

## Hidden Utilities and T<sub>E</sub>X

### Headers

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Pensieve header: Quantum  $\mathfrak{sl}_n$  following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal  $R$ -matrix for  $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type  $A_N$ ”.
2. We call our algebra  $U_q(\mathfrak{sl}_n)$ , not  $U_{\hbar}(\mathfrak{sl}(N+1))$ .
3. We do not have a name for Rosso’s  $X_i$  and  $Y_j$ . Rosso’s  $E_{\epsilon_i - \epsilon_{j+1}}$  is our  $x[i, j] \leftrightarrow x_{ij}$ . Likewise Rosso’s  $\eta_{\epsilon_i - \epsilon_{j+1}}$  is our  $y[i, j] \leftrightarrow y_{ij}$ .
4. Rosso’s  $H_i$  is our  $a_i$  and Rosso’s  $\xi_i$  is our  $b_i$ .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.  
Read more at <http://katlas.org/wiki/KnotTheory>.

pdf

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

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This version: April 2020. Original version: July 1994.

```
In[ ]:= MakePDF[]
```

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```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[_ $\mathcal{E}$ _, op_] := Collect[Normal@Series[_ $\mathcal{E}$ _, { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[_ $\mathcal{E}$ _] := SS[_ $\mathcal{E}$ _, Together];
```

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```
In[ ]:= CF[_ $\mathcal{E}$ _] := PPCF@If[$k ==  $\infty$ ,
  Expand[Expand[_ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor}^{op}}$ ],
  Expand[Expand[_ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor}^{op}}$  /.  $q^{n-} \rightarrow \sum_{k=0}^{k} \frac{(\hbar \epsilon \gamma n)^k}{k!}] /. e^{k-} /; k > $k \rightarrow 0$ 
];
```

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```
In[ ]:= x[i_, j_] := x<>i<>j; a[i_] := a<>i; A[i_] := A<>i;
ξ[i_, j_] := ξ<>i<>j; α[i_] := α<>i; ℳ[i_] := ℳ<>i;
```

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```
In[ ]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[]] := τ[s]; ℓ[s_[]] := ℓ[s];
```

```
In[ ]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[ ]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for  $R$  at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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```
In[ ]:= (* q~e^h γ ∈; Ai[n]~e^h ∈ ai; *)
```

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```
In[ ]:= $n = 5; $k = ∞; $E := {$n, $k}
```

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```
In[ ]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n],
    x[i, j]* = ξ[i, j]; ξ[i, j]* = x[i, j]; x[i, j]],
  Table[As[i, 1, $n - 1],
    A[i]* = ℳ[i]; ℳ[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i]; {a[i], A[i][_]}]
}
(#0 = U0@#) & /@ $EPBWGens;
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi_ → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

```
In[ ]:= #* & /@ $EPBWGens
```

Time: 0.02 seconds Out[ ]:=

```
{ξ12, ξ13, ξ23, α1, A1[_]*, α2, A2[_]*}
```

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```
In[*]:= UCF[ $\mathcal{E}$ _] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a@@L[AB], k], r]
  }],
  U[____], F
] /. F → CF]
```

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```
In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j[_] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}$ _] := ms[UCF[ $\mathcal{E}$ ]];
```

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```
In[*]:= Supp[ $\mathcal{E}$ _] := Union@Cases[{ $\mathcal{E}$ }, Ui[____] ⇒ i, ∞];
```

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```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σ[i][z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /.  $\epsilon^{k-}$  /. k > $k ⇒ 0];
  Do[z = mσ[i], i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];
```

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```
In[*]:= Δi→j,k[_] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk,k1→k@mj,j1→j[Δj,k[f] Δi→j1,k1[Ui[fs]]]
}];
Δi→j,k,l[_] :=  $\mathcal{E}$  /. Δi→j,k /. Δk→k,l
```

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```
In[*]:= Si[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i→i[Si[f] Si1[Ui1[fs]]]
}];
```

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## The $\mathbb{A}$ -algebra

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```
In[*]:= RRk[_ai_, _aj_] /; τ[_ai_] == τ[_aj_] == "a" := Uk[_aj_, _ai_];
RRk[_ai_, _Aj_[n_]] /; τ[_ai_] == "a" ∧ τ[_Aj_] == "A" := Uk[_Aj_[n], _ai_];
RRk[_Aj_[n_], _ai_] /; τ[_ai_] == "a" ∧ τ[_Aj_] == "A" := Uk[_ai_, _Aj_[n]];
RRk[_Ai_[m_], _Aj_[n_]] /; τ[_Ai_] == τ[_Aj_] == "A" := Uk[_Aj_[n], _Ai_[m]];
RRk[_am_, _xij_] /; τ[_am_] == "a" ∧ τ[_xij_] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@am;
    Uk[_xij_, _am_] + γ (δm,i - δm+1,i - δm,j + δm+1,j) Uk[_xij_] / 2
];
RRk[_Am_[n_], _xij_] /; τ[_Am_] == "A" ∧ τ[_xij_] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@Am; q-n (δn,i - δn+1,i - δn,j + δn+1,j) Uk[_xij_, _Am_[n]]
]
```

```
In[*]:= RR0[_a[1], _x[1, 2]]
```

Time: 0.05 seconds Out[ ]:=

```
γ U0[_x12] + U0[_x12, a1]
```

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We take the reduction rules RR for xij's from Yamane page

509:

(I)  $i = m < j < n$ , (II)  $i < m < n < j$ , (III)  $i < m < j = n$ ,

$$\frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n}$$

(IV)  $i < m < j < n$ , (V)  $i < j = m < n$ , (VI)  $i < j < m < n$ .

$$\frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n}$$

$$\begin{array}{ll} q^{-2} e_{ij} e_{mn} - e_{mn} e_{ij} = 0 & \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\ [e_{ij}, e_{mn}] = 0 & \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\ [e_{ij}, e_{mn}] = (q^2 - q^{-2}) e_{in} e_{mj} & \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\ q^2 e_{ij} e_{mn} - e_{mn} e_{ij} = q e_{in} & \text{if } ((i, j), (m, n)) \in C_{(V)}. \end{array}$$

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```

In[ ]:= RRk[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 Uk[xij, xmn], (* IVIII *)
    (i < m < j ∧ n < j) ∨ (j < m), Uk[xij, xmn], (* IIIVI *)
    i < m < j ∧ n > j, Uk[xij, xmn] + (q-2 - q2) Uk[x[i, n], x[m, j]], (* IV *)
    j == m, q2 Uk[xij, xmn] - q Uk[x[i, n]] (* V *)
  ]
]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.06 seconds Out:=
-q U2[x24] U3[x12] + q2 U2[x23, x34] U3[x12]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.05 seconds Out:=
q2 U0[x14] - q3 U0[x12, x24] - q3 U0[x13, x34] + q4 U0[x12, x23, x34]

In[ ]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
Time: 1.05 seconds Out:=
{}

In[ ]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[ ]:=
{}

```

```

Module[{Bas = $PBWGens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3 ->2 // m2,4 ->2 // m1,2 ->1) ==
      (P // m3,4 ->3 // m2,3 ->2 // m1,2 ->1) == (P // m1,2 ->1 // m3,4 ->3 // m1,3 ->1) ==
      (P // m1,2 ->1 // m1,3 ->1 // m1,4 ->1) == (P // m2,3 ->2 // m1,2 ->1 // m1,4 ->1)
  ]
]
» {x12, x13, x23, a1, a2}

```

Out[8]=

{ }

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## The Co-Product, Co-Associativity, and the Hopf Axiom

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In[8]:=

```

Do[As[i, 1, $n - 1],
  (* Warning: these initialization fail if $n is increased later on *)
  Δj_,k_ [a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj_,k_ [A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj_,k_ [x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj_,k_ [x[i, n]] = m[k, k1 -> k] [m[j, j1 -> j] [
    q Δj,k [x[i, n - 1]] Δj1,k1 [x[n - 1, n]] - q^-1 Δj,k [x[n - 1, n]] Δj1,k1 [x[i, n - 1]]
  ]
]

```

```

In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
]

Time: 0.06 seconds Out[*]=
Ui[x12] → Uj[A1[-1]] Uk[x12] + Uj[x12] Uk[A1[1]]
Ui[x13] → Uj[A1[-1], A2[-1]] Uk[x13] +
  (-1/q2 + q2) Uj[x12, A2[-1]] Uk[x23, A1[1]] + Uj[x13] Uk[A1[1], A2[1]]
Ui[x14] →
  Uj[A1[-1], A2[-1], A3[-1]] Uk[x14] + (-1/q2 + q2) Uj[x12, A2[-1], A3[-1]] Uk[x24, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1]] Uk[x34, A1[1], A2[1]] + Uj[x14] Uk[A1[1], A2[1], A3[1]]
Ui[x15] → Uj[A1[-1], A2[-1], A3[-1], A4[-1]] Uk[x15] +
  (-1/q2 + q2) Uj[x12, A2[-1], A3[-1], A4[-1]] Uk[x25, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1], A4[-1]] Uk[x35, A1[1], A2[1]] +
  (-1/q2 + q2) Uj[x14, A4[-1]] Uk[x45, A1[1], A2[1], A3[1]] +
  Uj[x15] Uk[A1[1], A2[1], A3[1], A4[1]]
Ui[x23] → Uj[A2[-1]] Uk[x23] + Uj[x23] Uk[A2[1]]
Ui[x24] → Uj[A2[-1], A3[-1]] Uk[x24] +
  (-1/q2 + q2) Uj[x23, A3[-1]] Uk[x34, A2[1]] + Uj[x24] Uk[A2[1], A3[1]]
Ui[x25] →
  Uj[A2[-1], A3[-1], A4[-1]] Uk[x25] + (-1/q2 + q2) Uj[x23, A3[-1], A4[-1]] Uk[x35, A2[1]] +
  (-1/q2 + q2) Uj[x24, A4[-1]] Uk[x45, A2[1], A3[1]] + Uj[x25] Uk[A2[1], A3[1], A4[1]]
Ui[x34] → Uj[A3[-1]] Uk[x34] + Uj[x34] Uk[A3[1]]
Ui[x35] → Uj[A3[-1], A4[-1]] Uk[x35] +
  (-1/q2 + q2) Uj[x34, A4[-1]] Uk[x45, A3[1]] + Uj[x35] Uk[A3[1], A4[1]]
Ui[x45] → Uj[A4[-1]] Uk[x45] + Uj[x45] Uk[A4[1]]
Ui[a1] → Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] → Uj[a2] Uk[] + Uj[] Uk[a2]
Ui[a3] → Uj[a3] Uk[] + Uj[] Uk[a3]
Ui[a4] → Uj[a4] Uk[] + Uj[] Uk[a4]

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
  ]]
]

» {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}

Out[*]=
{ }

```

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]
]
» {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}
Out[ ]:=
{ }

```

## Is $\Delta$ a quantization of the co-bracket?

```

In[ ]:= $k = 1
Time: 0.02 seconds Out[ ]:=
1

In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1 /@ Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[xs___, A-[n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[ ]:=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 ∷ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 ∷ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0

```

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## The Antipode

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```
In[*]:= Do[As[i, 1, $n - 1],
  Sj_ [a[i]] = -Uj[a[i]];
  Sj_ [A[i][n_]] = Uj[A[i][-n]];
  Sj_ [x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj_ [x[i, n]] = mj1, j→j [
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]
```

```
In[*]:= $k = 2
```

```
Time: 0.00 seconds Out[ ]=
2
```

```
In[*]:= Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
» {x12, x13, x23, a1, a2, A1[-2], A2[-1], A1[1], A2[-1], A1[-2], A2[1], A1[2], A2[0], A2[2], A2[0]}
```

```
Out[*]=
```

```
{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A1[-2] → U1[],
  A2[-1] → U1[], A1[1] → U1[], A2[-1] → U1[], A1[-2] → U1[],
  A2[1] → U1[], A1[2] → U1[], A2[0] → U1[], A2[2] → U1[], A2[0] → U1[]}
```

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## The Generating Functions for the $\mathbb{A}$ operations

```
In[*]:= BeginProfile[]
```

```
Time: 0.12 seconds Out[ ]=
ProfileRoot
```

```
In[*]:= $k = 1;
```

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## Ad, TT, Seepage

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```
In[*]:= PotQ[0] = True;
PotQ[c_. U_ [aj_] (EUs : U_ [] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

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```
Ad[x_] [y_] /; PotQ[x] ∧ (Head[y] != Plus) := PPAdPoty [eUB[y,x]/y y];
Ad[x_] [y_Plus] /; PotQ[x] := PPAdPotPlus [Ad[x] /@ y];
```

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```
Ad[x_] [y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil @Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]
```

```
In[*]:= $k = 2
```

```
Time: 0.00 seconds Out:=
2
```

```
In[*]:= Ad[ε ħ Uj [x12] Uk []] [Uj [x23] Uk [a4]]
```

```
Time: 0.06 seconds Out:=
(-ε ħ - γ ε2 ħ2) Uj [x13] Uk [a4] + Uj [x23] Uk [a4] + 2 γ ε2 ħ2 Uj [x12, x23] Uk [a4]
```

```
In[*]:= PotQ[7 ħ Uj [x12] Uk [a3]]
```

```
Time: 0.02 seconds Out:=
PotQ[7 ħ Uj [x12] Uk [a3]]
```

```
In[*]:= Ad[7 ħ Uj [x12] Uk [a3]] [Uj [x23] Uk [a4]]
```

```
Time: 0.09 seconds Out:=
Uj [x23] Uk [a4] + (-7 ħ - 7 γ ε ħ2 -  $\frac{7}{2}$  γ2 ε2 ħ3) Uj [x13] Uk [a3, a4] +
(14 γ ε ħ2 + 14 γ2 ε2 ħ3) Uj [x12, x23] Uk [a3, a4] - 98 γ2 ε2 ħ4 Uj [x12, x13] Uk [a3, a3, a4] +
98 γ2 ε2 ħ4 Uj [x12, x12, x23] Uk [a3, a3, a4] -  $\frac{686}{3}$  γ2 ε2 ħ5 Uj [x12, x12, x13] Uk [a3, a3, a3, a4]
```

```
In[*]:= Ad[π ħ Uj [a1]] [Uj [x23] Uk [a4]]
```

```
Time: 0.05 seconds Out:=
e $\frac{i\pi\gamma}{2}$  Uj [x23] Uk [a4]
```

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```
In[*]:= TTλ [] = 0; (* Translated Tangent *)
TTλ [xs___, x_] := PPTT @ UCF [Ad[x] [TTλ [xs]] + ∂λ x];
TTλ [xs_List] := TTλ @@ xs
```

```
In[ ]:= TTλ[λ ξ12 U0[x12] U1[], λ ξ23 U0[x23] U1[]]
```

Time: 0.08 seconds Out[ ]:=

```
ξ12 U0[x12] U1[] + λ ξ12 ξ23 U0[x13] U1[] + ξ23 U0[x23] U1[]]
```

pdf

```
In[ ]:= Oi[poly_] := Total[
  CoefficientRules[poly, $PBWgens] /.
    (p_ -> c_) -> c Ui @@ Flatten@MapThread[ConstantArray, {$PBWgens, p}]
]
```

```
In[ ]:= Oi[x122 x13 + 7 x2317]
```

Time: 0.03 seconds Out[ ]:=

```
Ui[x12, x12, x13] +
  7 Ui[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]
```

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```
In[ ]:= Seepagei[{ }, c_] = c;
Seepagei[ξs_List, P_] := Module[{lx},
  lx = $PBWgens[[Length@ξs]];
  CF@Collect[P, lx, (Ad[Last[ξs] Ui[lx]] [Oi[Seepagei[Most@ξs, #]]] /. Ui -> Times) &]
]
```

pdf

```
In[ ]:= Inverti[ε_] := Module[{A, B, β, j}, (* ε = A U[] + ε B *)
  A = Coefficient[ε, Ui[]) /. ε -> 0;
  B = UCF[(ε - A Ui[])];
  UCF[
    A-1 (Ui[] + Sum[As[j, 1, $k], (-1)j NonCommutativeMultiply @@ ConstantArray[B / A, j]])]
]
```

```
{ $k = 1, $n = 3 }
```

Time: 0.01 seconds Out[ ]:=

```
{ 2, 3 }
```

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$$\begin{aligned}
 L_\lambda &= \mathbb{C}\langle \lambda_{i_1}^1 x \rangle \mathbb{C}\langle \lambda_{i_2}^2 x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle f_i x_i \rangle \mathcal{O}(\text{Seepage}(\{f_i\}, A)) &= \mathcal{O}(e^{f_i x_i} A) \\
 TT(L_\lambda) &= L' \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f_i\}, B). \\
 R_\lambda &= \mathbb{C}\langle f+g \rangle, \quad f = f_i x_i, \quad g = \sum_{|w| \geq 1} g_w w & e^f = F, \quad e^g = G, \quad R_s = \mathcal{O}(FG), \quad \partial_\lambda F = F \partial_\lambda f \\
 TT(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathcal{O}(FG)' \mathcal{O}(\partial_\lambda FG) = S(G)' \mathcal{O}(F)' \mathcal{O}(F(\partial_\lambda f G + \partial_\lambda G)) = S(G)' S(\partial_\lambda f G + \partial_\lambda G).
 \end{aligned}$$

pdf

```
sol[kk_, nn_] := PP"sol"[kk, nn] @ Block[{ $k = kk, $n = nn }, Module[{ (*TTL, *) },
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
  TTL = ES["TTL"] @ PP"TTL" @ TTλ @
  Join[
```

```

Cases[$PBWGens, g_ := Which[
   $\tau[g] === "x", \lambda (g^*)_i U_k[g],$ 
   $\tau[g] === "a", (g^*)_i U_k[g]$ 
]],
Cases[$PBWGens, g_ := Which[
   $\tau[g] === "x", \lambda (g^*)_j U_k[g],$ 
   $\tau[g] === "a", (g^*)_j U_k[g]$ 
]]
];
mons[-1] = {};
mons[p_] := mons[p - 1]  $\cup$  Union@Cases[Coefficient[TTL,  $\epsilon$ , p], U_k[___],  $\infty$ ];
ff = Table[
  Sum[ $\epsilon^p f_{p,u}[\lambda]$ , {p, 0, $k}],
  {u, $PBWGens}
] /.  $f_{p,u}[\lambda] \rightarrow \tau[u] == "a" \rightarrow \text{If}[p == 0, (u^*)_i + (u^*)_j, 0]$ ;
fs = Flatten[Table[
  f_k, Times@u,
  {k, 0, $k}, {m, mons[k]}
]];
gg = Sum[
   $\epsilon^p f_{p, \text{Times}@@u}[\lambda] \text{Times} @@ u,$ 
  {p, 0, $k}, {u, Complement[mons[p], U_k/@$PBWGens]}
];
G = Series[egg, { $\epsilon$ , 0, $k}];
S_i_[B_] := UCF[SS@O_i[Seepage_i[ff, SS@UCF@B]]];
SG = ES["SG"] @ PPSG @ S_k[G];
TTR = ES["TTR"] @ PPTTR @ UCF[Invert_k[SG]  $\boxtimes$  S_k[( $\partial_\lambda$ (ff.$PBWGens) G +  $\partial_\lambda$ G)]];
diff = UCF[TTL - TTR];
eqns = Flatten[Table[
  Coefficient[Coefficient[diff, m],  $\epsilon$ , k] == 0,
  {k, 0, $k}, {m, mons[k]}
]];
inits = (#[0] == 0) & /@ fs;
PPDSolve @ Module[{},
  ss = {}; reqns = eqns; rinit = inits; rfs = fs;
  While[rfs != {},
    pos = FirstPosition[reqns, eq_ /; (Count[eq, f,  $\infty$ , Heads  $\rightarrow$  True] == 1), None, {1}];
    If[pos == None,
      Echo[rfs];
      {s} = DSolve[reqns  $\cup$  rinit, rfs,  $\lambda$ ];
      rfs = {},
      (* else *) {p} = pos;
      {s} = DSolve[reqns[[p]]  $\wedge$  rinit[[p]], rfs[[p]],  $\lambda$ ];
      reqns = Delete[reqns, p] /. s /.  $e_ == 0 \rightarrow \text{CF}[e] == 0$ ;
      rinit = Delete[rinit, p];

```

```

      rfs = Delete[rfs, p]
    ];
    ss = Join[ss, s /. F_Function -> ReplacePart[F, 2 -> CF[F[[2]]]]]
  ]
];
E[Total[fs] /. f_{p,u} -> e^p u (f_{p,u} /. ss)[1]] /. h | \gamma -> 1
]]

```

In[\*]:= sol[1, 3]

$$\begin{aligned}
 & \gg \text{TTL} \left( e^{-\frac{1}{2}\gamma(2\alpha_1+2\alpha_1\gamma-\alpha_2-\alpha_2\gamma)} \xi_{12_i} + e^{-\frac{1}{2}\gamma(2\alpha_1\gamma-\alpha_2\gamma)} \xi_{12_j} \right) U_k[x_{12}] + \langle\langle 5 \rangle\rangle + (\langle\langle 4 \rangle\rangle + \langle\langle 1 \rangle\rangle) U_k[x_{13}, x_{23}] \\
 & \gg \text{SG} U_k[] + e^{\langle\langle 1 \rangle\rangle} \in U_k[\langle\langle 1 \rangle\rangle] f_{1,\langle\langle 1 \rangle\rangle}[\lambda] + \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle + \\
 & \quad U_k[x_{13}, x_{23}] \left( e^{-\frac{3}{2}\gamma(\alpha_2+\alpha_2\gamma)} \in f_{\emptyset,x_{23}}[\lambda] f_{1,x_{12}x_{23}}[\lambda] + e^{-\frac{3}{2}\gamma(\alpha_2+\alpha_2\gamma)} \in f_{1,x_{13}x_{23}}[\lambda] \right) \\
 & \gg \text{TTR} U_k[x_{12}] \left( e^{-\frac{1}{2}\gamma(2\alpha_1+2\alpha_1\gamma-\alpha_2-\alpha_2\gamma)} f_{\emptyset,x_{12}'}[\lambda] + e^{-\frac{1}{2}\gamma(2\alpha_1+2\langle\langle 1 \rangle\rangle-\langle\langle 1 \rangle\rangle-\alpha_2\gamma)} \in f_{1,\langle\langle 3 \rangle\rangle}'[\lambda] \right) + \\
 & \quad \langle\langle 5 \rangle\rangle + U_k[x_{13}, x_{23}] (\langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle) \\
 & \gg \text{diff} U_k[x_{12}] \\
 & \quad \left( e^{-\frac{1}{2}\gamma(2\alpha_1+2\alpha_1\gamma-\alpha_2-\alpha_2\gamma)} \xi_{12_i} + e^{\langle\langle 1 \rangle\rangle} \xi_{12_j} - e^{\langle\langle 1 \rangle\rangle} \langle\langle 1 \rangle\rangle'[\lambda] - e^{-\frac{1}{2}\gamma(2\alpha_1+2\alpha_1\gamma-\langle\langle 2 \rangle\rangle-\alpha_2\gamma)} \in f_{1,\langle\langle 3 \rangle\rangle}'[\lambda] \right) + \\
 & \quad \langle\langle 5 \rangle\rangle + \langle\langle 1 \rangle\rangle (\langle\langle 1 \rangle\rangle) \\
 & \gg \text{eqns} \left\{ e^{-\frac{1}{2}\gamma(2\alpha_1+2\alpha_1\gamma-\alpha_2-\alpha_2\gamma)} \xi_{12_i} + e^{-\frac{1}{2}\gamma\langle\langle 1 \rangle\rangle\langle\langle 1 \rangle\rangle(2\langle\langle 1 \rangle\rangle-\langle\langle 1 \rangle\rangle)} \xi_{12_j} - e^{-\frac{1}{2}\gamma(2\alpha_1+2\alpha_1\gamma-\alpha_2-\alpha_2\gamma)} f_{\emptyset,x_{12}'}[\lambda] = 0, \right. \\
 & \quad \left. \langle\langle 1 \rangle\rangle = 0, \langle\langle 6 \rangle\rangle, \langle\langle 1 \rangle\rangle, \langle\langle 1 \rangle\rangle = 0 \right\}
 \end{aligned}$$

Out[\*]=

$$\begin{aligned}
 & E \left[ x_{12} \left( \xi_{12_i} + e^{\frac{1}{2}(2\alpha_1-\alpha_2)} \xi_{12_j} \right) - 2 e^{\frac{1}{2}(2\alpha_1-\alpha_2)} x_{12} x_{13} \in \xi_{12_j} \xi_{13_i} - \right. \\
 & \quad e^{\frac{1}{2}(2\alpha_1-\alpha_2)} x_{13} \in \xi_{12_j} \xi_{23_i} + 2 e^{\frac{1}{2}(2\alpha_1-\alpha_2)} x_{12} x_{23} \in \xi_{12_j} \xi_{23_i} - 2 e^{\frac{1}{2}(\alpha_1+\alpha_2)} x_{13} x_{23} \in \xi_{13_j} \xi_{23_i} + \\
 & \quad \left. x_{13} \left( \xi_{13_i} + e^{\frac{1}{2}(\alpha_1+\alpha_2)} \xi_{13_j} - e^{\frac{1}{2}(2\alpha_1-\alpha_2)} \xi_{12_j} \xi_{23_i} \right) + x_{23} \left( \xi_{23_i} + e^{\frac{1}{2}(-\alpha_1+2\alpha_2)} \xi_{23_j} \right) \right]
 \end{aligned}$$

In[\*]:= sol[1, 4]

» TTL  $\ll 1 \gg$

» SG  $U_k[\ ] + \ll 16 \gg +$

$$U_k[x_{24}, x_{34}] \left( e^{\frac{1}{2} \gamma (\alpha_1 + \alpha_1 - 3\alpha_3 - 3\alpha_3)} \in f_{\emptyset, x_{34}}[\lambda] f_{1, x_{23} x_{34}}[\lambda] + e^{\frac{1}{2} \gamma (\alpha_1 + \ll 5 \gg)} \in f_1 \ll 1 \gg x_{24} \ll 1 \gg \ll 3 \gg [\lambda] \right)$$

» TTR  $U_k[x_{12}] \left( e^{\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} f_{\emptyset, x_{12}}'[\lambda] + e^{\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \ll 1 \gg - \ll 1 \gg - \alpha_2)} \in f_{1, \ll 3 \gg}'[\lambda] \right) +$   
 $\ll 21 \gg + U_k[x_{24}, x_{34}] (\ll 1 \gg + \ll 1 \gg + \ll 1 \gg)$

» diff  $U_k[x_{12}]$

$$\left( e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} \xi_{12_i} + e^{\ll 1 \gg} \xi_{12_j} - e^{\ll 1 \gg} \ll 1 \gg'[\lambda] - e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \ll 2 \gg - \alpha_2)} \in f_{1, \ll 3 \gg}'[\lambda] \right) +$$

$$\ll 21 \gg + \ll 1 \gg (\ll 1 \gg)$$

» eqns  $\left\{ e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} \xi_{12_i} + e^{-\frac{1}{2} \ll 1 \gg \ll 1 \gg (2\alpha_1 - \ll 1 \gg)} \xi_{12_j} - e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} f_{\emptyset, x_{12}}'[\lambda] = 0, \right.$   
 $\left. \ll 1 \gg = 0, \ll 25 \gg, \ll 1 \gg, \ll 1 \gg = 0 \right\}$

Out[\*]=

$$\begin{aligned} & \mathbb{E} \left[ x_{12} \left( \xi_{12_i} + e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) - 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{13} \in \xi_{12_j} \xi_{13_i} - \right. \\ & 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{14} \in \xi_{12_j} \xi_{14_i} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{13} \in \xi_{12_j} \xi_{23_i} + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{23} \in \xi_{12_j} \xi_{23_i} - \\ & 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} x_{13} x_{23} \in \xi_{13_j} \xi_{23_i} + x_{13} \left( \xi_{13_i} + e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i} \right) + \\ & x_{23} \left( \xi_{23_i} + e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} \xi_{23_j} \right) + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{24} \in \xi_{12_j} \xi_{24_i} - \\ & 4 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} x_{14} x_{23} \in \xi_{13_j} \xi_{24_i} - 2 e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} x_{23} x_{24} \in \xi_{23_j} \xi_{24_i} + \\ & x_{13} x_{14} \in \left( -2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{14_i} + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{14_i} \xi_{23_i} + 2 e^{\frac{1}{2} (3\alpha_1 - \alpha_3)} \xi_{12_j} \xi_{13_j} \xi_{24_i} \right) + \\ & 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} x_{13} x_{34} \in \xi_{13_j} \xi_{34_i} - 2 e^{\frac{1}{2} (\alpha_1 + \alpha_3)} x_{14} x_{34} \in \xi_{14_j} \xi_{34_i} - \\ & e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} x_{24} \in \xi_{23_j} \xi_{34_i} + 2 e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} x_{23} x_{34} \in \xi_{23_j} \xi_{34_i} - \\ & 2 e^{\frac{1}{2} (-\alpha_1 + \alpha_2 + \alpha_3)} x_{24} x_{34} \in \xi_{24_j} \xi_{34_i} + x_{14} \in \left( -e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{24_i} - e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{34_i} \right) + \\ & x_{14} \left( \xi_{14_i} + e^{\frac{1}{2} (\alpha_1 + \alpha_3)} \xi_{14_j} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{24_i} - e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{34_i} \right) + \\ & x_{24} \left( \xi_{24_i} + e^{\frac{1}{2} (-\alpha_1 + \alpha_2 + \alpha_3)} \xi_{24_j} - e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} \xi_{23_j} \xi_{34_i} \right) + \\ & x_{13} x_{24} \in \left( -2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i} \xi_{24_i} - 2 e^{\frac{1}{2} (3\alpha_2 - 2\alpha_3)} \xi_{13_j} \xi_{23_j} \xi_{34_i} \right) + \\ & x_{14} x_{24} \in \left( -2 e^{\frac{1}{2} (\alpha_1 + \alpha_3)} \xi_{14_j} \xi_{24_i} + 2 e^{\alpha_2} \xi_{14_j} \xi_{23_j} \xi_{34_i} + 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{24_i} \xi_{34_i} \right) + \\ & x_{34} \left( \xi_{34_i} + e^{\frac{1}{2} (-\alpha_2 + 2\alpha_3)} \xi_{34_j} \right) \Big] \end{aligned}$$

In[\*]:= sol[2, 3]

» TTL  $\left( e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} \xi_{12_i} + e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) U_k[x_{12}] + \ll 14 \gg$   
 » SG  $U_k[] + \ll 20 \gg + U_k[x_{13}, x_{23}, x_{23}]$   
 $\left( 2 e^{\frac{1}{2} \gamma (\alpha_1 + \alpha_1 - 5\alpha_2 - 5\alpha_2)} \gamma \in^2 \hbar f_{\emptyset, x_{23}}[\lambda] f_{1, x_{13} x_{23}}[\lambda] + e^{\ll 1 \gg} \ll 2 \gg \ll 1 \gg + e^{\frac{1}{2} \gamma (\ll 1 \gg)} \in^2 f_{2, \ll 1 \gg}[\lambda] \right)$   
 » TTR  $\ll 1 \gg$   
 » diff  $\ll 1 \gg$   
 » eqns  $\left\{ e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} \xi_{12_i} + e^{-\frac{1}{2} \gamma (\ll 1 \gg \ll 1 \gg (2\ll 1 \gg - \ll 1 \gg))} \xi_{12_j} - e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} f_{\emptyset, x_{12}'}[\lambda] = 0, \right.$   
 $\left. \ll 1 \gg = 0, \ll 21 \gg, \ll 1 \gg, \ll 1 \gg = 0 \right\}$

Out[\*]=

$$\begin{aligned}
 & \mathbb{E} \left[ x_{12} \left( \xi_{12_i} + e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) - 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{13} \in \xi_{12_j} \xi_{13_i} - \right. \\
 & e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{13} \in \xi_{12_j} \xi_{23_i} + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{23} \in \xi_{12_j} \xi_{23_i} - \frac{1}{2} e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{13} \in^2 \xi_{12_j} \xi_{23_i} + \\
 & 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{23} \in^2 \xi_{12_j} \xi_{23_i} + 2 e^{2\alpha_1 - \alpha_2} x_{12}^2 x_{23} \in^2 \xi_{12_j}^2 \xi_{23_i} - \\
 & 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2)} x_{13} x_{23} \in \xi_{13_j} \xi_{23_i} + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{23}^2 \in^2 \xi_{12_j} \xi_{23_i}^2 + e^{2\alpha_1 - \alpha_2} x_{13}^2 \in^2 \xi_{12_j}^2 \xi_{23_i}^2 - \\
 & \frac{2}{9} e^{\frac{3}{2} (2\alpha_1 - \alpha_2)} x_{13}^3 \in^2 \xi_{12_j}^3 \xi_{23_i}^3 + x_{13} \left( \xi_{13_i} + e^{\frac{1}{2} (\alpha_1 + \alpha_2)} \xi_{13_j} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i} \right) + \\
 & x_{12} x_{13} \in^2 \left( 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{13_i} - 2 e^{2\alpha_1 - \alpha_2} \xi_{12_j}^2 \xi_{23_i} \right) + \\
 & x_{12}^2 x_{13} \in^2 \left( 2 e^{2\alpha_1 - \alpha_2} \xi_{12_j}^2 \xi_{13_i} - \frac{2}{3} e^{\frac{3}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j}^3 \xi_{23_i} \right) + \\
 & x_{12} x_{13} x_{23} \in^2 \left( -4 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{13_i} \xi_{23_i} - 4 e^{\frac{3\alpha_1}{2}} \xi_{12_j} \xi_{13_j} \xi_{23_i} \right) + \\
 & x_{13} x_{23} \in^2 \left( 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2)} \xi_{13_j} \xi_{23_i} - 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i}^2 \right) + \\
 & x_{12} x_{13}^2 \in^2 \left( 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{13_i}^2 + \frac{2}{3} e^{\frac{3}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j}^3 \xi_{23_i}^2 \right) + \\
 & x_{13} x_{23}^2 \in^2 \left( 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2)} \xi_{13_j} \xi_{23_i}^2 - \frac{2}{3} e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i}^3 \right) + \\
 & x_{13}^2 x_{23} \in^2 \left( 2 e^{\alpha_1 + \alpha_2} \xi_{13_j}^2 \xi_{23_i} + \frac{2}{3} e^{2\alpha_1 - \alpha_2} \xi_{12_j}^2 \xi_{23_i}^3 \right) + x_{23} \left( \xi_{23_i} + e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2)} \xi_{23_j} \right) \Big]
 \end{aligned}$$

In[\*]:= sol[2, 4]

$$\begin{aligned}
& \gg \text{TTL} \left( e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \xi_{12i} + e^{\frac{1}{2}\gamma} (2\alpha_{1j}-\alpha_{2j}) \xi_{12j} \right) U_k[x_{12}] + \\
& \left( e^{\frac{1}{2}\gamma} (\ll 1 \gg) \xi_{13i} + \ll 15 \gg + \frac{1}{2} \ll 6 \gg \xi_{23j} \right) \ll 1 \gg + \ll 71 \gg + \\
& \left( 2 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \epsilon^2 \lambda^2 \hbar^2 \xi_{24i} \xi_{34i}^2 + \frac{2}{3} e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \epsilon^2 \lambda^3 \hbar^2 \xi_{23i} \xi_{34i}^3 + \right. \\
& \left. \ll 10 \gg + \frac{2}{3} e^{-\frac{1}{2}\gamma} (\alpha_{1j}+\alpha_{2j}-5\alpha_{3j}) \gamma^2 \epsilon^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{34j}^3 \right) U_k[x_{24}, x_{34}, x_{34}] \\
& \gg \text{SG} U_k[] + \frac{1}{2} e^{\gamma} (3\alpha_{1i}+3\alpha_{1j}-\alpha_{3i}-\alpha_{3j}) \epsilon^2 U_k[x_{12}, x_{12}, x_{13}, x_{13}] f_{1,x_{12}x_{13}}[\lambda]^2 + \\
& \ll 165 \gg + U_k[x_{24}, x_{34}, x_{34}] \left( 2 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma \epsilon^2 \hbar f_{\theta,x_{34}}[\lambda] f_{1,x_{24}x_{34}}[\lambda] + \right. \\
& \left. e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \epsilon^2 f_{\theta,x_{34}}[\lambda] f_{2,x_{23}x_{34}^2}[\lambda] + e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \epsilon^2 f_{2,x_{24}x_{34}^2}[\lambda] \right) \\
& \gg \text{TTR} U_k[x_{12}] \\
& \left( e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) f_{\theta,x_{12}'}[\lambda] + e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \epsilon f_{1,x_{12}'}[\lambda] + e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \epsilon^2 f_{2,x_{12}'}[\lambda] \right) + \\
& \ll 72 \gg + U_k[x_{24}, x_{34}, x_{34}] \left( \frac{2}{3} e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \epsilon^2 \hbar^2 f_{\theta,x_{34}}[\lambda]^3 f_{\theta,x_{23}'}[\lambda] + \right. \\
& \left. 2 e^{\ll 1 \gg} \ll 3 \gg \ll 1 \gg^2 f_{\ll 1 \gg}'[\lambda] + \ll 1 \gg + \ll 1 \gg + e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\ll 1 \gg + \ll 1 \gg -5\alpha_{3i}-5\alpha_{3j}) \epsilon^2 f_{2,\ll 1 \gg}'[\lambda] \right) \\
& \gg \text{diff} U_k[x_{12}] \left( e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \xi_{12i} + e^{\frac{1}{2}\gamma} (2\alpha_{1j}-\alpha_{2j}) \xi_{12j} - e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) f_{\theta,x_{12}'}[\lambda] - \right. \\
& \left. e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \epsilon f_{1,x_{12}'}[\lambda] - e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \epsilon^2 f_{2,x_{12}'}[\lambda] \right) + \\
& U_k[x_{13}] \left( e^{\ll 1 \gg} \ll 1 \gg + \ll 42 \gg \right) + \ll 70 \gg + \ll 1 \gg + U_k[x_{24}, x_{34}, x_{34}] \\
& \left( 2 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\ll 2 \gg + \ll 1 \gg -5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \epsilon^2 \lambda^2 \hbar^2 \xi_{24i} \xi_{34i}^2 + \ll 1 \gg + \ll 21 \gg \right) \\
& \gg \text{eqns} \left\{ e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \xi_{12i} + e^{\frac{1}{2}\gamma} (2\alpha_{1j}-\alpha_{2j}) \xi_{12j} - e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) f_{\theta,x_{12}'}[\lambda] = 0, \right. \\
& e^{\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-\alpha_{3i}-\alpha_{3j}) \xi_{13i} + \ll 10 \gg = 0, \ll 1 \gg = 0, \ll 97 \gg, \\
& \ll 1 \gg = 0, -4 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\ll 7 \gg) \gamma^2 \lambda^2 \hbar^2 \xi_{24i} \xi_{24j} \xi_{34i} + \ll 47 \gg = 0, \\
& \left. 2 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \lambda^2 \hbar^2 \xi_{24i} \xi_{34i}^2 + \frac{2}{3} e^{\ll 1 \gg} \gamma^2 \lambda^3 \hbar^2 \xi_{23i} \xi_{34i}^3 + \ll 21 \gg = 0 \right\}
\end{aligned}$$

Out[ ]=

$$\begin{aligned}
& \{ f_{\theta,x_{12}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{12i} + e^{-\frac{1}{2}\gamma} (2\alpha_{1i}-\alpha_{2i}) \lambda \xi_{12j}], \\
& f_{\theta,x_{23}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{23i} + e^{\frac{1}{2}\gamma} (\alpha_{1i}-2\alpha_{2i}+\alpha_{3i}) \lambda \xi_{23j}], \\
& f_{\theta,x_{13}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{13i} + e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{2i}-\alpha_{3i}) \lambda \xi_{13j} - e^{-\frac{1}{2}\gamma} (2\alpha_{1i}-\alpha_{2i}) \lambda^2 \xi_{12j} \xi_{23i}], \\
& f_{\theta,x_{34}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{34i} + e^{\frac{1}{2}\gamma} (\alpha_{2i}-2\alpha_{3i}) \lambda \xi_{34j}], \\
& f_{\theta,x_{24}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{24i} + e^{\frac{1}{2}\gamma} (\alpha_{1i}-\alpha_{2i}-\alpha_{3i}) \lambda \xi_{24j} - e^{\frac{1}{2}\gamma} (\alpha_{1i}-2\alpha_{2i}+\alpha_{3i}) \lambda^2 \xi_{23j} \xi_{34i}], \\
& f_{\theta,x_{14}} \rightarrow \text{Function}[\{\lambda\}, \\
& \lambda \xi_{14i} + e^{\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{3i}) \lambda \xi_{14j} - e^{-\frac{1}{2}\gamma} (2\alpha_{1i}-\alpha_{2i}) \lambda^2 \xi_{12j} \xi_{24i} - e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{2i}-\alpha_{3i}) \lambda^2 \xi_{13j} \xi_{34i}], \\
& f_{1,x_{12}} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x_{23}} \rightarrow \text{Function}[\{\lambda\}, 0], \\
& f_{1,x_{13}} \rightarrow \text{Function}[\{\lambda\}, -e^{-\frac{1}{2}\gamma} (2\alpha_{1i}-\alpha_{2i}) \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i}], f_{1,x_{34}} \rightarrow \text{Function}[\{\lambda\}, 0],
\end{aligned}$$

$$\begin{aligned}
f_{1,x24} &\rightarrow \text{Function}\left[\{\lambda\}, -e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{23j} \xi_{34i}\right], \\
f_{1,x14} &\rightarrow \text{Function}\left[\{\lambda\}, -e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{24i} - e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{34i}\right], \\
f_{1,x12 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i}\right], \\
f_{1,x12 \times 13} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13i}\right], \\
f_{1,x12 \times 24} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{24i}\right], \\
f_{1,x12 \times 14} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{14i}\right], \\
f_{1,x13 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{23i}\right], f_{1,x13^2} \rightarrow \text{Function}[\{\lambda\}, 0], \\
f_{1,x23 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{23j} \xi_{34i}\right], \\
f_{1,x13 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{34i}\right], \\
f_{1,x23 \times 24} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \gamma \lambda^2 \hbar \xi_{23j} \xi_{24i}\right], f_{1,x13 \times 24} \rightarrow \\
&\quad \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^3 \hbar \xi_{12j} \xi_{23i} \xi_{24i} - 2 e^{-\frac{1}{2}\gamma(3\alpha_2-2\alpha_3)} \gamma \lambda^3 \hbar \xi_{13j} \xi_{23j} \xi_{34i}\right], \\
f_{1,x14 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, -4 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{24i}\right], \\
f_{1,x13 \times 14} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{14i} + \right. \\
&\quad \left. 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^3 \hbar \xi_{12j} \xi_{14i} \xi_{23i} + 2 e^{-\frac{1}{2}\gamma(3\alpha_1-\alpha_3)} \gamma \lambda^3 \hbar \xi_{12j} \xi_{13j} \xi_{24i}\right], \\
f_{1,x24 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{\frac{1}{2}\gamma(\alpha_1-\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{24j} \xi_{34i}\right], \\
f_{1,x14 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_3)} \gamma \lambda^2 \hbar \xi_{14j} \xi_{34i}\right], \\
f_{1,x24^2} &\rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x14 \times 24} \rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_3)} \gamma \lambda^2 \hbar \xi_{14j} \xi_{24i} + \right. \\
&\quad \left. 2 e^{-\gamma\alpha_2} \gamma \lambda^3 \hbar \xi_{14j} \xi_{23j} \xi_{34i} + 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^3 \hbar \xi_{13j} \xi_{24i} \xi_{34i}\right], \\
f_{1,x14^2} &\rightarrow \text{Function}[\{\lambda\}, 0], f_{2,x12} \rightarrow \text{Function}[\{\lambda\}, 0], f_{2,x23} \rightarrow \text{Function}[\{\lambda\}, 0], \\
f_{2,x13} &\rightarrow \text{Function}\left[\{\lambda\}, -\frac{1}{2} e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{23i}\right], f_{2,x34} \rightarrow \text{Function}[\{\lambda\}, 0], \\
f_{2,x24} &\rightarrow \text{Function}\left[\{\lambda\}, -\frac{1}{2} e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{23j} \xi_{34i}\right], \\
f_{2,x14} &\rightarrow \text{Function}\left[\{\lambda\}, -\frac{1}{2} e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{24i} - \frac{1}{2} e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{13j} \xi_{34i}\right], \\
f_{2,x12 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{23i}\right], \\
f_{2,x12 \times 13} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{13i} - 2 e^{-\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{23i}\right], \\
f_{2,x12 \times 24} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{24i}\right], \\
f_{2,x12 \times 14} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{14i} - 2 e^{-\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{24i}\right], \\
f_{2,x13 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23i}^2\right], \\
f_{2,x13^2} &\rightarrow \text{Function}[\{\lambda\}, e^{-\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{23i}^2], \\
f_{2,x23 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{23j} \xi_{34i}\right],
\end{aligned}$$

$$\begin{aligned}
& f_{2,x13 \times 34} \rightarrow \text{Function} \left[ \{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{13j} \xi_{34i} \right], \\
& f_{2,x23 \times 24} \rightarrow \text{Function} \left[ \{\lambda\}, 2 e^{-\frac{1}{2} \gamma (-\alpha_1 + 2\alpha_2 - \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{23j} \xi_{24i} - 2 e^{\gamma (\alpha_1 - 2\alpha_2 + \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{23j}^2 \xi_{34i} \right], \\
& f_{2,x13 \times 24} \rightarrow \\
& \quad \text{Function} \left[ \{\lambda\}, -4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23i} \xi_{24i} - 4 e^{-\frac{1}{2} \gamma (3\alpha_2 - 2\alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i} \right], \\
& f_{2,x14 \times 23} \rightarrow \text{Function} [\{\lambda\}, 0], f_{2,x13 \times 14} \rightarrow \text{Function} \left[ \{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{13j} \xi_{14i} + \right. \\
& \quad \left. 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{23i} \xi_{24i} - 2 e^{-\gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{34i} \right], f_{2,x24 \times 34} \rightarrow \\
& \quad \text{Function} \left[ \{\lambda\}, 2 e^{-\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 + \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{24j} \xi_{34i} - 2 e^{\frac{1}{2} \gamma (\alpha_1 - 2\alpha_2 + \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{34i}^2 \right], \\
& f_{2,x14 \times 34} \rightarrow \text{Function} \left[ \{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{14j} \xi_{34i} - 2 e^{\frac{1}{2} \gamma (-\alpha_1 - \alpha_2 + \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{34i}^2 \right], \\
& f_{2,x24^2} \rightarrow \text{Function} [\{\lambda\}, e^{-\gamma (-\alpha_1 + 2\alpha_2 - \alpha_3)} \gamma^2 \lambda^4 \hbar^2 \xi_{23j}^2 \xi_{34i}^2], \\
& f_{2,x14 \times 24} \rightarrow \text{Function} \left[ \{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{14j} \xi_{24i} - \right. \\
& \quad \left. 2 e^{\frac{1}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{24i}^2 + 2 e^{\frac{1}{2} \gamma (-3\alpha_2 + 2\alpha_3)} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i}^2 \right], \\
& f_{2,x14^2} \rightarrow \text{Function} [\{\lambda\}, e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{24i}^2 + e^{-\gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^4 \hbar^2 \xi_{13j}^2 \xi_{34i}^2], \\
& f_{2,x12^2 \times 23} \rightarrow \text{Function} [\{\lambda\}, 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{23i}], \\
& f_{2,x12^2 \times 13} \rightarrow \text{Function} \left[ \{\lambda\}, 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{13i} - \frac{2}{3} e^{\frac{3}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^3 \xi_{23i} \right], \\
& f_{2,x12^2 \times 24} \rightarrow \text{Function} [\{\lambda\}, 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{24i}], \\
& f_{2,x12^2 \times 14} \rightarrow \text{Function} \left[ \{\lambda\}, 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{14i} - \frac{2}{3} e^{\frac{3}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^3 \xi_{24i} \right], \\
& f_{2,x12 \times 23^2} \rightarrow \text{Function} [\{\lambda\}, 2 e^{\frac{1}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23i}^2], f_{2,x12 \times 13 \times 23} \rightarrow \\
& \quad \text{Function} \left[ \{\lambda\}, -4 e^{\frac{1}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13i} \xi_{23i} - 4 e^{-\frac{1}{2} \gamma (3\alpha_1 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13j} \xi_{23i} \right], \\
& f_{2,x12 \times 13^2} \rightarrow \text{Function} \left[ \{\lambda\}, 2 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13i}^2 + \frac{2}{3} e^{\frac{3}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^3 \xi_{23i}^2 \right], \\
& f_{2,x12 \times 23 \times 34} \rightarrow \text{Function} [\{\lambda\}, 0], f_{2,x12 \times 13 \times 34} \rightarrow \text{Function} [\{\lambda\}, 0], f_{2,x12 \times 23 \times 24} \rightarrow \\
& \quad \text{Function} [\{\lambda\}, 4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23i} \xi_{24i} - 4 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23j} \xi_{24i}], \\
& f_{2,x12 \times 13 \times 24} \rightarrow \text{Function} [\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13i} \xi_{24i} - \\
& \quad 4 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{23i} \xi_{24i}], f_{2,x12 \times 14 \times 23} \rightarrow \\
& \quad \text{Function} [\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{14i} \xi_{23i} - 8 e^{-\frac{1}{2} \gamma (3\alpha_1 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13j} \xi_{24i}], \\
& f_{2,x12 \times 13 \times 14} \rightarrow \text{Function} [\{\lambda\}, 4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13i} \xi_{14i} + 4 e^{-\frac{1}{2} \gamma (3\alpha_1 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \\
& \quad \xi_{13j} \xi_{14i} - 4 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{14i} \xi_{23i} + \frac{4}{3} e^{-\frac{3}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^3 \xi_{23i} \xi_{24i}], \\
& f_{2,x12 \times 24 \times 34} \rightarrow \text{Function} [\{\lambda\}, 0], f_{2,x12 \times 14 \times 34} \rightarrow \text{Function} [\{\lambda\}, 0], \\
& f_{2,x12 \times 24^2} \rightarrow \text{Function} [\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{24i}^2], \\
& f_{2,x12 \times 14 \times 24} \rightarrow \text{Function} [\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{14i} \xi_{24i} - \\
& \quad 4 e^{\frac{1}{2} \gamma (-3\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{14j} \xi_{24i} + 4 e^{\frac{1}{2} \gamma (-3\alpha_1 + \alpha_3)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{24i} \xi_{34i}],
\end{aligned}$$

$$\begin{aligned}
f_{2, \times 12 \times 14^2} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{14i}^2 + \frac{2}{3} e^{\frac{3}{2} \gamma (-2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^3 \xi_{24i}^2\right], \\
f_{2, \times 13 \times 23^2} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{23i}^2 - \frac{2}{3} e^{\frac{1}{2} \gamma (-2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{23i}^3\right], \\
f_{2, \times 13^2 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{23i} + \frac{2}{3} e^{-\gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{23i}^3\right], \\
f_{2, \times 13^3} &\rightarrow \text{Function}\left[\{\lambda\}, -\frac{2}{9} e^{-\frac{3}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^6 \hbar^2 \xi_{12j}^3 \xi_{23i}^3\right], \\
f_{2, \times 23^2 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{\gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j}^2 \xi_{34i}\right], f_{2, \times 13 \times 23 \times 34} \rightarrow \\
&\text{Function}\left[\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{23i} \xi_{34i} + 4 e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i}\right], \\
f_{2, \times 13^2 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{34i}\right], f_{2, \times 23^2 \times 24} \rightarrow \\
&\text{Function}\left[\{\lambda\}, 2 e^{-\gamma (-\alpha_{1i} + 2 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j}^2 \xi_{24i} - \frac{2}{3} e^{\frac{3}{2} \gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{23j}^3 \xi_{34i}\right], \\
f_{2, \times 13 \times 23 \times 24} &\rightarrow \text{Function}\left[\{\lambda\}, 4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{23i} \xi_{23j} \xi_{24i} + \right. \\
&\left. 4 e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{23i} \xi_{23j} \xi_{34i}\right], f_{2, \times 13^2 \times 24} \rightarrow \\
&\text{Function}\left[\{\lambda\}, 2 e^{-\gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{23i}^2 \xi_{24i} - 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + 4 \alpha_{2i} - 3 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j}^2 \xi_{23j} \xi_{34i}\right], \\
f_{2, \times 14 \times 23^2} &\rightarrow \text{Function}\left[\{\lambda\}, 0\right], \\
f_{2, \times 13 \times 14 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, 4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{14i} \xi_{23i} + \right. \\
&\left. 8 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{24i} + 4 e^{-\frac{1}{2} \gamma (3 \alpha_{1i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{23i} \xi_{24i}\right], f_{2, \times 13^2 \times 14} \rightarrow \\
&\text{Function}\left[\{\lambda\}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{14i} - 4 e^{-\frac{1}{2} \gamma (3 \alpha_{1i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{14i} \xi_{23i} + \right. \\
&\left. 2 e^{-\gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{14i} \xi_{23i}^2 - 2 e^{-\frac{1}{2} \gamma (4 \alpha_{1i} + \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j}^2 \xi_{24i} - \right. \\
&\left. \frac{2}{3} e^{\frac{3}{2} \gamma (-2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^6 \hbar^2 \xi_{12j}^3 \xi_{23i}^2 \xi_{24i} - \frac{2}{3} e^{-\frac{3}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j}^3 \xi_{34i}\right], \\
f_{2, \times 23 \times 34^2} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (-\alpha_{1i} + 2 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{34i}^2\right], \\
f_{2, \times 13 \times 34^2} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{34i}^2\right], \\
f_{2, \times 23 \times 24 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, -4 e^{\frac{1}{2} \gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{24i} \xi_{34i} - \right. \\
&\left. 4 e^{-\frac{1}{2} \gamma (-2 \alpha_{1i} + 3 \alpha_{2i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{24j} \xi_{34i}\right], f_{2, \times 13 \times 24 \times 34} \rightarrow \text{Function}\left[\{\lambda\}, \right. \\
&\left. -4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{24j} \xi_{34i} - 4 e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i}^2\right], f_{2, \times 14 \times 23 \times 34} \rightarrow \\
&\text{Function}\left[\{\lambda\}, -4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{23j} \xi_{34i} - 8 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{24i} \xi_{34i}\right], \\
f_{2, \times 13 \times 14 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{14i} \xi_{34i} - \right. \\
&\left. 4 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{14j} \xi_{34i} + 4 e^{-\frac{1}{2} \gamma (3 \alpha_{1i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{24i} \xi_{34i}\right], f_{2, \times 23 \times 24^2} \rightarrow \\
&\text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (-\alpha_{1i} + 2 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{24i}^2 + \frac{2}{3} e^{\frac{3}{2} \gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{23j}^3 \xi_{34i}^2\right], \\
f_{2, \times 13 \times 24^2} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{23i} \xi_{24i}^2 + \right. \\
&\left. 2 e^{\frac{1}{2} \gamma (\alpha_{1i} - 5 \alpha_{2i} + 3 \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j} \xi_{23j}^2 \xi_{34i}^2\right],
\end{aligned}$$

$$\begin{aligned}
& f_{2, \times 14 \times 23 \times 24} \rightarrow \text{Function} \left[ \{ \lambda \}, 4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{23j} \xi_{24i} + \right. \\
& \quad \left. 8 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{24i}^2 + 4 e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{23j} \xi_{24i} \xi_{34i} \right], \\
& f_{2, \times 13 \times 14 \times 24} \rightarrow \text{Function} \left[ \{ \lambda \}, 4 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{14i} \xi_{23i} \xi_{24i} + \right. \\
& \quad 4 e^{\frac{1}{2} \gamma (-3 \alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{14j} \xi_{23i} \xi_{24i} + 4 e^{\frac{1}{2} \gamma (-3 \alpha_{2i} + 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{14i} \xi_{23j} \xi_{34i} + \\
& \quad 4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + 3 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{14j} \xi_{23j} \xi_{34i} - 4 e^{\frac{1}{2} \gamma (-3 \alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j} \xi_{13j} \xi_{23i} \xi_{24i} \xi_{34i} - \\
& \quad \left. 4 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j} \xi_{13j} \xi_{23j} \xi_{24i} \xi_{34i} \right], f_{2, \times 14^2 \times 23} \rightarrow \text{Function} [ \{ \lambda \}, 0 ], f_{2, \times 13 \times 14^2} \rightarrow \\
& \text{Function} \left[ \{ \lambda \}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{14i}^2 - 2 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{14i}^2 \xi_{23i} - \right. \\
& \quad 4 e^{-\frac{1}{2} \gamma (3 \alpha_{1i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{14i} \xi_{24i} + 2 e^{\frac{1}{2} \gamma (-5 \alpha_{1i} + \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{13j} \xi_{24i}^2 - \\
& \quad \frac{2}{3} e^{\frac{3}{2} \gamma (-2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^6 \hbar^2 \xi_{12j}^3 \xi_{23i} \xi_{24i}^2 + \frac{2}{3} e^{-\frac{3}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j}^3 \xi_{34i}^2 \left. \right], f_{2, \times 24 \times 34^2} \rightarrow \\
& \text{Function} \left[ \{ \lambda \}, 2 e^{-\frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{24j} \xi_{34i}^2 - \frac{2}{3} e^{\frac{1}{2} \gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{23j} \xi_{34i}^3 \right], \\
& f_{2, \times 14 \times 34^2} \rightarrow \text{Function} \left[ \{ \lambda \}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{34i}^2 - \frac{2}{3} e^{\frac{1}{2} \gamma (-\alpha_{1i} - \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{34i}^3 \right], \\
& f_{2, \times 24^2 \times 34} \rightarrow \text{Function} \left[ \{ \lambda \}, 2 e^{-\gamma (-\alpha_{1i} + \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{24j}^2 \xi_{34i} + \frac{2}{3} e^{\gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{23j}^2 \xi_{34i}^3 \right], \\
& f_{2, \times 14 \times 24 \times 34} \rightarrow \text{Function} \left[ \{ \lambda \}, 4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{24i} \xi_{34i} + 4 e^{-\frac{1}{2} \gamma (\alpha_{2i} + 2 \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \right. \\
& \quad \left. \xi_{24j} \xi_{34i} - 4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^4 \hbar^2 \xi_{14j} \xi_{23j} \xi_{34i}^2 + \frac{4}{3} e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i}^3 \right], \\
& f_{2, \times 14^2 \times 34} \rightarrow \text{Function} \left[ \{ \lambda \}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j}^2 \xi_{34i} + \frac{2}{3} e^{\gamma (-\alpha_{1i} - \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j}^2 \xi_{34i}^3 \right], \\
& f_{2, \times 24^3} \rightarrow \text{Function} \left[ \{ \lambda \}, -\frac{2}{9} e^{-\frac{3}{2} \gamma (-\alpha_{1i} + 2 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^6 \hbar^2 \xi_{23j}^3 \xi_{34i}^3 \right], \\
& f_{2, \times 14 \times 24^2} \rightarrow \text{Function} \left[ \{ \lambda \}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{24i}^2 - \frac{2}{3} e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{24i}^3 - \right. \\
& \quad 4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^4 \hbar^2 \xi_{14j} \xi_{23j} \xi_{24i} \xi_{34i} - 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{24i}^2 \xi_{34i} + \\
& \quad \left. 2 e^{\frac{1}{2} \gamma (\alpha_{1i} - 4 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{14j} \xi_{23j}^2 \xi_{34i}^2 - \frac{2}{3} e^{\frac{1}{2} \gamma (\alpha_{1i} - 5 \alpha_{2i} + 3 \alpha_{3i})} \gamma^2 \lambda^6 \hbar^2 \xi_{13j} \xi_{23j}^2 \xi_{34i}^3 \right], \\
& f_{2, \times 14^2 \times 24} \rightarrow \text{Function} \left[ \{ \lambda \}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j}^2 \xi_{24i} + \frac{2}{3} e^{-\gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{24i}^3 - \right. \\
& \quad 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{14j}^2 \xi_{23j} \xi_{34i} - 4 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{14j} \xi_{24i} \xi_{34i} + \\
& \quad \left. 2 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j}^2 \xi_{24i} \xi_{34i}^2 - \frac{2}{3} e^{-\frac{1}{2} \gamma (\alpha_{1i} + 4 \alpha_{2i} - 3 \alpha_{3i})} \gamma^2 \lambda^6 \hbar^2 \xi_{13j}^2 \xi_{23j} \xi_{34i}^3 \right], \\
& f_{2, \times 14^3} \rightarrow \text{Function} \left[ \{ \lambda \}, -\frac{2}{9} e^{-\frac{3}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^6 \hbar^2 \xi_{12j}^3 \xi_{24i}^3 - \frac{2}{9} e^{-\frac{3}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^6 \hbar^2 \xi_{13j}^3 \xi_{34i}^3 \right] \}
\end{aligned}$$