

```
In[ ]:= SetDirectory["C:\\Users\\T15Roland\\Wiskunde\\Bn\\HigherRank"]
Once[<< KnotTheory`];
```

Time: 0.52 seconds Out[]:=

C:\\Users\\T15Roland\\Wiskunde\\Bn\\HigherRank

ParentDirectory: The argument File should be a positive machine-sized integer, a nonempty string or a File specification.

ParentDirectory: The argument File should be a positive machine-sized integer, a nonempty string or a File specification.

ToFileName: A string or list of strings is expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: A string or list of strings is expected at position 1 in ToFileName[{File, QuantumGroups}].

Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.

Read more at <http://katlas.org/wiki/KnotTheory>.

SetDelayed: Tag Diff in Diff[K_PD, rut_, ag_, n_, m_] is Protected.

Hidden Utilities and T_EX

As: Enables the likes of Table[As[n,6], As[k,0,n], Binomial[n,k]], as in pensieve://2017-01/As.nb.

```
In[ ]:= SetAttributes[As, HoldAll];
As /: (op : Do | Product | Sum | Table) [As[iter___], ε___] := op[ε, {iter}]
```

Have StringJoin handle Integer and Symbol inputs correctly:

```
In[ ]:= Unprotect[StringJoin];
StringJoin[l___, m_Integer, r___] := StringJoin[l, ToString[m], r];
StringJoin[x_Symbol, r___] := Symbol[SymbolName[x] <> StringJoin[r]];
Protect[StringJoin];
```

Time every Out line:

```
In[ ]:= CurrentValue[EvaluationNotebook[], EvaluationCompletionAction] = {"ShowTiming"};

SetOptions[EvaluationNotebook[], StyleDefinitions →
  Notebook[{Cell[StyleData[StyleDefinitions → "Default.nb"]], Cell[StyleData["Input"],
    CellEpilog → With[{time = AbsoluteCurrentValue["WindowStatusArea"]},
      If[CurrentValue[NextCell[], CellStyle] === {"Output"}, SetOptions[
        NextCell[], CellLabel → time <> " Out[" <> ToString[$Line - 1] <> "]" = "]]]],
    StyleDefinitions → "PrivateStylesheetFormatting.nb"]]
```

```
In[ ]:= PP_ = Identity;
```

tex

```
\documentclass[12pt,reqno]{amsart}
\usepackage[export]{adjustbox} % This also includes graphicx.
\usepackage{needspace}
\usepackage[textwidth=7.5in,textheight=10in,centering]{geometry}
\pagestyle{empty}
```

```

\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfEcho#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfPrint#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfMessage#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfSection#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfSubsection#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfSubsubsection#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfgraphInput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfgraphOutput#1{\vskip 1mm\par\noindent\includegraphics[width=1.5in]{#1}}

\begin{document}

```

nb2tex

```

In[ ]:= MakePDF [] := (
  Get [ "../nb2tex/nb2tex.m" ];
  $NotebookName = FileNameTake@FileNameTake@NotebookFileName[EvaluationNotebook[]];
  nb2tex[$NotebookName, PDFWidth -> 7.5, PDFFolder -> "Snips/" <> $NotebookName];
  texfile = OpenAppend[$NotebookName <> ".tex"];
  WriteLine[texfile, "\\end{document}"]; Close[texfile];
  Run@
    ("C:\\Users\\drorb\\AppData\\Local\\Programs\\MiKTeX\\miktex\\bin\\x64\\pdflatex.
      exe\" \" <> $NotebookName <> ".tex");
)

```

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

pdf

Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_{\hbar}(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

pdf

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

pdf

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

pdf

This version: April 2020. Original version: July 1994.

```
In[ ]:= MakePDF[]
```

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```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i,j];
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[ $\mathcal{E}$ _, op_] := Collect[Normal@Series[ $\mathcal{E}$ , { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[ $\mathcal{E}$ _] := SS[ $\mathcal{E}$ , Together];
```

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```
In[ ]:= $N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x<>i<>j; a[i_] := a<>i; A[i_] := A<>i;
 $\xi$ [i_, j_] :=  $\xi$ <>i<>j;  $\alpha$ [i_] :=  $\alpha$ <>i;  $\mathcal{A}$ [i_] :=  $\mathcal{A}$ <>i;
y[i_, j_] := y<>i<>j; b[i_] := b<>i; B[i_] := B<>i;
 $\eta$ [i_, j_] :=  $\eta$ <>i<>j;  $\beta$ [i_] :=  $\beta$ <>i;
 $\tau$ [s_Symbol] := First[Characters[SymbolName[s]]];
 $\mathcal{L}$ [s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
 $\tau$ [s_] :=  $\tau$ [s];  $\mathcal{L}$ [s_] :=  $\mathcal{L}$ [s];
```

```
In[ ]:= { $\tau$ [x[3, 4]],  $\mathcal{L}$ [x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

Greek - Latin duality:

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```

In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]* =  $\xi[i, j]$ ;
 $\xi[i, j]^* = x[i, j]$ ; y[i, j]* =  $\eta[i, j]$ ;
 $\eta[i, j]^* = y[i, j]$ ]
Do[As[i, 1, $N - 1], A[i]* =  $\mathcal{A}[i]$ ;  $\mathcal{A}[i]^* = A[i]$ ; a[i]* =  $\alpha[i]$ ;  $\alpha[i]^* = a[i]$ ;
b[i]* =  $\beta[i]$ ;  $\beta[i]^* = b[i]$ ]
(vs_List)* := (v  $\mapsto$  v*) /@ vs;
(u_{ $\alpha\beta$ })^* := (u^*)_{ $\alpha\beta$ };

```

```

In[ ]:= $N = 9

```

```

Time: 0.05 seconds Out[ ]:=
9

```

```

In[ ]:=  $\tau[\alpha 1]$ 

```

```

Time: 0.00 seconds Out[ ]:=
 $\alpha$ 

```

```

Flatten@Table[{ $\mathcal{A}[j]_{i-}^{p-} \mapsto e^{p\alpha[j]_i}$ ,  $B[j]_{i-}^{p-} \mapsto e^{-p\hbar b[j]_i}$ }, {j, 1, $N - 1}];
Join[{e^ $\chi$   $\mapsto$  e^{Expand@12U@ $\chi$ }},
Flatten@Table[{e^{ $c_- \cdot b[j]_{i-} + d_-} \mapsto B[j]_i^{-c/\hbar} e^d$ , e^{ $c_- \cdot \alpha[j]_{i-} + d_-} \mapsto \mathcal{A}[j]_i^c e^d$ }, {j, 1, $N - 1}]];

```

```

In[ ]:=  $\hbar = 1$ ;
U21[ $\mathcal{E}_-$ ] :=  $\mathcal{E}_- / . \{ \mathcal{A}1_{i-}^{p-} \mapsto e^{p\alpha 1_i}$ ,  $B1_{i-}^{p-} \mapsto e^{-p\hbar b1_i}$ ,  $\mathcal{A}2_{i-}^{p-} \mapsto e^{p\alpha 2_i}$ ,  $B2_{i-}^{p-} \mapsto e^{-p\hbar b2_i}$ ,  $\mathcal{A}3_{i-}^{p-} \mapsto e^{p\alpha 3_i}$ ,
 $B3_{i-}^{p-} \mapsto e^{-p\hbar b3_i}$ ,  $\mathcal{A}4_{i-}^{p-} \mapsto e^{p\alpha 4_i}$ ,  $B4_{i-}^{p-} \mapsto e^{-p\hbar b4_i}$ ,  $\mathcal{A}5_{i-}^{p-} \mapsto e^{p\alpha 5_i}$ ,  $B5_{i-}^{p-} \mapsto e^{-p\hbar b5_i}$ ,  $\mathcal{A}6_{i-}^{p-} \mapsto e^{p\alpha 6_i}$ ,
 $B6_{i-}^{p-} \mapsto e^{-p\hbar b6_i}$ ,  $\mathcal{A}7_{i-}^{p-} \mapsto e^{p\alpha 7_i}$ ,  $B7_{i-}^{p-} \mapsto e^{-p\hbar b7_i}$ ,  $\mathcal{A}8_{i-}^{p-} \mapsto e^{p\alpha 8_i}$ ,  $B8_{i-}^{p-} \mapsto e^{-p\hbar b8_i} \}$ 
12U[ $\mathcal{E}_-$ ] :=  $\mathcal{E}_- / . \{ e^{d_- + c_- \cdot b1_{i-}} \mapsto B1_i^{-\frac{c}{\hbar}} e^d$ , e^{ $d_- + c_- \cdot \alpha 1_{i-}} \mapsto \mathcal{A}1_i^c e^d$ ,
e^{ $d_- + c_- \cdot b2_{i-}} \mapsto B2_i^{-\frac{c}{\hbar}} e^d$ , e^{ $d_- + c_- \cdot \alpha 2_{i-}} \mapsto \mathcal{A}2_i^c e^d$ , e^{ $d_- + c_- \cdot b3_{i-}} \mapsto B3_i^{-\frac{c}{\hbar}} e^d$ ,
e^{ $d_- + c_- \cdot \alpha 3_{i-}} \mapsto \mathcal{A}3_i^c e^d$ , e^{ $d_- + c_- \cdot b4_{i-}} \mapsto B4_i^{-\frac{c}{\hbar}} e^d$ , e^{ $d_- + c_- \cdot \alpha 4_{i-}} \mapsto \mathcal{A}4_i^c e^d$ , e^{ $d_- + c_- \cdot b5_{i-}} \mapsto B5_i^{-\frac{c}{\hbar}} e^d$ ,
e^{ $d_- + c_- \cdot \alpha 5_{i-}} \mapsto \mathcal{A}5_i^c e^d$ , e^{ $d_- + c_- \cdot b6_{i-}} \mapsto B6_i^{-\frac{c}{\hbar}} e^d$ , e^{ $d_- + c_- \cdot \alpha 6_{i-}} \mapsto \mathcal{A}6_i^c e^d$ , e^{ $d_- + c_- \cdot b7_{i-}} \mapsto B7_i^{-\frac{c}{\hbar}} e^d$ ,
e^{ $d_- + c_- \cdot \alpha 7_{i-}} \mapsto \mathcal{A}7_i^c e^d$ , e^{ $d_- + c_- \cdot b8_{i-}} \mapsto B8_i^{-\frac{c}{\hbar}} e^d$ , e^{ $d_- + c_- \cdot \alpha 8_{i-}} \mapsto \mathcal{A}8_i^c e^d$ , e^ $\chi_- \mapsto$  Expand[12U[ $\chi_-$ ]]}
12U[r_Rule] := Module[{U = r[[1]] /. {b  $\rightarrow$  B, t  $\rightarrow$  T,  $\alpha \rightarrow \mathcal{A}$ }}, U  $\rightarrow$  12U[U21[U] /. r]];
AlsoUpper[rs_List] := rs  $\cup$  (12U /@ rs);

```

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Derivatives in the presence of exponentiated variables:

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```
In[ ] :=
Db1i[f_] := ∂b1i f - ħ B1i ∂B1i f;
Db2i[f_] := ∂b2i f - ħ B2i ∂B2i f;
Db3i[f_] := ∂b3i f - ħ B3i ∂B3i f;
Db4i[f_] := ∂b4i f - ħ B4i ∂B4i f;
Db5i[f_] := ∂b5i f - ħ B5i ∂B5i f;
Db6i[f_] := ∂b6i f - ħ B6i ∂B6i f;
Db7i[f_] := ∂b7i f - ħ B7i ∂B7i f;
Db8i[f_] := ∂b8i f - ħ B8i ∂B8i f;
Dα1i[f_] := ∂α1i f + A1i ∂A1i f;
Dα2i[f_] := ∂α2i f + A2i ∂A2i f;
Dα3i[f_] := ∂α3i f + A3i ∂A3i f;
Dα4i[f_] := ∂α4i f + A4i ∂A4i f;
Dα5i[f_] := ∂α5i f + A5i ∂A5i f;
Dα6i[f_] := ∂α6i f + A6i ∂A6i f;
Dα7i[f_] := ∂α7i f + A7i ∂A7i f;
Dα8i[f_] := ∂α8i f + A8i ∂A8i f;
Dv[f_] := ∂v f;
```

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```
In[ ] :=
LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
  Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars ∪ GreekVars))_;
```

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```
In[ ] :=
CF[ε_] := PPCF@Module[{s1},
  s1 = If[$k == ∞,
    Expand[ε] /. ep- → eFactor@p,
    Expand[ε] /. ep- → eFactor@p /. qn- → ∑k=0$k  $\frac{(\hbar \in \gamma n)^k}{k!}$  /. ek- /. k > $k → 0
  ]
  (*Expand@Collect[s1,Cases[s1,VarPattern,∞],CCF] /. CCF → Factor*)
]
```

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Weights:

```

In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N],
  Wt[x[i, j]] = {1, i - j}; Wt[y[i, j]] = {0, i - j};
  Wt[ξ[i, j]] = {0, j - i}; Wt[η[i, j]] = {1, j - i};
];
Do[As[i, 1, $N - 1],
  Wt[a[i]] = {1, 0}; Wt[b[i]] = {0, 0};
  Wt[α[i]] = {0, 0}; Wt[β[i]] = {1, 0};
];
Wt[u_] := Wt[u];

```

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```

In[ ]:= (* q~e^h γ €; Ai[n]~e^h ∈ ai; *)

```

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```

In[ ]:= $n = 3; $k = ∞; $E := {$n, $k}

```

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```

In[ ]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
  Table[As[i, 1, $n - 1], {a[i], A[i][_]}]
}
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi_ → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]

```

```

In[ ]:= $PBWGens*

```

Time: 0.01 seconds Out[]:=

```

{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}

```

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```

In[ ]:= UCF[ε_] := PPUCF@Module[{F, k}, Expand@Collect[
  ε /. {Ui_[L___, _[0], r___] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui_[L___, AB_[n_], r___] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence @@ Table[a @@ L[AB], k], r]
  }],
  U_[___], F
] /. F → CF]

```

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```

In[*]:= m_s__[0] = 0;
m_s__[x_Plus] := UCF[m_s /@ x];
m_{i→j}_[ε] := ε /. U_i → U_j;
m_{i,j→k}_[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_{i,j→k}_[c_. U_i_[ ] U_j_[y___]] := c U_k[y];
m_{i,j→k}_[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
  UCF[c U_i[xx, x[n1 + n2]] U_j[yy]] // m_{i,j→k};
m_{i,j→k}_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((c U_i[xx] RR_j[x, y] // UCF // m_{i,j→i}) U_j[yy] // UCF // m_{i,j→k}) // UCF
];
m_s__[ε_] := m_s[UCF[ε]];

```

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```

In[*]:= Supp[ε_] := Union@Cases[{ε}, U_i[___] :> i, ∞];

```

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```

In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PP_NCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = m_{i→σ[i]}[z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /. e^{k·} /; k > $k :> 0];
  Do[z = m_{σ[i], i→i}[z], {i, is}]; z];
UB[x_, y_] := PP_UB@UCF[x ⋈ y - y ⋈ x];

```

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```

In[*]:= Δ_{i→j,k}_[ε_] := UCF@Module[{j1, k1}, ε /. {
  U_i[] → U_j[] U_k[],
  U_i[f_, fs___] :> m_{k,k1→k} @ m_{j,j1→j}[Δ_{j,k}[f] Δ_{i→j1,k1}[U_i[fs]]]
}];
Δ_{i→j,k,l}_[ε_] := ε // Δ_{i→j,k} // Δ_{k→k,l}

```

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```

In[*]:= S_i_[ε_] := UCF@Module[{i1}, ε /. {
  U_i[f_, fs___] :> m_{i1,i→i}[S_i[f] S_{i1}[U_{i1}[fs]]]
}];
S̄_i_[ε_] := UCF@Module[{i1}, ε /. {
  U_i[f_, fs___] :> m_{i1,i→i}[S̄_i[f] S̄_{i1}[U_{i1}[fs]]]
}];

```

```

In[*]:= Δ_{1,2→3}[U_1[x12]]

```

Time: 0.02 seconds Out[]=

```

Δ_{1,2→3}[U_1[x12]]

```

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The $\mathbb{A}$ -algebra

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```
In[*]:=
RR_k[ai_, aj_] /;  $\tau[ai] == \tau[aj] == "a" := U_k[aj, ai];
RR_k[ai_, Aj_[n_]] /;  $\tau[ai] == "a" \wedge \tau[Aj] == "A" := U_k[Aj[n], ai];
RR_k[Aj_[n_], ai_] /;  $\tau[ai] == "a" \wedge \tau[Aj] == "A" := U_k[ai, Aj[n]];
RR_k[Ai_[m_], Aj_[n_]] /;  $\tau[Ai] == \tau[Aj] == "A" := U_k[Aj[n], Ai[m]];
RR_k[am_, xij_] /;  $\tau[am] == "a" \wedge \tau[xij] == "x" := Module[{i, j, m},
  {i, j} =  $\iota @ xij$ ; {m} =  $\iota @ am$ ;
   $U_k[xij, am] + \gamma (\delta_{m,i} - \delta_{m+1,i} - \delta_{m,j} + \delta_{m+1,j}) U_k[xij] / 2$ 
];
RR_k[Am_[n_], xij_] /;  $\tau[Am] == "A" \wedge \tau[xij] == "x" := Module[{i, j, m},
  {i, j} =  $\iota @ xij$ ; {m} =  $\iota @ Am$ ;  $q^{-n (\delta_{m,i} - \delta_{m+1,i} - \delta_{m,j} + \delta_{m+1,j})} U_k[xij, Am[n]]$ 
]$$$$$$ 
```

In[*]:= \$k

Time: 0.03 seconds Out[

1

In[*]:= RR_0[a[1], x[1, 2]]

Time: 0.05 seconds Out[

 $\gamma U_0[x_{12}] + U_0[x_{12}, a_1]$

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We take the reduction rules RR for xij's from Yamane page

509:

(I) $i = m < j < n$, (II) $i < m < n < j$, (III) $i < m < j = n$,

$$\frac{\overline{i} \quad \overline{j}}{\overline{m} \quad \overline{n}} \quad \frac{\overline{i} \quad \overline{j}}{\overline{m} \quad \overline{n}} \quad \frac{\overline{i} \quad \overline{j}}{\overline{m} \quad \overline{n}}$$

(IV) $i < m < j < n$, (V) $i < j = m < n$, (VI) $i < j < m < n$.

$$\frac{\overline{i} \quad \overline{j}}{\overline{m} \quad \overline{n}} \quad \frac{\overline{i} \quad \overline{j}}{\overline{m} \quad \overline{n}} \quad \frac{\overline{i} \quad \overline{j}}{\overline{m} \quad \overline{n}}$$

$$\begin{aligned} q^{-2} e_{ij} e_{mn} - e_{mn} e_{ij} &= 0 \\ [e_{ij}, e_{mn}] &= 0 \\ [e_{ij}, e_{mn}] &= (q^2 - q^{-2}) e_{in} e_{mj} \\ q^2 e_{ij} e_{mn} - e_{mn} e_{ij} &= q e_{in} \end{aligned}$$

$$\begin{aligned} &\text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\ &\text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\ &\text{if } ((i, j), (m, n)) \in C_{(IV)}. \\ &\text{if } ((i, j), (m, n)) \in C_{(V)}. \end{aligned}$$

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```

In[ ]:= RRk[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 Uk[xij, xmn], (* IVIII *)
    (i < m < j ∧ n < j) ∨ (j < m), Uk[xij, xmn], (* IIIVI *)
    i < m < j ∧ n > j, Uk[xij, xmn] + (q-2 - q2) Uk[x[i, n], x[m, j]], (* IV *)
    j == m, q2 Uk[xij, xmn] - q Uk[x[i, n]] (* V *)
  ]
]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.06 seconds Out:=
-q U2[x24] U3[x12] + q2 U2[x23, x34] U3[x12]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.05 seconds Out:=
q2 U0[x14] - q3 U0[x12, x24] - q3 U0[x13, x34] + q4 U0[x12, x23, x34]

In[ ]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
Time: 1.05 seconds Out:=
{}

In[ ]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
]]
>> {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[ ]:=
{}

```

```
Module[{Bas = $PBWGens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3 ->2 // m2,4 ->2 // m1,2 ->1) ==
      (P // m3,4 ->3 // m2,3 ->2 // m1,2 ->1) == (P // m1,2 ->1 // m3,4 ->3 // m1,3 ->1) ==
      (P // m1,2 ->1 // m1,3 ->1 // m1,4 ->1) == (P // m2,3 ->2 // m1,2 ->1 // m1,4 ->1)
  ]
]
» {x12, x13, x23, a1, a2}
```

Out[]:=

{ }

pdf

The Co-Product, Co-Associativity, and the Hopf Axiom

pdf

According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

pdf

```
In[ ]:= Do[As[i, 1, $n - 1],
  Δj,k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj,k[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj,k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj,k[x[i, n]] = mk,k1 ->k[mj,j1 ->j[
    q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
  ]
]
```

```
In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui / @ Bas], P -> Δi ->j,k[P]]
]
```

Time: 0.03 seconds Out[]:=

```
Ui[x12] -> Uj[x12] Uk[] - 2 ∈ ħ Uj[x12] Uk[a1] + Uj[] Uk[x12] + 2 ∈ ħ Uj[a1] Uk[x12]
Ui[x13] -> Uj[x13] Uk[] - 2 ∈ ħ Uj[x13] Uk[a1] - 2 ∈ ħ Uj[x13] Uk[a2] +
  Uj[] Uk[x13] + 2 ∈ ħ Uj[a1] Uk[x13] + 2 ∈ ħ Uj[a2] Uk[x13] + 4 ∇ ∈ ħ Uj[x12] Uk[x23]
Ui[x23] -> Uj[x23] Uk[] - 2 ∈ ħ Uj[x23] Uk[a2] + Uj[] Uk[x23] + 2 ∈ ħ Uj[a2] Uk[x23]
Ui[a1] -> Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] -> Uj[a2] Uk[] + Uj[] Uk[a2]
```

```
In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ -> True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→1,2 // Δ2→2,3) = (P // Δ1→1,3 // Δ1→1,2)
  ]]
]
» {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}
Out[ ]:=
{ }
```

```
In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ -> 0][Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]]
]
» {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}
Out[ ]:=
{ }
```

Is Δ a quantization of the co-bracket?

```
In[ ]:= $k = 1
Time: 0.02 seconds Out[ ]:=
1

In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1/@Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[_][xs___, A[_][n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[ ]:=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 γ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 γ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0
```

pdf

The Antipode

pdf

In[]:=

```

Do[As[i, 1, $n - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj-[x[i, n]] = mj1, j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]

Do[As[i, 1, $n - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj-[x[i, n]] = mj1, j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]

```

In[]:=

```

Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
» {x12, x13, x23, a1, a2, A2[-1], A2[-1], A2[1], A1[2], A1[-2], A1[-1], A1[-2], A1[1], A1[0], A1[-2]}

```

Out[]:=

```

{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A2[-1] → U1[],
 A2[-1] → U1[], A2[1] → U1[], A1[2] → U1[], A1[-2] → U1[],
 A1[-1] → U1[], A1[-2] → U1[], A1[1] → U1[], A1[0] → U1[], A1[-2] → U1[]}

```

```

In[*]:= $k = ∞
Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas],
  P = U1[X];
  X -> UCF[(P // S1 // S1) - P]
]]
]
Time: 0.05 seconds Out:=
∞

» {x12, x13, x23, a1, a2, A2[-1], A1[-2], A2[0], A1[-2], A1[2], A1[0], A1[0], A2[2], A1[0], A2[-1]}
Out[*]=
{}

```

pdf

The Generating Functions for the $\mathbb{A}$ operations

```

In[*]:= BeginProfile[]
Time: 0.03 seconds Out:=
BeginProfile[]

```

```

In[*]:= $k = 1;

```

pdf

Ad, TT, Seepage

pdf

```

In[*]:= PotQ[0] = True;
PotQ[c_. Ui[aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);

```

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```

In[*]:= Ad[x_] [y_] /; PotQ[x] ∧ (Head[y] != Plus) := PPAdPoty [If[y === 0, 0, eUB[y,x]/y y]];
Ad[x_] [y_Plus] /; PotQ[x] := PPAdPotPlus [Ad[x] /@ y];

```

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```

In[*]:= Ad[x_] [y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil @Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]

```

In[*]:= **Ad**[7 $\hbar$ U_j [**x12**] U_k [**a3**]] [U_j [**x23**] U_k [**a4**]]

Time: 0.09 seconds Out[=

$$U_j[x23] U_k[a4] + \left(-7 \hbar - 7 \gamma \in \hbar^2 - \frac{7}{2} \gamma^2 \epsilon^2 \hbar^3 \right) U_j[x13] U_k[a3, a4] + \\ \left(14 \gamma \in \hbar^2 + 14 \gamma^2 \epsilon^2 \hbar^3 \right) U_j[x12, x23] U_k[a3, a4] - 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x12, x13] U_k[a3, a3, a4] + \\ 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x12, x12, x23] U_k[a3, a3, a4] - \frac{686}{3} \gamma^2 \epsilon^2 \hbar^5 U_j[x12, x12, x13] U_k[a3, a3, a3, a4]$$

pdf

In[*]:= **TT** _{λ} [] = 0; (* Translated Tangent *)
TT _{λ} [**xs**____, **x**_] := **PP**_{TT}@**UCF**[**Ad**[**x**][**TT** _{λ} [**xs**]] + ∂_{λ} **x**];
TT _{λ} [**xs_List**] := **TT** _{λ} @@**xs**

In[*]:= **TT** _{λ} [λ ξ 12 U_0 [**x12**] U_1 [], λ ξ 23 U_0 [**x23**] U_1 []]

Time: 0.01 seconds Out[=

$$\xi 12 U_0[x12] U_1[] + \lambda \xi 12 \xi 23 U_0[x13] U_1[] + \xi 23 U_0[x23] U_1[]$$

pdf

In[*]:= **O** _{i} [**poly**_] := **Total**[
CoefficientRules[**poly**, # _{i} & /@ **\$PBWGens**] /.
(**p**_ → **c**_) => **c** **U** _{i} @@**Flatten**@**MapThread**[**ConstantArray**, {**\$PBWGens**, **p**}]
];
O _{i} ,**is**__[**poly**_] := **O** _{i} [**O** _{is} [**poly**]];
P _{i} [**$\mathcal{E}$** _] := **$\mathcal{E}$** /. **U** _{i} [**xs**____] => **Times**@@ (# _{i} & /@ {**xs**});
P _{i} ,**is**__[**poly**_] := **P** _{i} [**P** _{is} [**poly**]];

In[*]:= **O** _{i} [**x12** _{i} ² **x13** _{i} + 7 **x23** _{i} ¹⁷]

Time: 0.06 seconds Out[=

$$U_i[x12, x12, x13] + \\ 7 U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]$$

In[*]:= **O** _{i,j} [**x12** _{i} ² **x13** _{i} **x24** _{j} + 7 **x23** _{i} ¹⁷ **x25** _{j} ³]

Time: 0.03 seconds Out[=

$$U_i[x12, x12, x13] U_j[x24] + \\ 7 U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23] \\ U_j[x25, x25, x25]$$

In[*]:= **P** _{i,j} @**O** _{i,j} [**x12** _{i} ² **x13** _{i} **x24** _{j} + 7 **x23** _{i} ¹⁷ **x25** _{j} ³]

Time: 0.06 seconds Out[=

$$x12_i^2 x13_i x24_j + 7 x23_i^{17} x25_j^3$$

pdf

```
In[ ]:= Seepage_i_[{ }, c_] = c;
Seepage_i_[ss_List, P_] := Module[{lx},
  lx = $PBWgens[Length@ss];
  CF@Collect[P, lx_i, P_i[Ad[Last[ss] U_i[lx]] [0_i[Seepage_i[Most@ss, #]]]]
    &]
];
Seepage_i_,is_[ss_List, ss__List, P_] := Seepage_i_[ss, Seepage_is[ss, P]]
```

pdf

```
In[ ]:= Invert_is_[s_] := Module[{A, B, U1, j}, (* s=A U[] + e B *)
  U1 = Times@@ (U_#[] & /@ {is});
  A = Coefficient[s, U1] /. e -> 0;
  B = UCF[(s - A U1)];
  UCF[
    A^-1 (U1 + Sum[As[j, 1, $k], (-1)^j NonCommutativeMultiply@@ ConstantArray[B/A, j]])]
]
```

```
In[ ]:= $k = 3;
Invert_i,j[7 U_i[] U_j[] + e U_i[x12] U_j[x23]]
```

Time: 0.09 seconds Out[]:=

$$\frac{1}{7} U_i[] U_j[] - \frac{1}{49} e U_i[x12] U_j[x23] + \frac{1}{343} e^2 U_i[x12, x12] U_j[x23, x23] - \frac{e^3 U_i[x12, x12, x12] U_j[x23, x23, x23]}{2401}$$

pdf

```
In[ ]:= TriangularDSolve[eqns_, inits_, fs_, l_, CF_] :=
  PP_TriangularDSolve@Module[{ss = {}, reqns = eqns, rinit = inits, rfs = fs, s, p},
    While[rfs != {},
      {p} = FirstPosition[Count[#, Alternatives@@rfs, ∞, Heads -> True] & /@ reqns, 1];
      {{s}} = DSolve[reqns[[p]] ^ rinit[[p]], rfs[[p]], l];
      AppendTo[ss, s = s /. F_Function -> ReplacePart[F, 2 -> CF[F[[2]]]]];
      reqns = Delete[reqns, p] /. s /. e_ == 0 -> CF[e] == 0;
      rinit = Delete[rinit, p];
      rfs = Delete[rfs, p];
    ];
    ss
  ]
```

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The Generating Functions Engine

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Canonical Forms:

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```
In[ ]:= LogReduce[ $\mathcal{E}$ _] :=
   $\mathcal{E}$  /.  $c$ _ * Log[ $a$ _]  $\Rightarrow$  Log@Factor[ $a^c$ ] /. Log[ $a$ _] + Log[ $b$ _]  $\Rightarrow$  Log@Factor[ $a b$ ];
CF[ $\mathcal{E}$ _E] := CF /@ MapAt[LogReduce,  $\mathcal{E}$ , 1];
CF[ $\mathcal{E}$  : E_][_] := CF /@  $\mathcal{E}$ ;
```

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E operations:

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```
In[ ]:=  $\mathcal{E}$ _E[$_] := Length[ $\mathcal{E}$ ] - 1; E_[ $\mathcal{E}$ S_] [$_] := E[ $\mathcal{E}$ S] [$];
 $\mathcal{E}$ _E[k_Integer] :=  $\mathcal{E}$ [[k+1]]; E_[ $\mathcal{E}$ S_] [k_Integer] := { $\mathcal{E}$ S}[[k+1]];
E /:  $\mathcal{E}$ 1_E  $\equiv$   $\mathcal{E}$ 2_E := Inner[CF@#1 == CF@#2 &,  $\mathcal{E}$ 1,  $\mathcal{E}$ 2, And];
Ed1→r1[ $\mathcal{E}$ 1S_]  $\equiv$  Ed2→r2[ $\mathcal{E}$ 2S_] ^:= (d1 == d2)  $\wedge$  (r1 == r2)  $\wedge$  (E[ $\mathcal{E}$ 1S]  $\equiv$  E[ $\mathcal{E}$ 2S]);
E /:  $\mathcal{E}$ 1_E *  $\mathcal{E}$ 2_E := E @@ Table[CF[ $\mathcal{E}$ 1[kk] +  $\mathcal{E}$ 2[kk]], {kk, 0, Min[ $\mathcal{E}$ 1[$],  $\mathcal{E}$ 2[$]]}];
Ed1→r1[ $\mathcal{E}$ 1S_] Ed2→r2[ $\mathcal{E}$ 2S_] ^:= E(d1∪d2)→(r1∪r2) @@ (E[ $\mathcal{E}$ 1S] E[ $\mathcal{E}$ 2S]);
```

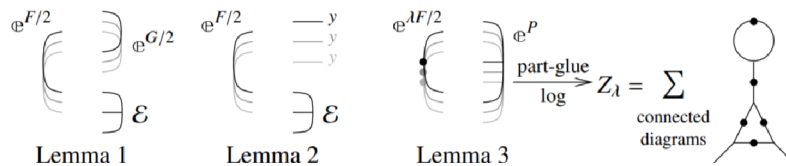
pdf

```
Ed1→r1[ $\mathcal{E}$ 1S_] // Ed2→r2[ $\mathcal{E}$ 2S_] := Module[{is = r1 ∩ d2, lvs, llvs},
  lvs = Cases[{ $\mathcal{E}$ 1S}, (Alternatives @@ LatinVars)_i_ /; MemberQ[is, i], ∞] ∪
  Cases[{ $\mathcal{E}$ 2S}, (Alternatives @@ GreekVars)_i_ /; MemberQ[is, i], ∞]*;
  llvs = lvs /. u_i_  $\Rightarrow$  u$@i // Echo;
  E(d1∪Complement[d2,is])→(r2∪Complement[r1,is]) @@
  (Zipllvs∪llvs*[{(llvs)*.(llvs), EchoLabel["zipinput"]][Times[
    E[ $\mathcal{E}$ 1S] /. Thread[lvs → llvs],
    E[ $\mathcal{E}$ 2S] /. Thread[lvs* → llvs*]
  ]]])
]
```

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Ziping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

pdf



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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii} ’s and the y_{ii} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

pdf

```

In[*]:= Zipvs [{ $\mathcal{F}$ _,  $\mathcal{E}$ _}] :=
Module[{dvs = Select[vs, (Wt[#] == {0, 0}  $\vee$  Wt[#] == {1, 0}) &] // Echo},
  { $\mathcal{F}$ ,  $\mathcal{E}$ } // Zip1vs (* // Zip2Complement[vs, dvs] *) // Zip2dvs //
  Zip3Complement[vs, dvs] // Zip3dvs
] // Last

```

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Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F: \mathcal{E} \otimes^{\frac{1}{2}} \sum_{i,j \in B} G_{ij} z_i z_j \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1}: \mathcal{E} \right\rangle_B$$

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```

In[*]:= Zip1{} = Identity;
Zip1vs @ { $\mathcal{F}$ _,  $\mathbb{E}[Q_, P\_]$ } := Module[{ $\mathcal{I}$ , F, G, u, v},
  Echo[{vs,  $\mathcal{F}$ , Q, P}];
   $\mathcal{I}$  = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v} \mathcal{F}$ , 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v} Q$ , 0], {u, vs}, {v, vs}];
  Echo@MatrixForm[ $\mathcal{I} - G.F$ ];
  {CF[(vs) . (F.Inverse[ $\mathcal{I} - G.F$ ]) . (vs) / 2],
    $\mathbb{E}$ [CF[Q - PowerExpand@Log[Det[ $\mathcal{I} - G.F$ ]] / 2 - vs.G.vs / 2], P]} // Echo
]

```

```

In[*]:= Wt /@ {x12$[2], y12$[1],  $\eta$ 12$[1],  $\xi$ 12$[2]}

```

Time: 0.05 seconds Out[] =

```
{ {1, -1}, {0, -1}, {1, 1}, {0, 1} }
```

```

In[*]:=  $\mathbb{E}_{\{ \} \rightarrow \{1,2\}}$  [x122 y121] //  $\mathbb{E}_{\{1,2\} \rightarrow \{ \}}$  [ (-1 + eb1i)  $\eta$ 121  $\xi$ 122]

```

```
» {x12$[2], y12$[1]}
```

```
» { }
```

```
» { {x12$[2], y12$[1],  $\eta$ 12$[1],  $\xi$ 12$[2]}, y12$[1]  $\eta$ 12$[1] + x12$[2]  $\xi$ 12$[2],
  x12$[2] y12$[1] -  $\eta$ 12$[1]  $\xi$ 12$[2] + eb1i  $\eta$ 12$[1]  $\xi$ 12$[2] }
```

```
»  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 
```

```
» {y12$[1]  $\eta$ 12$[1] + x12$[2]  $\xi$ 12$[2],  $\mathbb{E}$ [x12$[2] y12$[1] -  $\eta$ 12$[1]  $\xi$ 12$[2] + eb1i  $\eta$ 12$[1]  $\xi$ 12$[2]] }
```

```
» {Zip3, {x12$[2], y12$[1],  $\eta$ 12$[1],  $\xi$ 12$[2]},
  y12$[1]  $\eta$ 12$[1] + x12$[2]  $\xi$ 12$[2],  $\mathbb{E}$ [x12$[2] y12$[1] -  $\eta$ 12$[1]  $\xi$ 12$[2] + eb1i  $\eta$ 12$[1]  $\xi$ 12$[2]] }
```

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Getting rid of linear terms.

Lemma 2. $\left\langle F: \mathcal{E} \otimes^{\frac{1}{2}} \sum_{i \in B} F_{ij} y_i y_j \right\rangle_B = e^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F: \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

pdf

```

Zip2_{ } = Identity;
Zip2_{vs_} @ {F_, E[Q_, P_]} := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[∂_{u,v}F], 0], {u, vs}, {v, vs}];
  Y = Table[∂_v Q, {v, vs}] /. AlsoUpper@Table[v → 0, {v, vs}];
  CF /@ {F, E[Q - Y.vs + Y.F.Y / 2, P]} /. AlsoUpper@Thread[vs → vs + F.Y]
]

```

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Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathfrak{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} (\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

pdf

In[*]:=

```

Zip3_{vs_} @ {F_, E[E_]} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Echo["Zip3", vs, F, E];
  Do[Z[0, kk] = E[kk + 1], {kk, 0, E@$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v}F, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = D_u[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = D_v[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ E@$ , ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] === 0, 0, F[u, v] / (2 (m + 1)) (Z[m, kk, u, v] +
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
    F - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
    E@@Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, E@$}]
  }) /. AlsoUpper@Table[v → 0, {v, vs}]
]

```

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“Define” Code

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```

In[ ]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs]);
Define[op_is__ = ε_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[op_nisp, $E_, PPBoot@Block[{i, j, k}, op_isp, $E = ε; op_nis, $E]];
    SD[op_isp, op_{is}, $E]; SD[op_sis__, op_{sis}];
  ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
  } ] ]

```

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$$L_\lambda = \mathbb{E}(\lambda_{i_1}^1 x) \mathbb{E}(\lambda_{i_2}^2 x) \quad \text{Seepage is defined by: } \mathbb{E}_{\text{off}}(f; x_i) \mathbb{D}(\text{Seepage}(f_i, A)) = \mathbb{D}(e^{f; x_i} A)$$

$$\text{TT}(L_\lambda) = L' \partial_\lambda L_\lambda \quad S(B) = \text{Seepage}(\{f_i\}, B).$$

$$R_\lambda = \mathbb{E}(f+g), \quad f = f_i x_i, \quad g = \sum_{i \in I} g_i w_i, \quad e^f = F, \quad e^g = G, \quad R_\lambda = \mathbb{D}(FG), \quad \partial_\lambda F = F \partial_\lambda f$$

$$\text{TT}(R_\lambda) = R'_\lambda \partial_\lambda R_\lambda = \mathbb{D}(FG)' \mathbb{D}(\partial_\lambda FG) = S(G)' \mathbb{D}(F)' \mathbb{D}(F(\partial_\lambda f G + \partial_\lambda G)) = S(G)' S(\partial_\lambda f G + \partial_\lambda G)$$

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In[*]:=

```

Define[amti,j→k = PPamt[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ES[s_][ε_] := ε;
  TTL = ES["TTL"]@PPTTL@TTλ@
    Join[l1 = Table[If[τ[g] == "x", λ, 1] (g*)i Uk[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
  ff = Table[
    Sum[εp fp,uk[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp-,u-[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
  fs = Flatten[Table[
    fp,ℙk[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙk[u][λ] ℙk[u],
    {p, 0, $k}, {u, Complement[mons[p], Uk /@ $PBWGens]}
  ];
  G = Series[εgg, {ε, 0, $k}];
  S[B_] := UCF[SS@Ok[Seepagek[ff, SS@UCF@B]]];
  SG = ES["SG"]@PPSG@S[G];
  TTR = ES["TTR"]@PPTTR@UCF[Invertk[SG] ⊗ S[(∂λ (ff. (#k & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i,j}→{k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], (α[m]i + α[m]j) a[m]k], 0],
    Total[Cases[fs, fp,-]] /. fp-,u- => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]

```

pdf

In[]:=

```

Define[aΔti→j,k = PPaΔt[$k,$n] @Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
TTL = ES["TTL"] @
  TTλ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
mons[-1] = {};
mons[p_] := mons[p - 1] ∪
  Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] => Uj[x] Uk[y], ∞];
ffj = Table[
  Sum[εp fp,uj[λ], {p, 0, $k}],
  {u, $PBWGens}
] /. fp,uj[λ] /; τ[u] == "a" => If[p == 0, (u*)i, 0];
ffk = ffj /. j → k;
fs = Flatten[Table[
  fp,ℙj,k[m],
  {p, 0, $k}, {m, mons[p]}
]];
gg = Sum[
  εp fp,ℙj,k[u][λ] ℙj,k[u],
  {p, 0, $k}, {u, Complement[mons[p], Uj[] (Uk /@ $PBWGens) ∪ Uk[] (Uj /@ $PBWGens)]}
];
G = Series[egg, {ε, 0, $k}];
S[B_] := UCF[SS@Oj,k[Seepagej,k[ffj, ffk, SS@UCF@B]]];
SG = ES["SG"] @ PPSG @ S[G];
TTR = ES["TTR"] @ PPTTR @ UCF[
  Inverttj,k[SG] ⊗ S[(∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λ G)];
diff = UCF[TTL - TTR];
eqns = Flatten[Table[
  Coefficient[Coefficient[diff, m], ε, p] == 0,
  {p, 0, $k}, {m, mons[p]}
]];
inits = (#[0] == 0) & /@ fs;
ss = TriangularDSolve[eqns, inits, fs, λ, CF];
E{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
  If[p == 0, Sum[As[m, 1, $n - 1], α[m]i (a[m]j + a[m]k)], 0],
  Total[Cases[fs, fp,-] /. fp,u => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
]
]]

```

In[*]:=

```

Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] :=
    mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c_. Ui[x____] => Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,ui[λ] /; τ[u] == "a" => If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp,ℙi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙi[u][λ] ℙi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⊗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], -α[m]i a[m]i], 0],
    Total[Cases[fs, fp,u]] /. fp,u => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

```

In[*]:= Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] :=
    mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c_. Ui[x___] ⇒ Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,ui[λ] /; τ[u] == "a" ⇒ If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp,ℙi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙi[u][λ] ℙi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[εgg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⌗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], -α[m]i a[m]i], 0],
    Total[Cases[fs, fp,-]] /. fp,- ⇒ u (fp,u /. ss)[λ] /. λ | ℏ | γ → 1
  ]
]]

```

```

In[*]:= $k = 1; $n = 3;

```

```

In[*]:= aSt1 // aSt1

```

Time: 5.94 seconds Out[] =

```

E{1}→{1} [a11 α11 + a21 α21 + x121 ξ121 + x131 ξ131 + x231 ξ231, 0]

```

```
In[*]:= (*Splitter into B tensor A*)
SYi→j, k, l, m := E{i}→{j, k, l, m} [
  Sum[η[r, s]i y[r, s]j + ξ[r, s]i x[r, s]m, {r, 1, $n - 1}, {s, r + 1, $n}] +
  Sum[α[r]i a[r]l + β[r]i b[r]k, {r, 1, $n - 1}], Sequence @@ ConstantArray[0, $k]
]
```

```
In[*]:= SY1→2, 3, 4, 5
```

```
Time: 0.00 seconds Out[ ]=
```

$$E_{\{1\} \rightarrow \{2, 3, 4, 5\}} [a_{14} \alpha_{11} + a_{24} \alpha_{21} + b_{13} \beta_{11} + b_{23} \beta_{21} + y_{112} \eta_{111} + y_{122} \eta_{121} + y_{132} \eta_{131} + y_{222} \eta_{221} + y_{232} \eta_{231} + x_{115} \xi_{111} + x_{125} \xi_{121} + x_{135} \xi_{131} + x_{225} \xi_{221} + x_{235} \xi_{231}, 0]$$

P: A tensor B → k
R in B tensor A

```
In[*]:= Pi, j := E{i, j}→{} @@ PadRight [
  {Sum[η[r, s]j ξ[r, s]i, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[β[r]j α[r]i, {r, 1, $n - 1}]},
  $k + 1]
Ri, j := E{ }→{i, j} @@ PadRight [
  {Sum[y[r, s]i x[r, s]j, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[b[r]i a[r]j, {r, 1, $n - 1}]},
  $k + 1]
```

```
In[*]:= R1, 2 // P2, 3
R1, 2 // P5, 1
```

```
Time: 0.03 seconds Out[ ]=
```

$$E_{\{3\} \rightarrow \{1\}} [b_{11} \beta_{13} + b_{21} \beta_{23} + y_{121} \eta_{123} + y_{131} \eta_{133} + y_{231} \eta_{233}, 0]$$

```
Out[*]=
```

$$E_{\{5\} \rightarrow \{2\}} [a_{12} \alpha_{15} + a_{22} \alpha_{25} + x_{122} \xi_{125} + x_{132} \xi_{135} + x_{232} \xi_{235}, 0]$$

```
In[*]:= Define[bmtj, k→i = Module[{i0, i1, j1, k1}, (Ri0, i1 // aΔti1→j1, k1 // Pj1, j // Pk1, k) /. i0 → i],
  bΔtk→i, j = Module[{i1, j1, k1, k2}, ((Ri, i1 Rj, j1 // amti1, j1→k1) // Pk1, k2) /. k2 → k],
  bSti = Module[{i1, i2}, (Ri, i1 // aSti1 // Pi1, i2) /. i2 → i],
  bSti = Module[{i1, i2}, (Ri, i1 // aSti1 // Pi1, i2) /. i2 → i]]
Define[Ri, j = Ri, j // bSti]
```

```
In[*]:= amt1, 2→3 // l2U
```

```
Time: 0.03 seconds Out[ ]=
```

$$E_{\{1, 2\} \rightarrow \{3\}} \left[a_{13} (\alpha_{11} + \alpha_{12}) + a_{23} (\alpha_{21} + \alpha_{22}) + x_{123} \left(\xi_{121} + e^{\alpha_{11} - \frac{\alpha_{21}}{2}} \xi_{122} \right) + x_{133} \left(\xi_{131} + e^{\frac{\alpha_{11}}{2} + \frac{\alpha_{21}}{2}} \xi_{132} - e^{\alpha_{11} - \frac{\alpha_{21}}{2}} \xi_{122} \xi_{231} \right) + x_{233} \left(\xi_{231} + e^{-\frac{\alpha_{11}}{2} + \alpha_{21}} \xi_{232} \right), \right. \\ \left. - 2 e^{\alpha_{11} - \frac{\alpha_{21}}{2}} x_{123} x_{133} \xi_{122} \xi_{131} - e^{\alpha_{11} - \frac{\alpha_{21}}{2}} x_{133} \xi_{122} \xi_{231} + 2 e^{\alpha_{11} - \frac{\alpha_{21}}{2}} x_{123} x_{233} \xi_{122} \xi_{231} - 2 e^{\frac{\alpha_{11}}{2} + \frac{\alpha_{21}}{2}} x_{133} x_{233} \xi_{132} \xi_{231} \right]$$

```
In[*]:= $k = 0; $n = 2;
```

In[*]:= **bmt**_{1,2→3}

$$\gg \text{TTL } e^{-\gamma \alpha_{11}} \xi_{12_1} U_j[x_{12}] U_k[] + e^{-\gamma \alpha_{11}} \xi_{12_1} U_j[] U_k[x_{12}]$$

$$\gg \text{SG } U_j[] U_k[]$$

$$\gg \text{TTR } e^{-\gamma \alpha_{11}} U_j[x_{12}] U_k[] f_{\theta, x_{12_j}}'[\lambda] + e^{-\gamma \alpha_{11}} U_j[] U_k[x_{12}] f_{\theta, x_{12_k}}'[\lambda]$$

Out[*]=

$$E_{\{1,2\} \rightarrow \{3\}} [b_{13} \beta_{11} + b_{13} \beta_{12} + y_{123} \eta_{12_1} + y_{123} \eta_{12_2}]$$

In[*]:= **dm**_{1,2→3} // **Simplify**

Time: 0.30 seconds Out[*]=

$$E_{\{1,2\} \rightarrow \{3\}} \left[a_{13} (\alpha_{11} + \alpha_{12}) + b_{13} (\beta_{11} + \beta_{12}) + e^{-\alpha_{11}} (y_{123} (e^{\alpha_{11}} \eta_{12_1} + \eta_{12_2}) + (-1 + e^{b_{13}}) \eta_{12_2} \xi_{12_1} + e^{\alpha_{11}} x_{123} (\xi_{12_1} + e^{\alpha_{11}} \xi_{12_2})) \right]$$

In[*]:= **dm**_{1,2→3} // **l2u**

Time: 0.01 seconds Out[*]=

$$E_{\{1,2\} \rightarrow \{3\}} \left[a_{13} \alpha_{11} + a_{13} \alpha_{12} + b_{13} \beta_{11} + b_{13} \beta_{12} + y_{123} \eta_{12_1} + \frac{y_{123} \eta_{12_2}}{\mathcal{A}_{11}} + x_{123} \xi_{12_1} - \frac{\eta_{12_2} \xi_{12_1}}{\mathcal{A}_{11}} + \frac{\eta_{12_2} \xi_{12_1}}{B_{13} \mathcal{A}_{11}} + x_{123} \mathcal{A}_{11} \xi_{12_2} \right]$$

In[*]:= **E**_{{1}→{1,2}} [**x12**₁ **y12**₂] // **dm**_{1,2→3}

Time: 1.50 seconds Out[*]=

\$Aborted

In[*]:= **E**_{{1}→{1,2}} [**x12**₁ **y12**₂] // **E**_{{1,2}→{3}} [**a13** **α11** + **a13** **α12** + **b13** **β11** + **b13** **β12** +

$$y_{123} \eta_{12_1} + \frac{y_{123} \eta_{12_2}}{\mathcal{A}_{11}} + x_{123} \xi_{12_1} - \frac{\eta_{12_2} \xi_{12_1}}{\mathcal{A}_{11}} + \frac{\eta_{12_2} \xi_{12_1}}{B_{13} \mathcal{A}_{11}} + x_{123} \mathcal{A}_{11} \xi_{12_2}]$$

Time: 6.61 seconds Out[*]=

\$Aborted

In[*]:= **bΔt**_{1→2,3} // **l2u**

Time: 0.05 seconds Out[*]=

$$E_{\{1\} \rightarrow \{2,3\}} \left[b_{12} \beta_{11} + b_{13} \beta_{11} + b_{22} \beta_{21} + b_{23} \beta_{21} + y_{122} \eta_{12_1} + e^{b_{12} - \frac{b_{22}}{2}} y_{123} \eta_{12_1} + y_{132} \eta_{13_1} + e^{\frac{b_{12}}{2} + \frac{b_{22}}{2}} y_{133} \eta_{13_1} - e^{b_{12} - \frac{b_{22}}{2}} y_{123} y_{23_2} \eta_{13_1} + y_{23_2} \eta_{23_1} + e^{-\frac{b_{12}}{2} + b_{22}} y_{23_3} \eta_{23_1}, -e^{b_{12} - \frac{b_{22}}{2}} y_{123} y_{23_2} \eta_{13_1} - 2 e^{b_{12} - \frac{b_{22}}{2}} y_{123} y_{13_2} \eta_{12_1} \eta_{13_1} + 2 e^{b_{12} - \frac{b_{22}}{2}} y_{123} y_{23_2} \eta_{12_1} \eta_{23_1} - 2 e^{\frac{b_{12}}{2} + \frac{b_{22}}{2}} y_{13_3} y_{23_2} \eta_{13_1} \eta_{23_1} \right]$$

In[*]:= **E**_{{1}→{2,3}} [**b12** **β11** + **b13** **β11** + **b22** **β21** + **b23** **β21** + **y122** **η121** + **e**^{**b12** - $\frac{b_{22}}{2}$} **y123** **η121** + **y132** **η131** + **e** ^{$\frac{b_{12}}{2} + \frac{b_{22}}{2}$} **y133** **η131** - **e**^{**b12** - $\frac{b_{22}}{2}$} **y123** **y232** **η131** + **y232** **η231** + **e** ^{$-\frac{b_{12}}{2} + b_{22}$} **y233** **η231**, -**e**^{**b12** - $\frac{b_{22}}{2}$} **y123** **y232** **η131** - 2 **e**^{**b12** - $\frac{b_{22}}{2}$} **y123** **y132** **η121** **η131** + 2 **e**^{**b12** - $\frac{b_{22}}{2}$} **y123** **y232** **η121** **η231** - 2 **e**^{**b12**} **y133** **y232** **η131** **η231**] // **l2u**

Time: 0.02 seconds Out[*]=

$$E_{\{1\} \rightarrow \{2,3\}} \left[b_{12} \beta_{11} + b_{13} \beta_{11} + b_{22} \beta_{21} + b_{23} \beta_{21} + y_{122} \eta_{12_1} + e^{b_{12} - \frac{b_{22}}{2}} y_{123} \eta_{12_1} + y_{132} \eta_{13_1} + e^{\frac{b_{12}}{2} + \frac{b_{22}}{2}} y_{133} \eta_{13_1} - e^{b_{12} - \frac{b_{22}}{2}} y_{123} y_{23_2} \eta_{13_1} + y_{23_2} \eta_{23_1} + e^{-\frac{b_{12}}{2} + b_{22}} y_{23_3} \eta_{23_1}, -e^{b_{12} - \frac{b_{22}}{2}} y_{123} y_{23_2} \eta_{13_1} - 2 e^{b_{12} - \frac{b_{22}}{2}} y_{123} y_{13_2} \eta_{12_1} \eta_{13_1} + 2 e^{b_{12} - \frac{b_{22}}{2}} y_{123} y_{23_2} \eta_{12_1} \eta_{23_1} - 2 e^{b_{12}} y_{13_3} y_{23_2} \eta_{13_1} \eta_{23_1} \right]$$

In[*]:= 12U

Time: 0.00 seconds Out[*]=
12U

Testing A and B dually paired Hopf algebras

co-associativity

In[*]:= {(aΔt_{1→1,2} // aΔt_{2→2,3}) ≡ (aΔt_{1→2,3} // aΔt_{2→1,2}), (bΔt_{1→1,2} // bΔt_{2→2,3}) ≡ (bΔt_{1→2,3} // bΔt_{2→1,2})}

» TTL $e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_i} U_j[x_{12}] U_k[] +$
 $\left(e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \xi_{13_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \lambda \xi_{12_i} \xi_{23_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \gamma \in \lambda \hbar \xi_{12_i} \xi_{23_i} \right) U_j[x_{13}] U_k[] +$
 $\ll 31 \gg + 2 e^{-\frac{3\gamma\alpha_2}{2}} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} U_j[] U_k[x_{13}, x_{23}]$
 » SG $U_j[] U_k[] + e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \in U_j[x_{12}] U_k[a_1] f_{1,a_{1k}x_{12j}}[\lambda] + \ll 21 \gg +$
 $U_j[] U_k[x_{13}, x_{23}] \left(e^{-\frac{3\gamma\alpha_2}{2}} \in f_{\theta,x_{23k}}[\lambda] f_{1,x_{12k}x_{23k}}[\lambda] + e^{-\frac{3\gamma\alpha_2}{2}} \in f_{1,x_{13k}x_{23k}}[\lambda] \right)$
 » TTR $U_j[x_{12}] U_k[x_{12}] \left(-e^{-\gamma(2\alpha_1-\alpha_2)} \gamma \in f_{1,a_{1j}x_{12k}}[\lambda] f_{\theta,x_{12j}}'[\lambda] - e^{-\gamma(2\alpha_1-\alpha_2)} \gamma \in f_{1,a_{1k}x_{12j}}[\lambda] f_{\theta,x_{12k}}'[\lambda] \right) +$
 $\ll 1 \gg + \ll 27 \gg + \ll 1 \gg +$
 $U_j[] U_k[x_{13}, x_{23}] \left(2 e^{-\frac{3\gamma\alpha_2}{2}} \gamma \in \hbar f_{\theta,x_{23k}}[\lambda] f_{\theta,x_{13k}}'[\lambda] + \ll 1 \gg + e^{-\frac{\ll 1 \gg}{2}} \in f_{1,\ll 1 \gg}'[\lambda] \right)$

Out[*]=
{True, True}

algebra morphism

In[*]:= {(aΔt_{i→1,2} aΔt_{j→3,4} // amt_{1,3→i} // amt_{2,4→j}) ≡ (amt_{i,j→k} // aΔt_{k→i,j}),
 (bΔt_{i→1,2} bΔt_{j→3,4} // bmt_{1,3→i} // bmt_{2,4→j}) ≡ (bmt_{i,j→k} // bΔt_{k→i,j})}

Time: 0.50 seconds Out[*]=
{True, True}

associativity

In[*]:= {(amt_{1,2→k} // amt_{k,3→k}) ≡ (amt_{2,3→k} // amt_{1,k→k}), (bmt_{1,2→k} // bmt_{k,3→k}) ≡ (bmt_{2,3→k} // bmt_{1,k→k})}

Time: 0.12 seconds Out[*]=
{True, True}

antipode

In[*]:= {aΔt_{i→1,2} // aSt₁ // amt_{1,2→1}, aΔt_{i→1,2} // aSt₂ // amt_{1,2→1},
 bΔt_{i→1,2} // bSt₁ // bmt_{1,2→1}, bΔt_{i→1,2} // bSt₂ // bmt_{1,2→1}}

Time: 1.06 seconds Out[*]=
{E_{{i}→{1}}[0, 0], E_{{i}→{1}}[0, 0], E_{{i}→{1}}[0, 0], E_{{i}→{1}}[0, 0]}

inverse antipode

In[*]:= {aSt_i // aSt_i, bSt_i // bSt_i}

Time: 0.33 seconds Out[*]=
 {E_{{i}→{i}}[a_{1i} α_{1i} + a_{2i} α_{2i} + x_{12i} ξ_{12i} + x_{13i} ξ_{13i} + x_{23i} ξ_{23i}, 0],
 E_{{i}→{i}}[b_{1i} β_{1i} + b_{2i} β_{2i} + y_{12i} η_{12i} + y_{13i} η_{13i} + y_{23i} η_{23i}, 0]}

Pairing

```
In[*]:= { (amt1,2→3 // P3,-3) ≡ (bΔ-3→-1,-2 // P1,-1 // P2,-2),
  (bmt1,2→3 // P-3,3) ≡ (aΔ-3→-1,-2 // P-1,1 // P-2,2),
  (aSt1 // P1,-1) ≡ (bSt-1 // P1,-1), (aSt1 // P1,-1) ≡ (bSt-1 // P1,-1) }
```

Time: 0.20 seconds Out:=

```
{True, True, True, True}
```

```
In[*]:= Define [dmi,j→k = Module [ {i1, i2, i3, i4},
  ( (sYi→i4,i4,i1,i1 // aΔi1→i1,i2 // aΔi2→i2,i3 // aSti3) (sYj→-i1,-i1,-i4,-i4 // bΔ-i1→-i1,-i2 //
    bΔ-i2→-i2,-i3) ) // (Pi1,-i3 Pi3,-i1 amti2,-i4→k bmti4,-i2→k) ] ]
Define [dSi = sYi→1,1,2,2 // (bSt1 aSt2) // dm2,1→i,
  dSi = sYi→1,1,2,2 // (bSt1 aSt2) // dm2,1→i,
  dΔi→j,k = (bΔi→1,3 aΔi→4,2) // (dm3,4→k dm1,2→j) ]
(*Notice we take co-opposite on the A-side! to make the double*)
```

Testing Double

co-associativity

```
In[*]:= (dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→2,3 // dΔ2→1,2)
```

Time: 4.83 seconds Out:=

```
True
```

algebra morphism

```
In[*]:= (dΔi→1,2 dΔj→3,4 // dm1,3→i // dm2,4→j) ≡ (dmi,j→k // dΔk→i,j)
```

Time: 16.09 seconds Out:=

```
True
```

associativity

```
In[*]:= (dm1,2→k // dmk,3→k) ≡ (dm2,3→k // dm1,k→k)
```

Time: 6.73 seconds Out:=

```
True
```

antipode

```
In[*]:= dΔi→1,2
```

Time: 0.01 seconds Out:=

$$\begin{aligned} & E_{\{i\} \rightarrow \{1,2\}} \left[a_{11} \alpha_{11} + a_{12} \alpha_{11} + a_{21} \alpha_{21} + a_{22} \alpha_{21} + b_{11} \beta_{11} + b_{12} \beta_{11} + b_{21} \beta_{21} + b_{22} \beta_{21} + y_{121} \eta_{121} + \right. \\ & e^{\frac{1}{2} (2b_{11}-b_{21})} y_{122} \eta_{121} + y_{131} \eta_{131} + e^{\frac{1}{2} (b_{11}+b_{21})} y_{132} \eta_{131} - e^{\frac{1}{2} (2b_{11}-b_{21})} y_{122} y_{231} \eta_{131} + y_{231} \eta_{231} + \\ & e^{\frac{1}{2} (-b_{11}+2b_{21})} y_{232} \eta_{231} + x_{121} \xi_{121} + x_{122} \xi_{121} + x_{131} \xi_{131} + x_{132} \xi_{131} + x_{231} \xi_{231} + x_{232} \xi_{231}, \\ & - e^{\frac{1}{2} (2b_{11}-b_{21})} y_{122} y_{231} \eta_{131} - 2 e^{\frac{1}{2} (2b_{11}-b_{21})} y_{122} y_{131} \eta_{121} \eta_{131} + 2 e^{\frac{1}{2} (2b_{11}-b_{21})} y_{122} y_{231} \eta_{121} \eta_{231} - \\ & 2 e^{\frac{1}{2} (b_{11}+b_{21})} y_{132} y_{231} \eta_{131} \eta_{231} + 2 a_{12} x_{121} \xi_{121} - 2 a_{11} x_{122} \xi_{121} + 2 a_{12} x_{131} \xi_{131} + \\ & 2 a_{22} x_{131} \xi_{131} - 2 a_{11} x_{132} \xi_{131} - 2 a_{21} x_{132} \xi_{131} + 4 x_{122} x_{231} \xi_{131} - \\ & x_{122} x_{131} \xi_{121} \xi_{131} + x_{121} x_{132} \xi_{121} \xi_{131} + 2 a_{22} x_{231} \xi_{231} - 2 a_{21} x_{232} \xi_{231} + \\ & \left. x_{122} x_{231} \xi_{121} \xi_{231} - x_{121} x_{232} \xi_{121} \xi_{231} - x_{132} x_{231} \xi_{131} \xi_{231} + x_{131} x_{232} \xi_{131} \xi_{231} \right] \end{aligned}$$

In[*]:= $d\Delta_{i \rightarrow 1,2} // dS_1$

In[*]:= $\{d\Delta_{i \rightarrow 1,2} // dS_1 // dm_{1,2 \rightarrow 1}, d\Delta_{i \rightarrow 1,2} // dS_2 // dm_{1,2 \rightarrow 1}, dS_1 // \overline{dS_1}\}$

Time: 395.36 seconds Out[=

\$Aborted

quasi-triangular axioms

In[*]:= $(R_{1,3} // d\Delta_{1 \rightarrow 1,2}) \equiv (R_{1,3} R_{2,4} // dm_{3,4 \rightarrow 3})$

$(R_{1,3} // d\Delta_{3 \rightarrow 2,3}) \equiv (R_{1,3} R_{0,2} // dm_{1,0 \rightarrow 1})$

$(d\Delta_{i \rightarrow k,j} R_{1,2} // dm_{j,1 \rightarrow 1} // dm_{k,2 \rightarrow 2}) \equiv (R_{1,2} d\Delta_{i \rightarrow j,k} // dm_{1,j \rightarrow 1} // dm_{2,k \rightarrow 2})$

Time: 15.44 seconds Out[=

True

Out[*]=

True

Out[*]=

True

In[*]:= $(R_{1,2} // aSt_2) \equiv (\overline{R}_{1,2})$

Time: 0.01 seconds Out[=

True

Braid-like Reidemeister moves: R2,R3

$\{R_{1,2} \overline{R}_{3,4} // dm_{1,3 \rightarrow 1} // dm_{2,4 \rightarrow 2}, \overline{R}_{1,2} R_{3,4} // dm_{1,3 \rightarrow 1} // dm_{2,4 \rightarrow 2},$

$(R_{1,2} R_{4,3} R_{5,6} // dm_{1,4 \rightarrow 1} // dm_{2,5 \rightarrow 5} // dm_{3,6 \rightarrow 3}) \equiv (R_{1,6} R_{2,3} R_{4,5} // dm_{1,4 \rightarrow 1} // dm_{2,5 \rightarrow 5} // dm_{3,6 \rightarrow 3})\}$

Time: 29.61 seconds Out[=

$\{\mathbb{E}_{\{\} \rightarrow \{1,2\}}[\emptyset, \emptyset], \mathbb{E}_{\{\} \rightarrow \{1,2\}}[\emptyset, \emptyset], \text{True}\}$

Drinfeld element:

In[*]:= $\$k = 0; \$n = 2;$

In[*]:= $u_{i_} := \text{Module}[\{i1, i2\}, R_{i1,i2} // dS_{i2} // dm_{i2,i1 \rightarrow i}]$

In[*]:= u_2

Time: 7.39 seconds Out[=

\$Aborted

In[*]:= $R_{i1,i2} // dS_{i2}$

» $TTL e^{-\gamma \alpha_{1i}} \xi_{12i} U_j[x_{12}] U_k[] + e^{-\gamma \alpha_{1i}} \xi_{12i} U_j[] U_k[x_{12}]$

» $SG U_j[] U_k[]$

» $TTR e^{-\gamma \alpha_{1i}} U_j[x_{12}] U_k[] f_{0,x_{12j}}'[\lambda] + e^{-\gamma \alpha_{1i}} U_j[] U_k[x_{12}] f_{0,x_{12k}}'[\lambda]$

Out[*]=

$\mathbb{E}_{\{\} \rightarrow \{i1,i2\}}[-a_{1i2} b_{1i1} - e^{-b_{1i1}} x_{12i2} y_{12i1}]$

In[*]:= dS_{i2}

Time: 0.05 seconds Out[=

$\mathbb{E}_{\{i2\} \rightarrow \{i2\}}[-a_{1i2} \alpha_{1i2} - b_{1i2} \beta_{1i2} - e^{-b_{1i2} + \alpha_{1i2}} y_{12i2} \eta_{12i2} - e^{-\alpha_{1i2}} x_{12i2} \xi_{12i2} + \eta_{12i2} \xi_{12i2} - e^{-b_{1i2}} \eta_{12i2} \xi_{12i2}]$

In[*]:= **dm_{i2,i1→i}** // **Simplify**

Time: 0.17 seconds Out[]=

$$\mathbb{E}_{\{i2,i1\} \rightarrow \{i\}} \left[\mathbf{a}_{1i} (\alpha_{1i1} + \alpha_{1i2}) + \mathbf{b}_{1i} (\beta_{1i1} + \beta_{1i2}) + e^{-\alpha_{1i2}} \left(\mathbf{y}_{12i} (\eta_{12i1} + e^{\alpha_{1i2}} \eta_{12i2}) + (-1 + e^{\mathbf{b}_{1i}}) \eta_{12i1} \xi_{12i2} + e^{\alpha_{1i2}} \mathbf{x}_{12i} (e^{\alpha_{1i2}} \xi_{12i1} + \xi_{12i2}) \right) \right]$$

In[*]:= **E_{{i}→{1,2}}[**x₁₂** **y₁₂**]** // **E_{{1,2}→{i}}[**(-1 + e^{b_{1i}}) η₁₂ ξ₁₂**]**

Time: 8.47 seconds Out[]=

\$Aborted

In[*]:= **dm_{i2,i1→i}**

Time: 0.16 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i2,i1\} \rightarrow \{i\}} \left[\mathbf{a}_{1i} \alpha_{1i1} + \mathbf{a}_{1i} \alpha_{1i2} + \mathbf{a}_{2i} \alpha_{2i1} + \mathbf{a}_{2i} \alpha_{2i2} + \mathbf{b}_{1i} \beta_{1i1} + \mathbf{b}_{1i} \beta_{1i2} + \right. \\ & \mathbf{b}_{2i} \beta_{2i1} + \mathbf{b}_{2i} \beta_{2i2} + e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \mathbf{y}_{12i} \eta_{12i1} + \mathbf{y}_{12i} \eta_{12i2} + e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{13i} \eta_{13i1} + \\ & \mathbf{y}_{13i} \eta_{13i2} + e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} \mathbf{y}_{23i} \eta_{23i1} + \mathbf{y}_{23i} \eta_{23i2} + e^{\frac{1}{2}(2\alpha_{1i2} - \alpha_{2i2})} \mathbf{x}_{12i} \xi_{12i1} + \\ & \mathbf{x}_{12i} \xi_{12i2} - e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \eta_{12i1} \xi_{12i2} + e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} \eta_{12i1} \xi_{12i2} - \\ & e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{23i} \eta_{13i1} \xi_{12i2} + e^{\frac{1}{2}(\alpha_{1i2} + \alpha_{2i2})} \mathbf{x}_{13i} \xi_{13i1} + \mathbf{x}_{13i} \xi_{13i2} - \\ & e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \eta_{13i1} \xi_{13i2} + e^{\frac{1}{2}(\mathbf{b}_{1i} + \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \eta_{13i1} \xi_{13i2} + e^{\frac{1}{2}(-\alpha_{1i2} + 2\alpha_{2i2})} \mathbf{x}_{23i} \xi_{23i1} + \mathbf{x}_{23i} \xi_{23i2} + \\ & e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{12i} \eta_{13i1} \xi_{23i2} - e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} \eta_{23i1} \xi_{23i2} + e^{\frac{1}{2}(-\mathbf{b}_{1i} + 2\mathbf{b}_{2i} + \alpha_{1i2} - 2\alpha_{2i2})} \eta_{23i1} \xi_{23i2} - \\ & e^{\frac{1}{2}(2\alpha_{1i2} - \alpha_{2i2})} \mathbf{x}_{13i} \xi_{12i1} \xi_{23i2} - e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \eta_{13i1} \xi_{12i2} \xi_{23i2} + e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \eta_{13i1} \xi_{12i2} \xi_{23i2}, \\ & 2 e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \mathbf{y}_{12i} \beta_{1i2} \eta_{12i1} - 2 \mathbf{y}_{12i} \beta_{1i1} \eta_{12i2} + 2 e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{13i} \beta_{1i2} \eta_{13i1} + \\ & 2 e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{13i} \beta_{2i2} \eta_{13i1} - e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{12i} \mathbf{y}_{13i} \eta_{12i2} \eta_{13i1} - 2 \mathbf{y}_{13i} \beta_{1i1} \eta_{13i2} - \\ & 2 \mathbf{y}_{13i} \beta_{2i1} \eta_{13i2} + e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \mathbf{y}_{12i} \mathbf{y}_{13i} \eta_{12i1} \eta_{13i2} + 2 e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} \mathbf{y}_{23i} \beta_{2i2} \eta_{23i1} + \\ & 4 e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} \mathbf{y}_{13i} \eta_{12i2} \eta_{23i1} + e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} \mathbf{y}_{12i} \mathbf{y}_{23i} \eta_{12i2} \eta_{23i1} - \\ & e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} \mathbf{y}_{13i} \mathbf{y}_{23i} \eta_{13i2} \eta_{23i1} - 2 \mathbf{y}_{23i} \beta_{2i1} \eta_{23i2} - e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \mathbf{y}_{12i} \mathbf{y}_{23i} \eta_{12i1} \eta_{23i2} + \\ & e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{13i} \mathbf{y}_{23i} \eta_{13i1} \eta_{23i2} + 4 \mathbf{x}_{12i} \beta_{1i1} \xi_{12i2} - 2 e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \eta_{12i1} \xi_{12i2} + \\ & 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} \eta_{12i1} \xi_{12i2} - 2 e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \mathbf{a}_{1i} \eta_{12i1} \xi_{12i2} - \\ & 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} \mathbf{a}_{1i} \eta_{12i1} \xi_{12i2} + 2 e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \mathbf{x}_{12i} \mathbf{y}_{12i} \eta_{12i1} \xi_{12i2} - \\ & 2 e^{\frac{1}{2}(-2\alpha_{1i2} + \alpha_{2i2})} \beta_{1i1} \eta_{12i1} \xi_{12i2} + 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} \beta_{1i1} \eta_{12i1} \xi_{12i2} + \\ & 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 4\alpha_{1i2} + 2\alpha_{2i2})} \mathbf{y}_{12i} \eta_{12i1}^2 \xi_{12i2} - 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} \mathbf{y}_{12i} \eta_{12i1} \eta_{12i2} \xi_{12i2} + \\ & e^{\frac{1}{2}(-\alpha_{1i2} - \alpha_{2i2})} \mathbf{x}_{12i} \mathbf{y}_{13i} \eta_{13i1} \xi_{12i2} - 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{23i} \eta_{13i1} \xi_{12i2} + \\ & 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{a}_{1i} \mathbf{y}_{23i} \eta_{13i1} \xi_{12i2} - 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{23i} \beta_{1i1} \eta_{13i1} \xi_{12i2} - \\ & 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{23i} \beta_{2i2} \eta_{13i1} \xi_{12i2} - e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 3\alpha_{1i2})} \mathbf{y}_{13i} \eta_{12i1} \eta_{13i1} \xi_{12i2} - \\ & 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 3\alpha_{1i2})} \mathbf{y}_{12i} \mathbf{y}_{23i} \eta_{12i1} \eta_{13i1} \xi_{12i2} - 4 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{13i} \eta_{12i2} \eta_{13i1} \xi_{12i2} + \\ & e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{12i} \mathbf{y}_{23i} \eta_{12i2} \eta_{13i1} \xi_{12i2} - e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 2\alpha_{1i2} - 2\alpha_{2i2})} \mathbf{y}_{13i} \mathbf{y}_{23i} \eta_{13i1}^2 \xi_{12i2} - \\ & e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} \mathbf{y}_{13i} \eta_{12i1} \eta_{13i2} \xi_{12i2} + 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{13i} \mathbf{y}_{23i} \eta_{13i1} \eta_{13i2} \xi_{12i2} - \\ & e^{\frac{1}{2}(\alpha_{1i2} - 2\alpha_{2i2})} \mathbf{x}_{12i} \mathbf{y}_{23i} \eta_{23i1} \xi_{12i2} + e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{23i} \eta_{12i1} \eta_{23i1} \xi_{12i2} + \\ & e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 3\alpha_{2i2})} \mathbf{y}_{23i}^2 \eta_{13i1} \eta_{23i1} \xi_{12i2} + e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - 2\alpha_{1i2} + \alpha_{2i2})} \mathbf{y}_{23i} \eta_{12i1} \eta_{23i2} \xi_{12i2} - \\ & e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} - \alpha_{1i2} - \alpha_{2i2})} \mathbf{y}_{23i}^2 \eta_{13i1} \eta_{23i2} \xi_{12i2} - 2 \mathbf{x}_{12i} \eta_{12i1} \xi_{12i1} \xi_{12i2} - \\ & 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i})} \mathbf{x}_{12i} \eta_{12i1} \xi_{12i1} \xi_{12i2} + 2 e^{\frac{1}{2}(2\mathbf{b}_{1i} - \mathbf{b}_{2i} + \alpha_{1i2} - 2\alpha_{2i2})} \mathbf{x}_{12i} \mathbf{y}_{23i} \eta_{13i1} \xi_{12i1} \xi_{12i2} - \end{aligned}$$

$$\begin{aligned}
& 2 e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} x_{12i} \eta_{12i1} \xi_{12i2}^2 + 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} x_{12i} \eta_{12i1} \xi_{12i2}^2 + \\
& 2 e^{2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2}} \eta_{12i1}^2 \xi_{12i2}^2 - 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-4\alpha_{1i2}+2\alpha_{2i2})} \eta_{12i1}^2 \xi_{12i2}^2 - \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{12i} y_{23i} \eta_{13i1} \xi_{12i2}^2 - 3 e^{\frac{1}{2}(4b_{1i}-2b_{2i}-3\alpha_{1i2})} y_{23i} \eta_{12i1} \eta_{13i1} \xi_{12i2}^2 + \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2})} y_{23i} \eta_{12i1} \eta_{13i1} \xi_{12i2}^2 + e^{2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2}} y_{23i}^2 \eta_{13i1}^2 \xi_{12i2}^2 - \\
& e^{\frac{1}{2}(-\alpha_{1i2}+2\alpha_{2i2})} x_{13i} \eta_{12i1} \xi_{12i2} \xi_{13i1} - e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}+2\alpha_{2i2})} x_{13i} \eta_{12i1} \xi_{12i2} \xi_{13i1} + \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i})} x_{13i} y_{23i} \eta_{13i1} \xi_{12i2} \xi_{13i1} + 4 x_{13i} \beta_{1i1} \xi_{13i2} + 4 x_{13i} \beta_{2i1} \xi_{13i2} + \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} x_{23i} \eta_{12i1} \xi_{13i2} + e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} x_{13i} y_{12i} \eta_{12i1} \xi_{13i2} - \\
& 6 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \eta_{13i1} \xi_{13i2} + 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{13i1} \xi_{13i2} + 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} a_{1i} \eta_{13i1} \xi_{13i2} - 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} a_{1i} \eta_{13i1} \xi_{13i2} - 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} a_{2i} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} a_{2i} \eta_{13i1} \xi_{13i2} + 4 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{12i} y_{12i} \eta_{13i1} \xi_{13i2} + \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{13i} y_{13i} \eta_{13i1} \xi_{13i2} - 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{23i} y_{23i} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \beta_{1i1} \eta_{13i1} \xi_{13i2} + 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \beta_{1i1} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \beta_{2i1} \eta_{13i1} \xi_{13i2} + 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \beta_{2i1} \eta_{13i1} \xi_{13i2} + \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2})} y_{12i} \eta_{12i1} \eta_{13i1} \xi_{13i2} + e^{\frac{1}{2}(b_{1i}+b_{2i}-3\alpha_{1i2})} y_{12i} \eta_{12i1} \eta_{13i1} \xi_{13i2} - \\
& 2 e^{-\frac{3\alpha_{1i2}}{2}} y_{12i} \eta_{12i1} \eta_{13i1} \xi_{13i2} - e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{12i} \eta_{12i2} \eta_{13i1} \xi_{13i2} + \\
& 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{13i} \eta_{13i1}^2 \xi_{13i2} - 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{12i} y_{23i} \eta_{13i1}^2 \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{13i} \eta_{13i1} \eta_{13i2} \xi_{13i2} - 4 e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{12i} \eta_{23i1} \xi_{13i2} + \\
& e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{13i} y_{23i} \eta_{23i1} \xi_{13i2} + 4 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \eta_{12i1} \eta_{23i1} \xi_{13i2} - \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{12i1} \eta_{23i1} \xi_{13i2} + 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{2i2})} y_{23i} \eta_{13i1} \eta_{23i1} \xi_{13i2} - \\
& e^{\frac{1}{2}(b_{1i}+b_{2i}-3\alpha_{2i2})} y_{23i} \eta_{13i1} \eta_{23i1} \xi_{13i2} - e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{23i} \eta_{13i1} \eta_{23i2} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(2\alpha_{1i2}-\alpha_{2i2})} x_{12i} x_{13i} \xi_{12i1} \xi_{13i2} - 4 e^{\frac{1}{2}(2b_{1i}-b_{2i})} x_{13i} \eta_{12i1} \xi_{12i1} \xi_{13i2} - \\
& e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i1} \xi_{13i2} - e^{\frac{1}{2}(b_{1i}+b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i1} \xi_{13i2} + \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} x_{13i} y_{23i} \eta_{13i1} \xi_{12i1} \xi_{13i2} - 2 e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} x_{13i} \eta_{12i1} \xi_{12i2} \xi_{13i2} - \\
& 4 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i2} \xi_{13i2} + 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i2} \xi_{13i2} + \\
& 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i2} \xi_{13i2} - e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{13i} y_{23i} \eta_{13i1} \xi_{12i2} \xi_{13i2} + \\
& 4 e^{\frac{1}{2}(4b_{1i}-2b_{2i}-3\alpha_{1i2})} \eta_{12i1} \eta_{13i1} \xi_{12i2} \xi_{13i2} - 6 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2})} \eta_{12i1} \eta_{13i1} \xi_{12i2} \xi_{13i2} - \\
& 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-3\alpha_{1i2})} \eta_{12i1} \eta_{13i1} \xi_{12i2} \xi_{13i2} + 2 e^{\frac{3}{2}(b_{1i}-\alpha_{1i2})} \eta_{12i1} \eta_{13i1} \xi_{12i2} \xi_{13i2} + \\
& 2 e^{-\frac{3\alpha_{1i2}}{2}} \eta_{12i1} \eta_{13i1} \xi_{12i2} \xi_{13i2} - 3 e^{\frac{1}{2}(3b_{1i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{23i} \eta_{13i1}^2 \xi_{12i2} \xi_{13i2} + \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{23i} \eta_{13i1}^2 \xi_{12i2} \xi_{13i2} - 4 e^{2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2}} y_{23i} \eta_{13i1}^2 \xi_{12i2} \xi_{13i2} - \\
& 2 x_{13i} \eta_{13i1} \xi_{13i1} \xi_{13i2} - 2 e^{\frac{1}{2}(b_{1i}+b_{2i})} x_{13i} \eta_{13i1} \xi_{13i1} \xi_{13i2} - 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{13i} \eta_{13i1} \xi_{13i2}^2 + \\
& 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{13i} \eta_{13i1} \xi_{13i2}^2 - 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} \eta_{13i1}^2 \xi_{13i2}^2 + \\
& 2 e^{b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2}} \eta_{13i1}^2 \xi_{13i2}^2 + e^{-\frac{3}{2}(\alpha_{1i2}-\alpha_{2i2})} x_{23i} \eta_{12i1} \xi_{12i2} \xi_{23i1} + \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2}+3\alpha_{2i2})} x_{23i} \eta_{12i1} \xi_{12i2} \xi_{23i1} - e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} x_{23i} y_{23i} \eta_{13i1} \xi_{12i2} \xi_{23i1} - \\
& e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} x_{23i} \eta_{13i1} \xi_{13i2} \xi_{23i1} - e^{\frac{1}{2}(b_{1i}+b_{2i}-2\alpha_{1i2}+\alpha_{2i2})} x_{23i} \eta_{13i1} \xi_{13i2} \xi_{23i1} + \\
& 4 x_{23i} \beta_{2i1} \xi_{23i2} - e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} x_{23i} y_{12i} \eta_{12i1} \xi_{23i2} + 3 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} y_{12i} \eta_{13i1} \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} a_{2i} y_{12i} \eta_{13i1} \xi_{23i2} + e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{23i} y_{13i} \eta_{13i1} \xi_{23i2} +
\end{aligned}$$

$$\begin{aligned}
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} y_{12i} \beta_{1i2} \eta_{13i1} \xi_{23i2} + 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} y_{12i} \beta_{2i1} \eta_{13i1} \xi_{23i2} + \\
& e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} y_{12i} y_{13i} \eta_{13i1} \eta_{13i2} \xi_{23i2} - 2 e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} \eta_{23i1} \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} \eta_{23i1} \xi_{23i2} - 2 e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} a_{2i} \eta_{23i1} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} a_{2i} \eta_{23i1} \xi_{23i2} + 2 e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{23i} y_{23i} \eta_{23i1} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} \beta_{2i1} \eta_{23i1} \xi_{23i2} + 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} \beta_{2i1} \eta_{23i1} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} y_{12i} \eta_{12i1} \eta_{23i1} \xi_{23i2} - e^{\frac{1}{2}(-b_{1i}+2b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{12i} \eta_{12i1} \eta_{23i1} \xi_{23i2} + \\
& e^{\frac{1}{2}(-b_{1i}+2b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} y_{12i} \eta_{12i2} \eta_{23i1} \xi_{23i2} + e^{\frac{1}{2}(-b_{1i}+2b_{2i}-3\alpha_{2i2})} y_{13i} \eta_{13i1} \eta_{23i1} \xi_{23i2} + \\
& 2 e^{-\frac{3\alpha_{2i2}}{2}} y_{13i} \eta_{13i1} \eta_{23i1} \xi_{23i2} - e^{\frac{1}{2}(-b_{1i}+2b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} y_{13i} \eta_{13i2} \eta_{23i1} \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+2\alpha_{1i2}-4\alpha_{2i2})} y_{23i} \eta_{23i1}^2 \xi_{23i2} - e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} y_{12i} y_{23i} \eta_{13i1} \eta_{23i2} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} y_{23i} \eta_{23i1} \eta_{23i2} \xi_{23i2} - e^{\frac{1}{2}(2\alpha_{1i2}-\alpha_{2i2})} x_{13i} \xi_{12i1} \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(2\alpha_{1i2}-\alpha_{2i2})} x_{12i} x_{23i} \xi_{12i1} \xi_{23i2} - 4 e^{\frac{1}{2}(2\alpha_{1i2}-\alpha_{2i2})} x_{13i} \beta_{2i1} \xi_{12i1} \xi_{23i2} + \\
& x_{13i} y_{12i} \eta_{12i1} \xi_{12i1} \xi_{23i2} - e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{12i} y_{12i} \eta_{13i1} \xi_{12i1} \xi_{23i2} - \\
& e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{13i} y_{13i} \eta_{13i1} \xi_{12i1} \xi_{23i2} + e^{\frac{1}{2}(-b_{1i}+2b_{2i}+3\alpha_{1i2}-3\alpha_{2i2})} x_{12i} \eta_{23i1} \xi_{12i1} \xi_{23i2} + \\
& e^{\frac{3}{2}(\alpha_{1i2}-\alpha_{2i2})} x_{12i} \eta_{23i1} \xi_{12i1} \xi_{23i2} - 2 e^{\frac{3}{2}(\alpha_{1i2}-\alpha_{2i2})} x_{13i} y_{23i} \eta_{23i1} \xi_{12i1} \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(-2\alpha_{1i2}+\alpha_{2i2})} x_{23i} \eta_{12i1} \xi_{12i2} \xi_{23i2} - 5 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \eta_{13i1} \xi_{12i2} \xi_{23i2} + \\
& 5 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{13i1} \xi_{12i2} \xi_{23i2} - 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} a_{1i} \eta_{13i1} \xi_{12i2} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} a_{1i} \eta_{13i1} \xi_{12i2} \xi_{23i2} - 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} a_{2i} \eta_{13i1} \xi_{12i2} \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} a_{2i} \eta_{13i1} \xi_{12i2} \xi_{23i2} + 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{12i} y_{12i} \eta_{13i1} \xi_{12i2} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{23i} y_{23i} \eta_{13i1} \xi_{12i2} \xi_{23i2} - 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \beta_{1i1} \eta_{13i1} \xi_{12i2} \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \beta_{1i1} \eta_{13i1} \xi_{12i2} \xi_{23i2} - 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \beta_{2i1} \eta_{13i1} \xi_{12i2} \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \beta_{2i1} \eta_{13i1} \xi_{12i2} \xi_{23i2} + 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2})} y_{12i} \eta_{12i1} \eta_{13i1} \xi_{12i2} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{12i} \eta_{12i2} \eta_{13i1} \xi_{12i2} \xi_{23i2} - e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{13i} \eta_{13i1}^2 \xi_{12i2} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{12i} y_{23i} \eta_{13i1}^2 \xi_{12i2} \xi_{23i2} - e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{13i} \eta_{13i1} \eta_{13i2} \xi_{12i2} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} x_{12i} \eta_{23i1} \xi_{12i2} \xi_{23i2} + 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} \eta_{12i1} \eta_{23i1} \xi_{12i2} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{12i1} \eta_{23i1} \xi_{12i2} \xi_{23i2} - 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{12i1} \eta_{23i1} \xi_{12i2} \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}-\alpha_{1i2}-\alpha_{2i2})} \eta_{12i1} \eta_{23i1} \xi_{12i2} \xi_{23i2} - e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{2i2})} y_{23i} \eta_{13i1} \eta_{23i1} \xi_{12i2} \xi_{23i2} + \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} y_{23i} \eta_{13i1} \eta_{23i2} \xi_{12i2} \xi_{23i2} + 2 e^{\frac{1}{2}(2b_{1i}-b_{2i})} x_{13i} \eta_{12i1} \xi_{12i1} \xi_{12i2} \xi_{23i2} - \\
& e^{\frac{1}{2}(\alpha_{1i2}-2\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i1} \xi_{12i2} \xi_{23i2} - 3 e^{\frac{1}{2}(2b_{1i}-b_{2i}+\alpha_{1i2}-2\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i1} \xi_{12i2} \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i2}^2 \xi_{23i2} + 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{12i} \eta_{13i1} \xi_{12i2}^2 \xi_{23i2} + \\
& 4 e^{\frac{1}{2}(4b_{1i}-2b_{2i}-3\alpha_{1i2})} \eta_{12i1} \eta_{13i1} \xi_{12i2}^2 \xi_{23i2} - 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{1i2})} \eta_{12i1} \eta_{13i1} \xi_{12i2}^2 \xi_{23i2} + \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{23i} \eta_{13i1}^2 \xi_{12i2}^2 \xi_{23i2} - 3 e^{2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2}} y_{23i} \eta_{13i1}^3 \xi_{12i2}^2 \xi_{23i2} - \\
& 2 e^{\frac{1}{2}(\alpha_{1i2}+\alpha_{2i2})} x_{13i} x_{23i} \xi_{13i1} \xi_{23i2} + x_{13i} y_{12i} \eta_{13i1} \xi_{13i1} \xi_{23i2} - \\
& e^{\frac{1}{2}(2\alpha_{1i2}-\alpha_{2i2})} x_{13i} \eta_{23i1} \xi_{13i1} \xi_{23i2} - e^{\frac{1}{2}(-b_{1i}+2b_{2i}+2\alpha_{1i2}-\alpha_{2i2})} x_{13i} \eta_{23i1} \xi_{13i1} \xi_{23i2} - \\
& 2 x_{13i} \eta_{13i1} \xi_{12i2} \xi_{13i1} \xi_{23i2} - 2 e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{23i} \eta_{13i1} \xi_{13i2} \xi_{23i2} + \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i2}-\alpha_{2i2})} x_{23i} \eta_{13i1} \xi_{13i2} \xi_{23i2} + e^{\frac{1}{2}(-\alpha_{1i2}-\alpha_{2i2})} x_{13i} y_{12i} \eta_{13i1} \xi_{13i2} \xi_{23i2} + \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{12i} \eta_{13i1}^2 \xi_{13i2} \xi_{23i2} + e^{\frac{1}{2}(b_{1i}+b_{2i}-2\alpha_{1i2}-2\alpha_{2i2})} y_{12i} \eta_{13i1}^2 \xi_{13i2} \xi_{23i2} -
\end{aligned}$$

$$\begin{aligned}
& 2 e^{-\alpha_{1i_2}-\alpha_{2i_2}} y_{12_i} \eta_{13_{i1}}^2 \xi_{13_{i2}} \xi_{23_{i2}} + 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+\alpha_{1i_2}-2\alpha_{2i_2})} x_{13_i} \eta_{23_{i1}} \xi_{13_{i2}} \xi_{23_{i2}} - \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}-3\alpha_{2i_2})} \eta_{13_{i1}} \eta_{23_{i1}} \xi_{13_{i2}} \xi_{23_{i2}} + 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-3\alpha_{2i_2})} \eta_{13_{i1}} \eta_{23_{i1}} \xi_{13_{i2}} \xi_{23_{i2}} - \\
& 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}-3\alpha_{2i_2})} \eta_{13_{i1}} \eta_{23_{i1}} \xi_{13_{i2}} \xi_{23_{i2}} + 2 e^{\frac{3}{2}(b_{2i}-\alpha_{2i_2})} \eta_{13_{i1}} \eta_{23_{i1}} \xi_{13_{i2}} \xi_{23_{i2}} + \\
& 2 e^{-\frac{3\alpha_{2i_2}}{2}} \eta_{13_{i1}} \eta_{23_{i1}} \xi_{13_{i2}} \xi_{23_{i2}} + 3 e^{\frac{1}{2}(\alpha_{1i_2}-2\alpha_{2i_2})} x_{13_i} \eta_{13_{i1}} \xi_{12_{i1}} \xi_{13_{i2}} \xi_{23_{i2}} - \\
& 4 e^{\frac{1}{2}(2b_{1i}-b_{2i}+\alpha_{1i_2}-2\alpha_{2i_2})} x_{13_i} \eta_{13_{i1}} \xi_{12_{i1}} \xi_{13_{i2}} \xi_{23_{i2}} + e^{\frac{1}{2}(b_{1i}+b_{2i}+\alpha_{1i_2}-2\alpha_{2i_2})} x_{13_i} \eta_{13_{i1}} \xi_{12_{i1}} \xi_{13_{i2}} \xi_{23_{i2}} - \\
& 2 e^{\frac{1}{2}(-\alpha_{1i_2}-\alpha_{2i_2})} x_{13_i} \eta_{13_{i1}} \xi_{12_{i2}} \xi_{13_{i2}} \xi_{23_{i2}} + 2 e^{\frac{1}{2}(3b_{1i}-2\alpha_{1i_2}-2\alpha_{2i_2})} \eta_{13_{i1}}^2 \xi_{12_{i2}} \xi_{13_{i2}} \xi_{23_{i2}} - \\
& 6 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i_2}-2\alpha_{2i_2})} \eta_{13_{i1}}^2 \xi_{12_{i2}} \xi_{13_{i2}} \xi_{23_{i2}} - 2 e^{\frac{1}{2}(b_{1i}+b_{2i}-2\alpha_{1i_2}-2\alpha_{2i_2})} \eta_{13_{i1}}^2 \xi_{12_{i2}} \xi_{13_{i2}} \xi_{23_{i2}} + \\
& 2 e^{-\alpha_{1i_2}-\alpha_{2i_2}} \eta_{13_{i1}}^2 \xi_{12_{i2}} \xi_{13_{i2}} \xi_{23_{i2}} + 4 e^{2b_{1i}-b_{2i}-\alpha_{1i_2}-\alpha_{2i_2}} \eta_{13_{i1}}^2 \xi_{12_{i2}} \xi_{13_{i2}} \xi_{23_{i2}} + \\
& 2 e^{\frac{1}{2}(-2\alpha_{1i_2}+\alpha_{2i_2})} x_{23_i} y_{12_i} \eta_{13_{i1}} \xi_{23_{i1}} \xi_{23_{i2}} - 2 x_{23_i} \eta_{23_{i1}} \xi_{23_{i1}} \xi_{23_{i2}} - \\
& 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i})} x_{23_i} \eta_{23_{i1}} \xi_{23_{i1}} \xi_{23_{i2}} - e^{\frac{1}{2}(-2\alpha_{1i_2}+\alpha_{2i_2})} x_{23_i} \eta_{13_{i1}} \xi_{12_{i2}} \xi_{23_{i1}} \xi_{23_{i2}} + \\
& 3 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i_2}+\alpha_{2i_2})} x_{23_i} \eta_{13_{i1}} \xi_{12_{i2}} \xi_{23_{i1}} \xi_{23_{i2}} + e^{\frac{1}{2}(-\alpha_{1i_2}-\alpha_{2i_2})} x_{23_i} y_{12_i} \eta_{13_{i1}} \xi_{23_{i2}}^2 - \\
& 2 e^{\frac{1}{2}(\alpha_{1i_2}-2\alpha_{2i_2})} x_{23_i} \eta_{23_{i1}} \xi_{23_{i2}}^2 + 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+\alpha_{1i_2}-2\alpha_{2i_2})} x_{23_i} \eta_{23_{i1}} \xi_{23_{i2}}^2 + \\
& e^{\frac{1}{2}(-b_{1i}+2b_{2i}-3\alpha_{2i_2})} y_{12_i} \eta_{13_{i1}} \eta_{23_{i1}} \xi_{23_{i2}}^2 - 2 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+2\alpha_{1i_2}-4\alpha_{2i_2})} \eta_{23_{i1}}^2 \xi_{23_{i2}}^2 + \\
& 2 e^{-b_{1i}+2b_{2i}+\alpha_{1i_2}-2\alpha_{2i_2}} \eta_{23_{i1}}^2 \xi_{23_{i2}}^2 - 3 e^{\frac{1}{2}(-b_{1i}+2b_{2i}+3\alpha_{1i_2}-3\alpha_{2i_2})} x_{13_i} \eta_{23_{i1}} \xi_{12_{i1}} \xi_{23_{i2}}^2 + \\
& e^{\frac{3}{2}(\alpha_{1i_2}-\alpha_{2i_2})} x_{13_i} \eta_{23_{i1}} \xi_{12_{i1}} \xi_{23_{i2}}^2 + 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-\alpha_{1i_2}-\alpha_{2i_2})} x_{23_i} \eta_{13_{i1}} \xi_{12_{i2}} \xi_{23_{i2}}^2 + \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i_2}-2\alpha_{2i_2})} y_{12_i} \eta_{13_{i1}}^2 \xi_{12_{i2}} \xi_{23_{i2}}^2 + e^{\frac{1}{2}(\alpha_{1i_2}-2\alpha_{2i_2})} x_{13_i} \eta_{13_{i1}} \xi_{12_{i1}} \xi_{12_{i2}} \xi_{23_{i2}}^2 + \\
& e^{\frac{1}{2}(2b_{1i}-b_{2i}+\alpha_{1i_2}-2\alpha_{2i_2})} x_{13_i} \eta_{13_{i1}} \xi_{12_{i1}} \xi_{12_{i2}} \xi_{23_{i2}}^2 - \\
& 2 e^{\frac{1}{2}(2b_{1i}-b_{2i}-2\alpha_{1i_2}-2\alpha_{2i_2})} \eta_{13_{i1}}^2 \xi_{12_{i2}}^2 \xi_{23_{i2}}^2 + 2 e^{2b_{1i}-b_{2i}-\alpha_{1i_2}-\alpha_{2i_2}} \eta_{13_{i1}}^2 \xi_{12_{i2}}^2 \xi_{23_{i2}}^2 \Big]
\end{aligned}$$

```

In[*]:= Block[{ $k = 0 },
  Echo[dSi2 // dmi2,i1→i];
  Ri1,i2 // dSi2 // dmi2,i1→i]
» E{i1,i2}→{i} [ a1i α1i1 - a1i α1i2 + b1i β1i1 - b1i β1i2 + eα1i2 y12i η12i1 - e-b1i+α1i2 y12i η12i2 +
  e-α1i2 x12i ξ12i1 - e-α1i2 x12i ξ12i2 + η12i1 ξ12i2 - eb1i η12i1 ξ12i2 + η12i2 ξ12i2 - e-b1i η12i2 ξ12i2 ]
Out[*]=
$Aborted

```

Ziping fails to compute the Drinfeld element u . Why?

We still need to find C such that R2c holds:

According to PG.pdf the ribbon element v is a square root of $v^2=uS(u)$. Then the pivot (a.k.a. distinguished group-like element, spinner) is $C = uv^{-1}$.

```

In[*]:= R3,2 R̄1,6 C4 // dm1,3→1 // dm2,4→2 // dm2,6→2
C2

```