

Hidden Utilities and T_EX

As: Enables the likes of Table[As[n,6], As[k,0,n], Binomial[n,k]], as in pensieve://2017-01/As.nb.

```
In[*]:= SetAttributes[As, HoldAll];
As /: (op : Do | Product | Sum | Table) [As[iter___],  $\varepsilon$ ___] := op[ $\varepsilon$ , {iter}]
```

Have StringJoin handle Integer and Symbol inputs correctly:

```
In[*]:= Unprotect[StringJoin];
StringJoin[l___, m_Integer, r___] := StringJoin[l, ToString[m], r];
StringJoin[x_Symbol, r___] := Symbol[SymbolName[x] <> StringJoin[r]];
Protect[StringJoin];
```

Time every Out line:

```
In[*]:= CurrentValue[EvaluationNotebook[], EvaluationCompletionAction] = {"ShowTiming"};

SetOptions[EvaluationNotebook[], StyleDefinitions →
  Notebook[{Cell[StyleData[StyleDefinitions → "Default.nb"]], Cell[StyleData["Input"],
    CellEpilog → With[{time = AbsoluteCurrentValue["WindowStatusArea"]},
      If[CurrentValue[NextCell[], CellStyle] === {"Output"}, SetOptions[
        NextCell[], CellLabel → time <> " Out[" <> ToString[$Line - 1] <> "]" = "]]]],
    StyleDefinitions → "PrivateStylesheetFormatting.nb"]]
```

```
In[*]:= PP_ = Identity;
```

tex

```
\documentclass[12pt,reqno]{amsart}
\usepackage[export]{adjustbox} % This also includes graphicx.
\usepackage{needspace}
\usepackage[textwidth=7.5in,textheight=10in,centering]{geometry}
\pagestyle{empty}

\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfEcho#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfPrint#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfMessage#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfSection#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfSubsection#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfSubsubsection#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfgraphInput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfgraphOutput#1{\vskip 1mm\par\noindent\includegraphics[width=1.5in]{#1}}
```

```
\begin{document}
```

nb2tex

```
In[ ]:= MakePDF[] := (
  Get["../nb2tex/nb2tex.m"];
  $NotebookName = FileName@FileNameTake@NotebookFileName[EvaluationNotebook[]];
  nb2tex[$NotebookName, PDFWidth -> 7.5, PDFFolder -> "Snips/" <> $NotebookName];
  texfile = OpenAppend[$NotebookName <> ".tex"];
  WriteLine[texfile, "\\end{document}"]; Close[texfile];
  Run@
    ("\"C:\\Users\\drorb\\AppData\\Local\\Programs\\MiKTeX\\miktex\\bin\\x64\\pdflatex.
     exe\" \" <> $NotebookName <> ".tex");
)
```

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_{\hbar}(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_j is our b_j .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

pdf

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

pdf

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

pdf

This version: April 2020. Original version: July 1994.

```
In[ ]:= MakePDF[]
```

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```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[ $\mathcal{E}$ _, op_] := Collect[Normal@Series[ $\mathcal{E}$ , { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[ $\mathcal{E}$ _] := SS[ $\mathcal{E}$ , Together];
```

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```
In[ ]:= $N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x[<>i<>j]; a[i_] := a[<>i]; A[i_] := A[<>i];
 $\xi$ [i_, j_] :=  $\xi$ [<>i<>j];  $\alpha$ [i_] :=  $\alpha$ [<>i];  $\mathcal{A}$ [i_] :=  $\mathcal{A}$ [<>i];
y[i_, j_] := y[<>i<>j]; b[i_] := b[<>i]; B[i_] := B[<>i];
 $\eta$ [i_, j_] :=  $\eta$ [<>i<>j];  $\beta$ [i_] :=  $\beta$ [<>i];
 $\tau$ [s_Symbol] := First[Characters[SymbolName[s]]];
 $\mathcal{L}$ [s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
 $\tau$ [s_] :=  $\tau$ [s];  $\mathcal{L}$ [s_] :=  $\mathcal{L}$ [s];
```

```
In[ ]:= { $\tau$ [x[3, 4]],  $\mathcal{L}$ [x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

Greek - Latin duality:

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```
In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]* =  $\xi$ [i, j];
 $\xi$ [i, j]* = x[i, j]; y[i, j]* =  $\eta$ [i, j];
 $\eta$ [i, j]* = y[i, j]]
Do[As[i, 1, $N - 1], A[i]* =  $\mathcal{A}$ [i];  $\mathcal{A}$ [i]* = A[i]; a[i]* =  $\alpha$ [i];  $\alpha$ [i]* = a[i];
b[i]* =  $\beta$ [i];  $\beta$ [i]* = b[i]]
(vs_List)* := (v  $\mapsto$  v*) /@ vs;
( $u_{\alpha\beta}$ )* := ( $u^*$ ) $_{\alpha\beta}$ ;
```

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```
In[ ]:= LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars  $\cup$  GreekVars))_;
```

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```

In[ ]:= CF[ $\mathcal{E}$ _] := PPCF@Module[{s1},
  s1 = If[$k ==  $\infty$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \mapsto e^{\text{Factor}^p}$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \mapsto e^{\text{Factor}^p}$  /.  $q^{n-} \mapsto \sum_{k=0}^{k} \frac{(\hbar \in \gamma n)^k}{k!}$  /.  $e^{k-}$  /.  $k > $k \mapsto 0$ 
  ]
  (*Expand@Collect[s1,Cases[s1,VarPattern, $\infty$ ],CCF]/.CCF→Factor*)
]

```

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Weights:

```

In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N],
  Wt[x[i, j]] = {1, i - j}; Wt[y[i, j]] = {0, i - j};
  Wt[ $\xi$ [i, j]] = {0, j - i}; Wt[ $\eta$ [i, j]] = {1, j - i};
];
Do[As[i, 1, $N - 1],
  Wt[a[i]] = {1, 0}; Wt[b[i]] = {0, 0};
  Wt[ $\alpha$ [i]] = {0, 0}; Wt[ $\beta$ [i]] = {1, 0};
];
Wt[u_] := Wt[u];

```

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```

In[ ]:= (* q~ $e^{\hbar \gamma \epsilon}$ ; Ai[n]~ $e^{\hbar \in ai}$ ; *)

```

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```

In[ ]:= $n = 3; $k =  $\infty$ ; $E := {$n, $k}

```

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```

In[ ]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
  Table[As[i, 1, $n - 1], {a[i], A[i][_]}]
}
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y}] /. $EPBWRule; x_ < y_ := ! OrderedQ[{y, x}] /. $EPBWRule;
$PBWGens := DeleteCases[$EPBWGens, _[_]]

```

```

In[ ]:= $PBWGens*

```

Time: 0.01 seconds Out[]:=

```

{ $\xi$ 12,  $\xi$ 13,  $\xi$ 14,  $\xi$ 15,  $\xi$ 23,  $\xi$ 24,  $\xi$ 25,  $\xi$ 34,  $\xi$ 35,  $\xi$ 45,  $\alpha$ 1,  $\alpha$ 2,  $\alpha$ 3,  $\alpha$ 4}

```

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```
In[*]:= UCF[ $\mathcal{E}_-$ ] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a@@L[AB], k], r]
  }]],
  U[____], F
] /. F → CF]
```

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```
In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j[_] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}_-$ ] := ms[UCF[ $\mathcal{E}$ ]];
```

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```
In[*]:= Supp[ $\mathcal{E}_-$ ] := Union@Cases[{ $\mathcal{E}$ }, Ui[____] ⇒ i, ∞];
```

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```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PPNCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /.  $\epsilon^{k-}$  /. k > $k ⇒ 0];
  Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];
```

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```
In[*]:= Δi→j,k[_] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk,k1→k@mj,j1→j[Δj,k[f] Δi→j1,k1[Ui[fs]]]
}];
Δi→j,k,l[_] :=  $\mathcal{E}$  /. Δi→j,k /. Δk→k,l
```

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```
In[*]:= Si[ $\mathcal{E}$ ] := UCF@Module[{i1},  $\mathcal{E}$  /. {
    Ui[f_, fs___]  $\Rightarrow$  mi1,i $\rightarrow$ i[Si[f] Si1[Ui1[fs]]]
}];
```

```
In[*]:=  $\Delta_{1,2\rightarrow 3}$ [U1[x12]]
```

```
Time: 0.02 seconds Out[ ]=
```

```
 $\Delta_{1,2\rightarrow 3}$ [U1[x12]]
```

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The $\mathbb{A}$ -algebra

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```
In[*]:= RRk[ai_, aj_] /;  $\tau$ [ai] ==  $\tau$ [aj] == "a" := Uk[aj, ai];
RRk[ai_, Aj_[n_]] /;  $\tau$ [ai] == "a"  $\wedge$   $\tau$ [Aj] == "A" := Uk[Aj[n], ai];
RRk[Aj_[n_], ai_] /;  $\tau$ [ai] == "a"  $\wedge$   $\tau$ [Aj] == "A" := Uk[ai, Aj[n]];
RRk[Ai_[m_], Aj_[n_]] /;  $\tau$ [Ai] ==  $\tau$ [Aj] == "A" := Uk[Aj[n], Ai[m]];
RRk[am_, xij_] /;  $\tau$ [am] == "a"  $\wedge$   $\tau$ [xij] == "x" := Module[{i, j, m},
    {i, j} =  $\mathcal{L}$ @xij; {m} =  $\mathcal{L}$ @am;
    Uk[xij, am] +  $\gamma$  ( $\delta_{m,i} - \delta_{m+1,i} - \delta_{m,j} + \delta_{m+1,j}$ ) Uk[xij] / 2
];
RRk[Am_[n_], xij_] /;  $\tau$ [Am] == "A"  $\wedge$   $\tau$ [xij] == "x" := Module[{i, j, m},
    {i, j} =  $\mathcal{L}$ @xij; {m} =  $\mathcal{L}$ @Am;  $q^{-n} (\delta_{m,i} - \delta_{m+1,i} - \delta_{m,j} + \delta_{m+1,j})$  Uk[xij, Am[n]]
]
```

```
In[*]:= RR0[a[1], x[1, 2]]
```

```
Time: 0.05 seconds Out[ ]=
```

```
 $\gamma$  U0[x12] + U0[x12, a1]
```

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We take the reduction rules RR for xij's from Yamane page

509:

(I) $i = m < j < n$, (II) $i < m < n < j$, (III) $i < m < j = n$,

$$\frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n}$$

(IV) $i < m < j < n$, (V) $i < j = m < n$, (VI) $i < j < m < n$.

$$\frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n}$$

$$\begin{array}{ll} q^{-2} e_{ij} e_{mn} - e_{mn} e_{ij} = 0 & \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\ [e_{ij}, e_{mn}] = 0 & \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\ [e_{ij}, e_{mn}] = (q^2 - q^{-2}) e_{in} e_{mj} & \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\ q^2 e_{ij} e_{mn} - e_{mn} e_{ij} = q e_{in} & \text{if } ((i, j), (m, n)) \in C_{(V)}. \end{array}$$

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```

In[ ]:= RRk[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 Uk[xij, xmn], (* IVIII *)
    (i < m < j ∧ n < j) ∨ (j < m), Uk[xij, xmn], (* IIIVI *)
    i < m < j ∧ n > j, Uk[xij, xmn] + (q-2 - q2) Uk[x[i, n], x[m, j]], (* IV *)
    j == m, q2 Uk[xij, xmn] - q Uk[x[i, n]] (* V *)
  ]
]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.06 seconds Out[ ]=
-q U2[x24] U3[x12] + q2 U2[x23, x34] U3[x12]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.05 seconds Out[ ]=
q2 U0[x14] - q3 U0[x12, x24] - q3 U0[x13, x34] + q4 U0[x12, x23, x34]

In[ ]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
Time: 1.05 seconds Out[ ]=
{}

In[ ]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[ ]:=
{}

```

```
Module[{Bas = $PBWGens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3 ->2 // m2,4 ->2 // m1,2 ->1) ==
      (P // m3,4 ->3 // m2,3 ->2 // m1,2 ->1) == (P // m1,2 ->1 // m3,4 ->3 // m1,3 ->1) ==
      (P // m1,2 ->1 // m1,3 ->1 // m1,4 ->1) == (P // m2,3 ->2 // m1,2 ->1 // m1,4 ->1)
  ]
]
» {x12, x13, x23, a1, a2}
```

Out[]:=

{ }

pdf

The Co-Product, Co-Associativity, and the Hopf Axiom

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According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

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```
In[ ]:= Do[As[i, 1, $n - 1],
  Δj,k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj,k[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj,k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj,k[x[i, n]] = mk,k1 ->k[mj,j1 ->j[
    q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
  ]
]
```

```
In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui / @ Bas], P -> Δi ->j,k[P]]
]
```

Time: 0.03 seconds Out[]:=

```
Ui[x12] -> Uj[x12] Uk[] - 2 ∈ ħ Uj[x12] Uk[a1] + Uj[] Uk[x12] + 2 ∈ ħ Uj[a1] Uk[x12]
Ui[x13] -> Uj[x13] Uk[] - 2 ∈ ħ Uj[x13] Uk[a1] - 2 ∈ ħ Uj[x13] Uk[a2] +
  Uj[] Uk[x13] + 2 ∈ ħ Uj[a1] Uk[x13] + 2 ∈ ħ Uj[a2] Uk[x13] + 4 ∇ ∈ ħ Uj[x12] Uk[x23]
Ui[x23] -> Uj[x23] Uk[] - 2 ∈ ħ Uj[x23] Uk[a2] + Uj[] Uk[x23] + 2 ∈ ħ Uj[a2] Uk[x23]
Ui[a1] -> Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] -> Uj[a2] Uk[] + Uj[] Uk[a2]
```

```
In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→1,2 // Δ2→2,3) = (P // Δ1→1,3 // Δ1→1,2)
  ]]
]
» {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}
Out[ ]:=
{ }
```

```
In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0][Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]]
]
» {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}
Out[ ]:=
{ }
```

Is Δ a quantization of the co-bracket?

```
In[ ]:= $k = 1
Time: 0.02 seconds Out[ ]:=
1

In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1/@Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[_xs_, A[_n_], as_] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[ ]:=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 γ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 γ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0
```

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The Antipode

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```
In[ ]:= Do[As[i, 1, $n - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj-[x[i, n]] = mj1,j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]
```

```
In[ ]:= x[1, 2]
```

```
Time: 0.00 seconds Out[ ]=
x12
```

```
In[ ]:= S1[x12]
```

```
Time: 0.00 seconds Out[ ]=
-  $\frac{U_1[x12]}{q^2}$ 
```

```
In[ ]:= Δ1,2→3[U1[x12]]
```

```
Time: 0.00 seconds Out[ ]=
Δ1,2→3[U1[x12]]
```

```
In[ ]:= $k = 2
```

```
Time: 0.00 seconds Out[ ]=
2
```

```
In[ ]:= Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
```

```
>> {x12, x13, x23, a1, a2, A2[-1], A2[-1], A2[1], A1[2], A1[-2], A1[-1], A1[-2], A1[1], A1[0], A1[-2]}
```

```
Out[ ]:=
```

```
{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A2[-1] → U1[],
  A2[-1] → U1[], A2[1] → U1[], A1[2] → U1[], A1[-2] → U1[],
  A1[-1] → U1[], A1[-2] → U1[], A1[1] → U1[], A1[0] → U1[], A1[-2] → U1[]}
```

```
In[ ]:= $n
```

```
Time: 0.02 seconds Out[ ]=
2
```

pdf

The Generating Functions for the $\mathbb{A}$ operations

```
In[ ]:= BeginProfile[ ]
```

```
Time: 0.03 seconds Out[ ]:=
BeginProfile[ ]
```

```
In[ ]:= $k = 1;
```

pdf

Ad, TT, Seepage

pdf

```
In[ ]:= PotQ[0] = True;
PotQ[c_ . U_i_[aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

pdf

```
In[ ]:= Ad[x_][y_] /; PotQ[x] ∧ (Head[y] != Plus) := PPAdPoty[eUB[y,x]/y y];
Ad[x_][y_Plus] /; PotQ[x] := PPAdPotPlus[Ad[x] /@ y];
```

pdf

```
In[ ]:= Ad[x_][y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil@Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]
```

```
In[ ]:= $k
```

```
Time: 0.03 seconds Out[ ]:=
1
```

```
In[ ]:= $k = 2
```

```
Time: 0.00 seconds Out[ ]:=
2
```

```
In[ ]:= Ad[ε ħ U_j[x12] U_k[]][U_j[x23] U_k[a4]]
```

```
Time: 0.03 seconds Out[ ]:=
-ε ħ U_j[x13] U_k[a4] + U_j[x23] U_k[a4]
```

```
In[ ]:= PotQ[7 ħ U_j[x12] U_k[a3]]
```

```
Time: 0.02 seconds Out[ ]:=
PotQ[7 ħ U_j[x12] U_k[a3]]
```

In[*]:= **Ad**[7 $\hbar$ U_j [x12] U_k [a3]] [U_j [x23] U_k [a4]]

Time: 0.09 seconds Out[]=

$$U_j[x23] U_k[a4] + \left(-7 \hbar - 7 \gamma \in \hbar^2 - \frac{7}{2} \gamma^2 \epsilon^2 \hbar^3 \right) U_j[x13] U_k[a3, a4] + \\ \left(14 \gamma \in \hbar^2 + 14 \gamma^2 \epsilon^2 \hbar^3 \right) U_j[x12, x23] U_k[a3, a4] - 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x12, x13] U_k[a3, a3, a4] + \\ 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x12, x12, x23] U_k[a3, a3, a4] - \frac{686}{3} \gamma^2 \epsilon^2 \hbar^5 U_j[x12, x12, x13] U_k[a3, a3, a3, a4]$$

In[*]:= **Ad**[π U_j [a1]] [U_j [x23] U_k [a4]]

Time: 0.05 seconds Out[]=

$$e^{\frac{i \pi \gamma}{2}} U_j[x23] U_k[a4]$$

pdf

In[*]:= **TT** _{λ} [] = 0; (* Translated Tangent *)
TT _{λ} [**xs**___, **x**_] := **PP**_{TT}@**UCF**[**Ad**[**x**] [**TT** _{λ} [**xs**]] + ∂_{λ} **x**];
TT _{λ} [**xs_List**] := **TT** _{λ} @@**xs**

In[*]:= **TT** _{λ} [λ ξ 12 U_0 [x12] U_1 [], λ ξ 23 U_0 [x23] U_1 []]

Time: 0.03 seconds Out[]=

$$\xi_{12} U_0[x12] U_1[] + (\lambda \xi_{12} \xi_{23} + \gamma \in \lambda \xi_{12} \xi_{23} \hbar) U_0[x13] U_1[] + \\ \xi_{23} U_0[x23] U_1[] - 2 \gamma \in \lambda \xi_{12} \xi_{23} \hbar U_0[x12, x23] U_1[]$$

In[*]:= **TT** _{λ} [0]

Time: 0.00 seconds Out[]=

$$\text{Ad}[0][0]$$

In[*]:= **\$k**

Time: 0.00 seconds Out[]=

$$1$$

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In[*]:= **O** _{i} [**poly**_] := **Total**[
CoefficientRules[**poly**, # _{i} & /@ **\$PBWGens**] /.
(**p**_ → **c**_) => **c** **U** _{i} @@**Flatten**@**MapThread**[**ConstantArray**, {**\$PBWGens**, **p**}]
];
O _{i} ,**is**__ [**poly**_] := **O** _{i} [**O** _{is} [**poly**]];
P _{i} [**e**_] := **e** /. **U** _{i} [**xs**___] => **Times**@@ (# _{i} & /@ {**xs**});
P _{i} ,**is**__ [**poly**_] := **P** _{i} [**P** _{is} [**poly**]];

In[*]:= **O** _{i} [x12² x13 _{i} + 7 x23 _{i} ¹⁷]

Time: 0.06 seconds Out[]=

$$U_i[x12, x12, x13] + \\ 7 U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]$$

In[*]:= $O_{i,j} [x_{12_i}^2 x_{13_i} x_{24_j} + 7 x_{23_i}^{17} x_{25_j}^3]$

Time: 0.03 seconds Out[]=

$U_i [x_{12}, x_{12}, x_{13}] U_j [x_{24}] +$
 $7 U_i [x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}]$
 $U_j [x_{25}, x_{25}, x_{25}]$

In[*]:= $P_{i,j} @ O_{i,j} [x_{12_i}^2 x_{13_i} x_{24_j} + 7 x_{23_i}^{17} x_{25_j}^3]$

Time: 0.06 seconds Out[]=

$x_{12_i}^2 x_{13_i} x_{24_j} + 7 x_{23_i}^{17} x_{25_j}^3$

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```
In[*]:= Seepage_i_[{ }, c_] = c;
Seepage_i_[ξs_List, P_] := Module[{lx},
  lx = $PBWgens[Length@ξs];
  CF@Collect[P, lx_i, P_i[Ad[Last[ξs] U_i[lx]] [O_i[Seepage_i[Most@ξs, #]]]]
    &]
];
Seepage_i_,is_[ξs_List, ξss_List, P_] := Seepage_i[ξs, Seepage_is[ξss, P]]
```

In[*]:= \$k = 1

Time: 0.00 seconds Out[]=

1

In[*]:= \$PBWgens*

Time: 0.01 seconds Out[]=

{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}

In[*]:= Seepage_i[\$PBWgens*, x35_i]

Time: 0.16 seconds Out[]=

$e^{\frac{1}{2}(\alpha_2 - \alpha_3 - \alpha_4)} \gamma x_{35_i} + 2 e^{\frac{1}{2}(\alpha_2 - 3\alpha_4)} \gamma \gamma \in \xi_{45} \hbar x_{35_i} x_{45_i}$

In[*]:= Seepage_i,j[#i & /@ \$PBWgens*, #j & /@ \$PBWgens*, x35_i x45_j]

Time: 0.25 seconds Out[]=

$e^{\frac{1}{2}(\alpha_2 - \alpha_3 + \alpha_3_j - \alpha_4 - 2\alpha_4_j)} x_{35_i} x_{45_j} + 2 e^{-\frac{1}{2}(\alpha_2 - \alpha_3_j + 3\alpha_4 + 2\alpha_4_j)} \gamma \gamma \in \hbar x_{35_i} x_{45_i} x_{45_j} \xi_{45_i}$

pdf

```
In[*]:= Invert_is__[E_] := Module[{A, B, U1, j}, (* E=A U[] + E B *)
  U1 = Times@@(U_#[] & /@ {is});
  A = Coefficient[E, U1] /. E -> 0;
  B = UCF[(E - A U1)];
  UCF[
    A^-1 (U1 + Sum[As[j, 1, $k], (-1)^j NonCommutativeMultiply@@ConstantArray[B/A, j]])]
]
```

In[*]:= **\$k = 3;**

Invert_{i,j}[7 U_i[] U_j[] + ε U_i[x12] U_j[x23]]

Time: 0.09 seconds Out[]:=

$$\frac{1}{7} U_i[] U_j[] - \frac{1}{49} \epsilon U_i[x12] U_j[x23] + \frac{1}{343} \epsilon^2 U_i[x12, x12] U_j[x23, x23] - \frac{\epsilon^3 U_i[x12, x12, x12] U_j[x23, x23, x23]}{2401}$$

{ \$k = 1, \$n = 3 }

Time: 0.01 seconds Out[]:=

{ 2, 3 }

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In[*]:=

```
TriangularDSolve[eqns_, inits_, fs_, λ_, CF_] :=
  PPTriangularDSolve@Module[{ss = {}, reqns = eqns, rinit = inits, rfs = fs, s, p},
    While[rfs != {},
      {p} = FirstPosition[Count[#, Alternatives @@ rfs, ∞, Heads → True] & /@ reqns, 1];
      {{s}} = DSolve[reqns[[p]] ∧ rinit[[p]], rfs[[p]], λ];
      AppendTo[ss, s /. F_Function → ReplacePart[F, 2 → CF[F[[2]]]]];
      reqns = Delete[reqns, p] /. s /. e_ == 0 → CF[e] == 0;
      rinit = Delete[rinit, p];
      rfs = Delete[rfs, p];
    ];
    ss
  ]
```

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“Define” Code

pdf

In[*]:=

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs]);
Define[op_is_ = ε_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[op_nisp, $E_, PPBoot@Block[{i, j, k}, op_isp, $E = ε; op_nis, $E]];
    SD[op_isp, op_{is}, $E]; SD[op_sis_, op_{sis}];
  ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
  } ] ]
```

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$$\begin{aligned}
L_\lambda &= \mathbb{C}\langle \lambda_{i_1,1} x \rangle \mathbb{C}\langle \lambda_{i_2,2} x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle f_i; x_i \rangle \mathbb{C}\langle \text{Seepage}(f_i; A) \rangle &= \mathbb{C}\langle e^{f_i}; A \rangle \\
\text{TTL}(L_\lambda) &= L'_\lambda \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f_i\}, B). \\
R_\lambda &= \mathbb{C}\langle f+g \rangle, \quad f=f_i x_i, \quad g=\sum_{(u,v)} g_{u,v} w & e^f = F e^g = G \quad R_\lambda = \mathbb{C}\langle FG \rangle \quad \partial_\lambda F = F \partial_\lambda f \\
\text{TTL}(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle' \mathbb{C}\langle F(\partial_\lambda f G + \partial_\lambda G) \rangle = S(G)' S(\partial_\lambda f G + \partial_\lambda G)
\end{aligned}$$

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In[*]:=

```

Define[amti,j→k = PPamt[$k,$n] @Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε; ES[s_][ε_] := ε;
  TTL = ES["TTL"] @PPTTL@TTλ@
    Join[l1 = Table[If[τ[g] == "x", λ, 1] (g*)i Uk[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
  ff = Table[
    Sum[εp fp,uk[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,uk[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
  fs = Flatten[Table[
    fp,ℙk[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙk[u][λ] ℙk[u],
    {p, 0, $k}, {u, Complement[mons[p], Uk /@ $PBWGens]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Ok[Seepagek[ff, SS@UCF@B]]];
  SG = ES["SG"] @PPSG@S[G];
  TTR = ES["TTR"] @PPTTR@UCF[Invertk[SG] ⊗ S[(∂λ (ff.(#k & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  ℙ{i,j}→{k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], (α[m]i + α[m]j) a[m]k], 0],
    Total[Cases[fs, fp,u]] /. fp,u => u (fp,u /. ss)[λ] /. λ | ℏ | γ → 1]
  ]
]

```

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```

In[ ]:= Define[ aΔti→j,k = PPaΔt[$k,$n] @Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
  TTL = ES["TTL"] @
    TTλ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p-1] ∪
    Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] => Uj[x] Uk[y], ∞];
  ffj = Table[
    Sum[εp fp,uj[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,uj[λ] /; τ[u] == "a" => If[p == 0, (u*)i, 0];
  ffk = ffj /. j → k;
  fs = Flatten[Table[
    fp,ℙj,k[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙj,k[u][λ] ℙj,k[u],
    {p, 0, $k}, {u, Complement[mons[p], Uj[] (Uk /@ $PBWGens) ∪ Uk[] (Uj /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oj,k[Seepagej,k[ffj, ffk, SS@UCF@B]]];
  SG = ES["SG"] @PPSG@S[G];
  TTR = ES["TTR"] @PPTTR@UCF[
    Inverttj,k[SG] ⓧ S[(∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λ G)];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], α[m]i (a[m]j + a[m]k)], 0],
    Total[Cases[fs, fp,-] /. fp,u => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

In[]:= \$k = 1; \$n = 2;

In[]:= \$PBWGens

Time: 0.03 seconds Out[]=

{x12, a1}

In[*]:= **S_i**[**U_i**[**x12**]]

Time: 1.16 seconds Out[]=

\$Aborted

In[*]:= **Table**[**UCF**[**If**[**τ**[**g**] === "x", **λ**, 1] (**g**^{*})_{**i**} **S_i**[**g**]], {**g**, **\$PBWGens**}]

Time: 0.03 seconds Out[]=

$\{-\lambda (1 - 2 \gamma \in \hbar) \xi_{12_i} U_i[x12], (-\lambda \xi_{13_i} + 2 \gamma \in \lambda \hbar \xi_{13_i}) U_i[x13] + 4 \gamma \in \lambda \hbar \xi_{13_i} U_i[x12, x23],$
 $-\lambda (1 - 2 \gamma \in \hbar) \xi_{23_i} U_i[x23], -\alpha_{1_i} U_i[a1], -\alpha_{2_i} U_i[a2]\}$

In[*]:= **TTL** = **TT_λ**@**Table**[**UCF**[**If**[**τ**[**g**] === "x", **λ**, 1] (**g**^{*})_{**i**} **S_i**[**U_i**[**g**]]], {**g**, **\$PBWGens**}]

Time: 0.16 seconds Out[]=

$\left(-e^{\frac{1}{2} \gamma (2 \alpha_{1_i} - \alpha_{2_i})} \xi_{12_i} + 2 e^{\frac{1}{2} \gamma (2 \alpha_{1_i} - \alpha_{2_i})} \gamma \in \hbar \xi_{12_i}\right) U_i[x12] +$
 $\left(-e^{\frac{1}{2} \gamma (\alpha_{1_i} + \alpha_{2_i})} \xi_{13_i} + 2 e^{\frac{1}{2} \gamma (\alpha_{1_i} + \alpha_{2_i})} \gamma \in \hbar \xi_{13_i} +\right.$
 $\left.e^{\frac{1}{2} \gamma (\alpha_{1_i} + \alpha_{2_i})} \lambda \xi_{12_i} \xi_{23_i} - 3 e^{\frac{1}{2} \gamma (\alpha_{1_i} + \alpha_{2_i})} \gamma \in \lambda \hbar \xi_{12_i} \xi_{23_i}\right) U_i[x13] +$
 $\left(-e^{-\frac{1}{2} \gamma (\alpha_{1_i} - 2 \alpha_{2_i})} \xi_{23_i} + 2 e^{-\frac{1}{2} \gamma (\alpha_{1_i} - 2 \alpha_{2_i})} \gamma \in \hbar \xi_{23_i}\right) U_i[x23] - 2 e^{\frac{3 \gamma \alpha_{1_i}}{2}} \gamma \in \lambda \hbar \xi_{12_i} \xi_{13_i} U_i[x12, x13] +$
 $\left(4 e^{\frac{1}{2} \gamma (\alpha_{1_i} + \alpha_{2_i})} \gamma \in \hbar \xi_{13_i} - 2 e^{\frac{1}{2} \gamma (\alpha_{1_i} + \alpha_{2_i})} \gamma \in \lambda \hbar \xi_{12_i} \xi_{23_i}\right) U_i[x12, x23] +$
 $2 e^{\gamma (\alpha_{1_i} + \alpha_{2_i})} \gamma \in \lambda^2 \hbar \xi_{12_i} \xi_{13_i} \xi_{23_i} U_i[x13, x13] - 2 e^{\frac{3 \gamma \alpha_{2_i}}{2}} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} U_i[x13, x23]$

```

Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}] ×
  mons[-1] = {};
  mons[p-] :=
  mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c-. Ui[x---] ⇒ Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp-,u-[λ] /; τ[u] == "a" ⇒ If[p == 0, (u*)i, 0];
  fs = Flatten[Table[
    fp,ℙi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙi[u][λ] ℙi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B-] := UCF[SS@0i[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⌗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], α[m]i (a[m]j + a[m]k)], 0],
    Total[Cases[fs, fp,-]] /. fp-,u- ⇒ u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
]

```

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In[]:=

```

Define[aΔti→j,k = PPaΔt[$k,$n] @Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
TTL = ES["TTL"] @
  TTλ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
mons[-1] = {};
mons[p_] := mons[p-1] ∪
  Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] => Uj[x] Uk[y], ∞];
ffj = Table[
  Sum[εp fp,uj[λ], {p, 0, $k}],
  {u, $PBWGens}
] /. fp,uj[λ] /; τ[u] == "a" => If[p == 0, (u*)i, 0];
ffk = ffj /. j → k;
fs = Flatten[Table[
  fp,ℙj,k[m],
  {p, 0, $k}, {m, mons[p]}
]];
gg = Sum[
  εp fp,ℙj,k[u][λ] ℙj,k[u],
  {p, 0, $k}, {u, Complement[mons[p], Uj[] (Uk /@ $PBWGens) ∪ Uk[] (Uj /@ $PBWGens)]}
];
G = Series[egg, {ε, 0, $k}];
S[B_] := UCF[SS@Oj,k[Seepagej,k[ffj, ffk, SS@UCF@B]]];
SG = ES["SG"] @ PPSG @ S[G];
TTR = ES["TTR"] @ PPTTR @ UCF[
  Invertj,k[SG] ⊗ S[(∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λ G)];
diff = UCF[TTL - TTR];
eqns = Flatten[Table[
  Coefficient[Coefficient[diff, m], ε, p] == 0,
  {p, 0, $k}, {m, mons[p]}
]];
inits = (#[0] == 0) & /@ fs;
ss = TriangularDSolve[eqns, inits, fs, λ, CF];
E{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
  If[p == 0, Sum[As[m, 1, $n-1], α[m]i (a[m]j + a[m]k)], 0],
  Total[Cases[fs, fp,-] /. fp,u => u (fp,u /. ss)[λ] /. λ | h | γ → 1]
]
]]

```

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The Generating Functions Engine

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Canonical Forms:

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```
In[*]:= LogReduce[ $\mathcal{E}$ _] :=
   $\mathcal{E}$  /.  $c$ _ * Log[ $a$ _]  $\Rightarrow$  Log@Factor[ $a^c$ ] /. Log[ $a$ _] + Log[ $b$ _]  $\Rightarrow$  Log@Factor[ $a b$ ];
CF[ $\mathcal{E}$ _E] := CF /@ MapAt[LogReduce,  $\mathcal{E}$ , 1];
CF[ $\mathcal{E}$  : E_][_] := CF /@  $\mathcal{E}$ ;
```

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E operations:

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```
In[*]:=  $\mathcal{E}$ _E[$_] := Length[ $\mathcal{E}$ ] - 1; E_[ $\mathcal{E}$ s_] [$_] := E[ $\mathcal{E}$ s] [$];
 $\mathcal{E}$ _E[k_Integer] :=  $\mathcal{E}$ [[k+1]]; E_[ $\mathcal{E}$ s_] [k_Integer] := { $\mathcal{E}$ s}[[k+1]];
E /:  $\mathcal{E}1$ _E  $\equiv$   $\mathcal{E}2$ _E := Inner[CF@#1 == CF@#2 &,  $\mathcal{E}1$ ,  $\mathcal{E}2$ , And];
 $\mathbb{E}_{d1 \rightarrow r1}$ [ $\mathcal{E}1$ s_]  $\equiv$   $\mathbb{E}_{d2 \rightarrow r2}$ [ $\mathcal{E}2$ s_]  $\wedge$  := ( $d1 == d2$ )  $\wedge$  ( $r1 == r2$ )  $\wedge$  ( $\mathbb{E}[\mathcal{E}1$ s]  $\equiv$   $\mathbb{E}[\mathcal{E}2$ s]);
E /:  $\mathcal{E}1$ _E *  $\mathcal{E}2$ _E := E @@ Table[CF[ $\mathcal{E}1$ [[kk]] +  $\mathcal{E}2$ [[kk]]], {kk, 0, Min[Length[ $\mathcal{E}1$ ], Length[ $\mathcal{E}2$ ]]};
 $\mathbb{E}_{d1 \rightarrow r1}$ [ $\mathcal{E}1$ s_]  $\mathbb{E}_{d2 \rightarrow r2}$ [ $\mathcal{E}2$ s_]  $\wedge$  :=  $\mathbb{E}_{(d1 \cup d2) \rightarrow (r1 \cup r2)}$  @@ ( $\mathbb{E}[\mathcal{E}1$ s]  $\mathbb{E}[\mathcal{E}2$ s]);
```

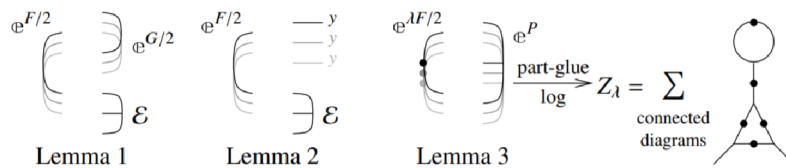
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```
In[*]:=  $\mathbb{E}_{d1 \rightarrow r1}$ [ $\mathcal{E}1$ s_] //  $\mathbb{E}_{d2 \rightarrow r2}$ [ $\mathcal{E}2$ s_] := Module[{is =  $r1 \cap d2$ , lvs, llvs},
  lvs = Cases[{ $\mathcal{E}1$ s}, (Alternatives @@ LatinVars)_i_ /; MemberQ[is, i],  $\infty$ ]  $\cup$ 
    Cases[{ $\mathcal{E}2$ s}, (Alternatives @@ GreekVars)_i_ /; MemberQ[is, i],  $\infty$ ]*;
  llvs = lvs /.  $u_{i\_}$   $\Rightarrow$   $u_{\$i}$ ;
   $\mathbb{E}_{(d1 \cup \text{Complement}[d2, is]) \rightarrow (r2 \cup \text{Complement}[r1, is])}$  @@ (Zip[llvs, llvs*][{ (llvs)*. (llvs), Times[
     $\mathbb{E}[\mathcal{E}1$ s] /. Thread[lvs  $\rightarrow$  llvs],
     $\mathbb{E}[\mathcal{E}2$ s] /. Thread[lvs*  $\rightarrow$  llvs*]
  ]}]
]
```

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Zippping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii} ’s and the y_{ii} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

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```

In[*]:= Zipvs [{ $\mathcal{F}$ _,  $\mathcal{E}$ _}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0}  $\vee$  Wt[#] == {1, 0}) &]},
  { $\mathcal{F}$ _,  $\mathcal{E}$ _} // Zip1vs (* // Zip2Complement[vs,dvs] *) // Zip2dvs //
  Zip3Complement[vs,dvs] // Zip3dvs
] // Last

```

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Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F: \mathcal{E} \otimes^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1}: \mathcal{E} \right\rangle_B$$

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```

In[*]:= Zip1{} = Identity;
Zip1vs @ { $\mathcal{F}$ _,  $\mathbb{E}[Q_, P\_]$ } := Module[{ $\mathcal{I}$ , F, G, u, v},
   $\mathcal{I}$  = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v} \mathcal{F}$ , 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v} Q$ , 0], {u, vs}, {v, vs}];
  {CF[(vs) . (F.Inverse[ $\mathcal{I}$  - G.F]) . (vs) / 2],
   $\mathbb{E}[CF[Q - \text{PowerExpand}@\text{Log}[\text{Det}[\mathcal{I} - G.F]] / 2 - \text{vs} \cdot G \cdot \text{vs} / 2], P]}$ 
]

```

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Getting rid of linear terms.

Lemma 2. $\left\langle F: \mathcal{E} \otimes^{\sum_{i \in B} y_i z_i} \right\rangle_B = e^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F: \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

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```

In[*]:= Zip2{} = Identity;
Zip2vs @ { $\mathcal{F}$ _,  $\mathbb{E}[Q_, P\_]$ } := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[ $\partial_{u,v} \mathcal{F}$ ], 0], {u, vs}, {v, vs}];
  Y = Table[ $\partial_v Q$ , {v, vs}] /. Table[v  $\rightarrow$  0, {v, vs}];
  CF /@ { $\mathcal{F}$ _,  $\mathbb{E}[Q - Y \cdot \text{vs} + Y \cdot F \cdot Y / 2, P]}$  /. Thread[vs  $\rightarrow$  vs + F.Y]}
]

```

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Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F: \mathbb{P}]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} (\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

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```

In[*]:= Zip3vs_@{ $\mathcal{F}_$ ,  $\mathcal{E}_\mathcal{E}$ } := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0,  $\mathcal{E}$ @$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v}\mathcal{F}$ , 0];
  Z[m_, kk_, u_] := Z[m, kk, u] =  $\partial_u Z[m, kk]$ ;
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] =  $\partial_v Z[m, kk, u]$ ;
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤  $\mathcal{E}$ @$, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2 (m + 1)}$  (Z[m, kk, u, v] +
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
     $\mathcal{F}$  - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
    E@@Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0,  $\mathcal{E}$ @$}]
  } /. Table[v → 0, {v, vs}])
]

```

```

In[*]:= {$k = 1, $n = 2}; amti,j→k

```

Time: 0.19 seconds Out[]=

$$E_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + x_{12k} \left(\xi_{12i} + e^{\alpha_{1i}} \xi_{12j} \right), 0 \right]$$

```

In[*]:= (amt1,2→2 // amt2,3→1) ≡ (amt2,3→2 // amt1,2→1)

```

Time: 0.08 seconds Out[]=

True

```

In[*]:= {$k = 1, $n = 3}; amti,j→k

```

```

(amt1,2→4 // amt4,3→5) ≡ (amt2,3→4 // amt1,4→5)

```

Time: 0.09 seconds Out[]=

$$\begin{aligned}
E_{\{i,j\} \rightarrow \{k\}} & \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\
& x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \\
& -2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \\
& \left. 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i} \right]
\end{aligned}$$

Out[]=

True

In[*]:= { \$k = 2, \$n = 3 }; amt_{i,j→k}

(amt_{1,2→4} // amt_{4,3→5}) ≡ (amt_{2,3→4} // amt_{1,4→5})

Time: 83.27 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\ & \quad x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \\ & \quad - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i}, \\ & \quad - \frac{1}{2} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} + 2 e^{2\alpha_{1i} - \alpha_{2i}} x_{12k}^2 x_{23k} \xi_{12j}^2 \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k}^2 \xi_{12j} \xi_{23i}^2 + e^{2\alpha_{1i} - \alpha_{2i}} x_{13k}^2 \xi_{12j}^2 \xi_{23i}^2 - \frac{2}{9} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k}^3 \xi_{12j}^3 \xi_{23i}^3 + \\ & \quad x_{12k} x_{13k} \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} - 2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i} \right) + \\ & \quad x_{12k}^2 x_{13k} \left(2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{13i} - \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i} \right) + \\ & \quad x_{12k} x_{13k} x_{23k} \left(-4 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} \xi_{23i} - 4 e^{\frac{3\alpha_{1i}}{2}} \xi_{12j} \xi_{13j} \xi_{23i} \right) + \\ & \quad x_{13k} x_{23k} \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^2 \right) + \\ & \quad x_{12k} x_{13k}^2 \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i}^2 + \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i}^2 \right) + \\ & \quad x_{13k} x_{23k}^2 \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i}^2 - \frac{2}{3} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^3 \right) + \\ & \quad \left. x_{13k}^2 x_{23k} \left(2 e^{\alpha_{1i} + \alpha_{2i}} \xi_{13j}^2 \xi_{23i} + \frac{2}{3} e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i}^3 \right) \right] \end{aligned}$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 2 }; aΔt_{i→j,k}

(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})

Time: 0.06 seconds Out[]=

$$\mathbb{E}_{\{i\} \rightarrow \{j,k\}} [(a_{1j} + a_{1k}) \alpha_{1i} + x_{12j} \xi_{12i} + x_{12k} \xi_{12i}, -2 a_{1k} x_{12j} \xi_{12i} + 2 a_{1j} x_{12k} \xi_{12i}]$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 3 }; aΔt_{i→j,k}

(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})

Time: 0.08 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i\} \rightarrow \{j,k\}} [\\ & \quad (a_{1j} + a_{1k}) \alpha_{1i} + (a_{2j} + a_{2k}) \alpha_{2i} + x_{12j} \xi_{12i} + x_{12k} \xi_{12i} + x_{13j} \xi_{13i} + x_{13k} \xi_{13i} + x_{23j} \xi_{23i} + x_{23k} \xi_{23i}, \\ & \quad - 2 a_{1k} x_{12j} \xi_{12i} + 2 a_{1j} x_{12k} \xi_{12i} - 2 a_{1k} x_{13j} \xi_{13i} - 2 a_{2k} x_{13j} \xi_{13i} + 2 a_{1j} x_{13k} \xi_{13i} + \\ & \quad 2 a_{2j} x_{13k} \xi_{13i} + x_{12k} x_{13j} \xi_{12i} \xi_{13i} - x_{12j} x_{13k} \xi_{12i} \xi_{13i} - 2 a_{2k} x_{23j} \xi_{23i} + 2 a_{2j} x_{23k} \xi_{23i} - \\ & \quad x_{12k} x_{23j} \xi_{12i} \xi_{23i} + x_{13k} x_{23j} \xi_{13i} \xi_{23i} - x_{13j} x_{23k} \xi_{13i} \xi_{23i} + x_{12j} x_{23k} (4 \xi_{13i} + \xi_{12i} \xi_{23i})] \end{aligned}$$

Out[*]=

True

In[*]:= { \$k = 2, \$n = 3 }; aΔt_{i→j,k}

(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})

» TTL $e^{-\frac{1}{2}\gamma(2\alpha_{1i}-\alpha_{2i})} \xi_{12i} U_j[x_{12}] U_k[] + \langle\langle 125 \rangle\rangle +$

$\left(2 e^{\frac{1}{2}\gamma(\alpha_{1i}-5\alpha_{2i})} \gamma^2 \epsilon^2 \lambda^2 \hbar^2 \xi_{13i} \xi_{23i}^2 + \frac{2}{3} e^{\langle\langle 1 \rangle\rangle} \gamma^2 \epsilon^2 \langle\langle 1 \rangle\rangle \hbar^2 \xi_{12i} \xi_{23i}^3 \right) \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle$

In[*]:= { \$k = 1, \$n = 2 }; (amt_{1,2→1} // aΔt_{1→1,2}) ≡ (aΔt_{2→3,4} // aΔt_{1→1,2} // amt_{1,3→1} // amt_{2,4→2})

Time: 0.06 seconds Out[] =

True

In[*]:= { \$k = 1, \$n = 3 }; (amt_{1,2→1} // aΔt_{1→1,2}) ≡ (aΔt_{2→3,4} // aΔt_{1→1,2} // amt_{1,3→1} // amt_{2,4→2})

Time: 7.16 seconds Out[] =

True