i} \eta_{13_{11}}^2 \xi_{12_{12}} \xi_{23_{12}}^2 + e^{\frac{1}{2} (\alpha_{12} - 2 \alpha_{22})} x_{13_i} \eta_{13_{11}} \xi_{12_{11}} \xi_{12_{12}} \xi_{23_{12}}^2 + \\
& e^{\frac{1}{2} (2 b_1 - b_2 + \alpha_{12} - 2 \alpha_{22})} x_{13_i} \eta_{13_{11}} \xi_{12_{11}} \xi_{12_{12}} \xi_{23_{12}}^2 - \\
& 2 e^{\frac{1}{2} (2 b_1 - b_2 - 2 \alpha_{12} - 2 \alpha_{22})} \eta_{13_{11}}^2 \xi_{12_{12}}^2 \xi_{23_{12}}^2 + 2 e^{2 b_1 - b_2 - \alpha_{12} - \alpha_{22}} \eta_{13_{11}}^2 \xi_{12_{12}}^2 \xi_{23_{12}}^2 \Big]
\end{aligned}$$

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In[*]:= Block[{ $k = 0 },
Echo[dS12 // dm12,i1-i];
R11,i2 // dS12 // dm12,i1-i]

```

$$\begin{aligned}
& \gg \text{TTL } e^{-\frac{1}{2} \gamma (2 \alpha_{11} - \alpha_{21})} \xi_{12_i} U_j [x_{12}] U_k [] + \left(e^{-\frac{1}{2} \gamma (\alpha_{11} + \alpha_{21})} \xi_{13_i} + e^{-\frac{1}{2} \gamma (\alpha_{11} + \alpha_{21})} \lambda \xi_{12_i} \xi_{23_i} \right) U_j [x_{13}] U_k [] + \\
& e^{\langle\langle 1 \rangle\rangle} \xi_{23_i} U_j [x_{23}] U_k [] + \langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle + e^{\frac{1}{2} \gamma (\alpha_{11} - 2 \alpha_{21})} \xi_{23_i} U_j [] U_k [x_{23}] \\
& \gg \text{SG } U_j [] U_k [] \\
& \gg \text{TTR } e^{-\frac{1}{2} \gamma (2 \alpha_{11} - \alpha_{21})} U_j [x_{12}] U_k [] f_{\theta, x_{12_j}}' [\lambda] + e^{-\frac{1}{2} \gamma (2 \alpha_{11} - \alpha_{21})} U_j [] U_k [x_{12}] f_{\theta, \langle\langle 3 \rangle\rangle_k}' [\lambda] + \langle\langle 1 \rangle\rangle \\
& \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle + e^{\frac{1}{2} \gamma (\langle\langle 2 \rangle\rangle_1 - \langle\langle 1 \rangle\rangle)} U_j [x_{23}] \langle\langle 1 \rangle\rangle f_{\langle\langle 1 \rangle\rangle}' [\lambda] + e^{\frac{1}{2} \gamma (\alpha_{11} - 2 \alpha_{21})} U_j [] U_k [x_{23}] f_{\theta, x_{23_k}}' [\lambda] \\
& \gg E_{\{i_1, i_2\} \rightarrow \{i\}} \left[a_{1_i} \alpha_{1_{i1}} - a_{1_i} \alpha_{1_{i2}} + a_{2_i} \alpha_{2_{i1}} - a_{2_i} \alpha_{2_{i2}} + b_{1_i} \beta_{1_{i1}} - b_{1_i} \beta_{1_{i2}} + b_{2_i} \beta_{2_{i1}} - \right. \\
& b_{2_i} \beta_{2_{i2}} + e^{\frac{1}{2} (2 \alpha_{1_{i2}} - \alpha_{2_{i2}})} y_{12_i} \eta_{12_{i1}} - e^{\frac{1}{2} (-2 b_{1_i} + b_{2_i} + 2 \alpha_{1_{i2}} - \alpha_{2_{i2}})} y_{12_i} \eta_{12_{i2}} + e^{\frac{1}{2} (\alpha_{1_{i2}} + \alpha_{2_{i2}})} y_{13_i} \eta_{13_{i1}} - \\
& e^{\frac{1}{2} (-b_{1_i} - b_{2_i} + \alpha_{1_{i2}} + \alpha_{2_{i2}})} y_{13_i} \eta_{13_{i2}} - e^{\frac{1}{2} (-b_{1_i} - b_{2_i} + \alpha_{1_{i2}} + \alpha_{2_{i2}})} y_{12_i} y_{23_i} \eta_{13_{i2}} + e^{\frac{1}{2} (-\alpha_{1_{i2}} + 2 \alpha_{2_{i2}})} y_{23_i} \eta_{23_{i1}} - \\
& e^{\frac{1}{2} (b_{1_i} - 2 b_{2_i} - \alpha_{1_{i2}} + 2 \alpha_{2_{i2}})} y_{23_i} \eta_{23_{i2}} + e^{\frac{1}{2} (-2 \alpha_{1_{i2}} + \alpha_{2_{i2}})} x_{12_i} \xi_{12_{i1}} - e^{\frac{1}{2} (-2 \alpha_{1_{i2}} + \alpha_{2_{i2}})} x_{12_i} \xi_{12_{i2}} + \eta_{12_{i1}} \xi_{12_{i2}} - \\
& e^{\frac{1}{2} (2 b_{1_i} - b_{2_i})} \eta_{12_{i1}} \xi_{12_{i2}} + \eta_{12_{i2}} \xi_{12_{i2}} - e^{\frac{1}{2} (-2 b_{1_i} + b_{2_i})} \eta_{12_{i2}} \xi_{12_{i2}} + e^{\frac{1}{2} (2 b_{1_i} - b_{2_i} - \alpha_{1_{i2}} + 2 \alpha_{2_{i2}})} y_{23_i} \eta_{13_{i1}} \xi_{12_{i2}} - \\
& e^{\frac{1}{2} (-b_{1_i} - b_{2_i} - \alpha_{1_{i2}} + 2 \alpha_{2_{i2}})} y_{23_i} \eta_{13_{i2}} \xi_{12_{i2}} + e^{\frac{1}{2} (-\alpha_{1_{i2}} - \alpha_{2_{i2}})} x_{13_i} \xi_{13_{i1}} - e^{\frac{1}{2} (-\alpha_{1_{i2}} - \alpha_{2_{i2}})} x_{13_i} \xi_{13_{i2}} + \\
& \eta_{13_{i1}} \xi_{13_{i2}} - e^{\frac{1}{2} (b_{1_i} + b_{2_i})} \eta_{13_{i1}} \xi_{13_{i2}} + \eta_{13_{i2}} \xi_{13_{i2}} - e^{\frac{1}{2} (-b_{1_i} - b_{2_i})} \eta_{13_{i2}} \xi_{13_{i2}} + e^{\frac{1}{2} (\alpha_{1_{i2}} - 2 \alpha_{2_{i2}})} x_{23_i} \xi_{23_{i1}} - \\
& e^{\frac{1}{2} (\alpha_{1_{i2}} - 2 \alpha_{2_{i2}})} x_{23_i} \xi_{23_{i2}} - e^{\frac{1}{2} (2 \alpha_{1_{i2}} - \alpha_{2_{i2}})} y_{12_i} \eta_{13_{i1}} \xi_{23_{i2}} + e^{\frac{1}{2} (-2 b_{1_i} + b_{2_i} + 2 \alpha_{1_{i2}} - \alpha_{2_{i2}})} y_{12_i} \eta_{13_{i2}} \xi_{23_{i2}} + \\
& \eta_{23_{i1}} \xi_{23_{i2}} - e^{\frac{1}{2} (-b_{1_i} + 2 b_{2_i})} \eta_{23_{i1}} \xi_{23_{i2}} + \eta_{23_{i2}} \xi_{23_{i2}} - e^{\frac{1}{2} (b_{1_i} - 2 b_{2_i})} \eta_{23_{i2}} \xi_{23_{i2}} + \\
& e^{\frac{1}{2} (-\alpha_{1_{i2}} - \alpha_{2_{i2}})} x_{13_i} \xi_{12_{i1}} \xi_{23_{i2}} - e^{\frac{1}{2} (-\alpha_{1_{i2}} - \alpha_{2_{i2}})} x_{13_i} \xi_{12_{i2}} \xi_{23_{i2}} + e^{\frac{1}{2} (2 b_{1_i} - b_{2_i})} \eta_{13_{i1}} \xi_{12_{i2}} \xi_{23_{i2}} - \\
& e^{\frac{1}{2} (b_{1_i} + b_{2_i})} \eta_{13_{i1}} \xi_{12_{i2}} \xi_{23_{i2}} - e^{\frac{1}{2} (-b_{1_i} - b_{2_i})} \eta_{13_{i2}} \xi_{12_{i2}} \xi_{23_{i2}} + e^{\frac{1}{2} (-2 b_{1_i} + b_{2_i})} \eta_{13_{i2}} \xi_{12_{i2}} \xi_{23_{i2}} \left. \right]
\end{aligned}$$

Out[*]=

$$\begin{aligned}
& E_{\{i\} \rightarrow \{i\}} \left[\text{Log} \left[e^{-b_{1_i} - b_{2_i}} \right] - a_{1_i} b_{1_i} - a_{2_i} b_{2_i} - e^{\frac{1}{2} (-2 b_{1_i} + b_{2_i})} x_{12_i} y_{12_i} - \right. \\
& e^{\frac{1}{2} (-b_{1_i} - b_{2_i})} x_{13_i} y_{13_i} - e^{\frac{1}{2} (b_{1_i} - 2 b_{2_i})} x_{23_i} y_{23_i} - e^{\frac{1}{2} (-b_{1_i} - b_{2_i})} x_{13_i} y_{12_i} y_{23_i} \left. \right]
\end{aligned}$$

Zipping fails to compute the Drinfeld element u . Why?We still need to find C such that R2c holds:

According to PG.pdf the ribbon element v is a square root of $v^2 = uS(u)$. Then the pivot (a.k.a. distinguished group-like element, spinner) is $C = uv^{-1}$.

$$\text{In[*]} := \mathbf{R}_{3,2} \bar{\mathbf{R}}_{1,6} \mathbf{C}_4 // \mathbf{dm}_{1,3 \rightarrow 1} // \mathbf{dm}_{2,4 \rightarrow 2} // \mathbf{dm}_{2,6 \rightarrow 2}$$

 $\mathbf{C}_2$

Time: 0.22 seconds Out[*]=

Out[*]=

 $\mathbf{C}_2$