

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_{\hbar}(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

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This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

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This version: April 2020. Original version: July 1994.

```
In[ ]:= MakePDF[]
```

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```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[_ $\mathcal{E}$ _, op_] := Collect[Normal@Series[_ $\mathcal{E}$ _, { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[_ $\mathcal{E}$ _] := SS[_ $\mathcal{E}$ _, Together];
```

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```
In[ ]:= CF[_ $\mathcal{E}$ _] := PPCF@If[$k ==  $\infty$ ,
  Expand[Expand[_ $\mathcal{E}$ ] /.  $e^{p-} \mapsto e^{\text{Factor}^{op}}$ ],
  Expand[Expand[_ $\mathcal{E}$ ] /.  $e^{p-} \mapsto e^{\text{Factor}^{op}}$  /.  $q^{n-} \mapsto \sum_{k=0}^{k} \frac{(\hbar \epsilon \gamma n)^k}{k!}$ ] /.  $e^{k-} \mapsto$  ;  $k > $k \mapsto 0$ 
];
```

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```
In[ ]:= x[i_, j_] := x<>i<>j; a[i_] := a<>i; A[i_] := A<>i;
ξ[i_, j_] := ξ<>i<>j; α[i_] := α<>i; ℳ[i_] := ℳ<>i;
```

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```
In[ ]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[]] := τ[s]; ℓ[s_[]] := ℓ[s];
```

```
In[ ]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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```
In[ ]:= (* q~e^h γ ∈; Ai[n]~e^h ∈ ai; *)
```

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```
In[ ]:= $n = 5; $k = ∞; $E := {$n, $k}
```

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```
In[ ]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n],
    x[i, j]* = ξ[i, j]; ξ[i, j]* = x[i, j]; x[i, j]],
  Table[As[i, 1, $n - 1],
    A[i]* = ℳ[i]; ℳ[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i]; {a[i], A[i][_]}]
  }
ℓ_List* := #* & /@ ℓ;
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, g_i → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

```
In[ ]:= $PBWGens*
```

Time: 0.01 seconds Out[]:=

```
{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}
```

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```

In[*]:= UCF[ $\mathcal{E}_-$ ] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a @ L[AB], k], r]
  }],
  U[____], F
] /. F → CF]

```

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```

In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j_-[ $\mathcal{E}_-$ ] :=  $\mathcal{E}$  /. Ui → Uj;
mi→j→k_-[c_. Ui[x____] Uj[_]] := c Uk[x];
mi→j→k_-[c_. Ui[_] Uj[y____]] := c Uk[y];
mi→j→k_-[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi→j→k;
mi→j→k_-[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi→j→i) Uj[yy] /. UCF /. mi→j→k) /. UCF
];
ms__[ $\mathcal{E}_-$ ] := ms[UCF[ $\mathcal{E}$ ]];

```

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```

In[*]:= Supp[ $\mathcal{E}_-$ ] := Union@Cases[{ $\mathcal{E}$ }, Ui[____] ⇒ i, ∞];

```

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```

In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PPNCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /.  $\epsilon^{k-}$  /. k > $k ⇒ 0];
  Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];

```

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```

In[*]:= Δi→j, k_-[ $\mathcal{E}_-$ ] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk, k1→k @ mj, j1→j [Δj, k[f] Δi→j1, k1[Ui[fs]]]
}];
Δi→j, k, l_-[ $\mathcal{E}_-$ ] :=  $\mathcal{E}$  /. Δi→j, k /. Δk→k, l

```

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```
In[*]:= Si[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i[Si[f] Si1[Ui1[fs]]]
}];
```

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The $\mathbb{A}$ -algebra

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```
In[*]:= RRk[ai_, aj_] /; τ[ai] == τ[aj] == "a" := Uk[aj, ai];
RRk[ai_, Aj_[n_]] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[Aj[n], ai];
RRk[Aj_[n_], ai_] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[ai, Aj[n]];
RRk[Ai_[m_], Aj_[n_]] /; τ[Ai] == τ[Aj] == "A" := Uk[Aj[n], Ai[m]];
RRk[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@am;
    Uk[xij, am] + γ (δm,i - δm+1,i - δm,j + δm+1,j) Uk[xij] / 2
];
RRk[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@Am; q-n (δn,i - δm+1,i - δm,j + δm+1,j) Uk[xij, Am[n]]
]
```

```
In[*]:= RR0[a[1], x[1, 2]]
```

Time: 0.05 seconds Out[]:=

```
γ U0[x12] + U0[x12, a1]
```

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We take the reduction rules RR for xij's from Yamane page

509:

$$\begin{array}{ccc} \text{(I)} & i = m < j < n, & \text{(II)} \quad i < m < n < j, & \text{(III)} \quad i < m < j = n, \\ \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} \end{array}$$

$$\begin{array}{ccc} \text{(IV)} & i < m < j < n, & \text{(V)} \quad i < j = m < n, & \text{(VI)} \quad i < j < m < n. \\ \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} \end{array}$$

$$\begin{array}{ll} q^{-2} e_{ij} e_{mn} - e_{mn} e_{ij} = 0 & \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\ [e_{ij}, e_{mn}] = 0 & \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\ [e_{ij}, e_{mn}] = (q^2 - q^{-2}) e_{in} e_{mj} & \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\ q^2 e_{ij} e_{mn} - e_{mn} e_{ij} = q e_{in} & \text{if } ((i, j), (m, n)) \in C_{(V)}. \end{array}$$

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```

In[ ]:= RRk[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 Uk[xij, xmn], (* IVIII *)
    (i < m < j ∧ n < j) ∨ (j < m), Uk[xij, xmn], (* IIIVI *)
    i < m < j ∧ n > j, Uk[xij, xmn] + (q-2 - q2) Uk[x[i, n], x[m, j]], (* IV *)
    j == m, q2 Uk[xij, xmn] - q Uk[x[i, n]] (* V *)
  ]
]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.06 seconds Out[ ]=
-q U2[x24] U3[x12] + q2 U2[x23, x34] U3[x12]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.05 seconds Out[ ]=
q2 U0[x14] - q3 U0[x12, x24] - q3 U0[x13, x34] + q4 U0[x12, x23, x34]

In[ ]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
Time: 1.05 seconds Out[ ]=
{}

In[ ]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[ ]:=
{}

```

```

Module[{Bas = $PBWgens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3 ->2 // m2,4 ->2 // m1,2 ->1) ==
      (P // m3,4 ->3 // m2,3 ->2 // m1,2 ->1) == (P // m1,2 ->1 // m3,4 ->3 // m1,3 ->1) ==
      (P // m1,2 ->1 // m1,3 ->1 // m1,4 ->1) == (P // m2,3 ->2 // m1,2 ->1 // m1,4 ->1)
  ]
]
» {x12, x13, x23, a1, a2}

```

Out[8]=

{ }

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The Co-Product, Co-Associativity, and the Hopf Axiom

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In[8]:=

```

Do[As[i, 1, $n - 1],
  (* Warning: these initialization fail if $n is increased later on *)
  Δj_,k_ [a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj_,k_ [A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj_,k_ [x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj_,k_ [x[i, n]] = m[k, k1 -> k] [m[j, j1 -> j] [
    q Δj,k [x[i, n - 1]] Δj1,k1 [x[n - 1, n]] - q^-1 Δj,k [x[n - 1, n]] Δj1,k1 [x[i, n - 1]]
  ]
]

```

```

In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
]
Time: 0.06 seconds Out[*]=
Ui[x12] → Uj[A1[-1]] Uk[x12] + Uj[x12] Uk[A1[1]]
Ui[x13] → Uj[A1[-1], A2[-1]] Uk[x13] +
  (-1/q2 + q2) Uj[x12, A2[-1]] Uk[x23, A1[1]] + Uj[x13] Uk[A1[1], A2[1]]
Ui[x14] →
  Uj[A1[-1], A2[-1], A3[-1]] Uk[x14] + (-1/q2 + q2) Uj[x12, A2[-1], A3[-1]] Uk[x24, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1]] Uk[x34, A1[1], A2[1]] + Uj[x14] Uk[A1[1], A2[1], A3[1]]
Ui[x15] → Uj[A1[-1], A2[-1], A3[-1], A4[-1]] Uk[x15] +
  (-1/q2 + q2) Uj[x12, A2[-1], A3[-1], A4[-1]] Uk[x25, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1], A4[-1]] Uk[x35, A1[1], A2[1]] +
  (-1/q2 + q2) Uj[x14, A4[-1]] Uk[x45, A1[1], A2[1], A3[1]] +
  Uj[x15] Uk[A1[1], A2[1], A3[1], A4[1]]
Ui[x23] → Uj[A2[-1]] Uk[x23] + Uj[x23] Uk[A2[1]]
Ui[x24] → Uj[A2[-1], A3[-1]] Uk[x24] +
  (-1/q2 + q2) Uj[x23, A3[-1]] Uk[x34, A2[1]] + Uj[x24] Uk[A2[1], A3[1]]
Ui[x25] →
  Uj[A2[-1], A3[-1], A4[-1]] Uk[x25] + (-1/q2 + q2) Uj[x23, A3[-1], A4[-1]] Uk[x35, A2[1]] +
  (-1/q2 + q2) Uj[x24, A4[-1]] Uk[x45, A2[1], A3[1]] + Uj[x25] Uk[A2[1], A3[1], A4[1]]
Ui[x34] → Uj[A3[-1]] Uk[x34] + Uj[x34] Uk[A3[1]]
Ui[x35] → Uj[A3[-1], A4[-1]] Uk[x35] +
  (-1/q2 + q2) Uj[x34, A4[-1]] Uk[x45, A3[1]] + Uj[x35] Uk[A3[1], A4[1]]
Ui[x45] → Uj[A4[-1]] Uk[x45] + Uj[x45] Uk[A4[1]]
Ui[a1] → Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] → Uj[a2] Uk[] + Uj[] Uk[a2]
Ui[a3] → Uj[a3] Uk[] + Uj[] Uk[a3]
Ui[a4] → Uj[a4] Uk[] + Uj[] Uk[a4]

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
  ]]
]
» {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}

Out[*]=
{ }

```

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]
]
» {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}
Out[ ]:=
{ }

```

Is Δ a quantization of the co-bracket?

```

In[ ]:= $k = 1
Time: 0.02 seconds Out[ ]:=
1

In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1 /@ Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[xs___, A-[n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[ ]:=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 γ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 γ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0

```

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The Antipode

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```
In[*]:= Do[As[i, 1, $n - 1],
  Sj_ [a[i]] = -Uj[a[i]];
  Sj_ [A[i][n_]] = Uj[A[i][-n]];
  Sj_ [x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj_ [x[i, n]] = mj1,j→j [
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]
```

```
In[*]:= $k = 2
```

```
Time: 0.00 seconds Out[ ]=
2
```

```
In[*]:= Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
» {x12, x13, x23, a1, a2, A1[-2], A2[-1], A1[1], A2[-1], A1[-2], A2[1], A1[2], A2[0], A2[2], A2[0]}
```

```
Out[*]=
```

```
{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A1[-2] → U1[],
  A2[-1] → U1[], A1[1] → U1[], A2[-1] → U1[], A1[-2] → U1[],
  A2[1] → U1[], A1[2] → U1[], A2[0] → U1[], A2[2] → U1[], A2[0] → U1[]}
```

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The Generating Functions for the $\mathbb{A}$ operations

```
In[*]:= BeginProfile[]
```

```
Time: 0.06 seconds Out[ ]=
ProfileRoot
```

```
In[*]:= $k = 1;
```

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Ad, TT, Seepage

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```
In[*]:= PotQ[0] = True;
PotQ[c_. U_ [aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

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```
In[*]:= Ad[x_] [y_] /; PotQ[x] ∧ (Head[y] != Plus) := PPAdPoty[eUB[y,x]/y y];
Ad[x_] [y_Plus] /; PotQ[x] := PPAdPotPlus[Ad[x] /@ y];
```

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```
In[*]:= Ad[x_] [y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil@Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]
```

```
In[*]:= $k = 2
```

Time: 0.00 seconds Out:=

2

```
In[*]:= Ad[ε ħ Uj[x12] Uk[]] [Uj[x23] Uk[a4]]
```

Time: 0.06 seconds Out:=

$$\left(-\epsilon \hbar - \gamma \epsilon^2 \hbar^2\right) U_j[x_{13}] U_k[a_4] + U_j[x_{23}] U_k[a_4] + 2 \gamma \epsilon^2 \hbar^2 U_j[x_{12}, x_{23}] U_k[a_4]$$

```
In[*]:= PotQ[7 ħ Uj[x12] Uk[a3]]
```

Time: 0.02 seconds Out:=

$$\text{PotQ}[7 \hbar U_j[x_{12}] U_k[a_3]]$$

```
In[*]:= Ad[7 ħ Uj[x12] Uk[a3]] [Uj[x23] Uk[a4]]
```

Time: 0.09 seconds Out:=

$$U_j[x_{23}] U_k[a_4] + \left(-7 \hbar - 7 \gamma \epsilon \hbar^2 - \frac{7}{2} \gamma^2 \epsilon^2 \hbar^3\right) U_j[x_{13}] U_k[a_3, a_4] + \\ \left(14 \gamma \epsilon \hbar^2 + 14 \gamma^2 \epsilon^2 \hbar^3\right) U_j[x_{12}, x_{23}] U_k[a_3, a_4] - 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{13}] U_k[a_3, a_3, a_4] + \\ 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{12}, x_{23}] U_k[a_3, a_3, a_4] - \frac{686}{3} \gamma^2 \epsilon^2 \hbar^5 U_j[x_{12}, x_{12}, x_{13}] U_k[a_3, a_3, a_3, a_4]$$

```
In[*]:= Ad[π ħ Uj[a1]] [Uj[x23] Uk[a4]]
```

Time: 0.05 seconds Out:=

$$e^{\frac{i \pi \gamma}{2}} U_j[x_{23}] U_k[a_4]$$

pdf

```
In[*]:= TTλ[] = 0; (* Translated Tangent *)
TTλ[xs___, x_] := PPTT@UCF[Ad[x] [TTλ[xs]] + ∂λ x];
TTλ[xs_List] := TTλ@@ xs
```

In[*]:= $\text{TT}_\lambda[\lambda \xi_{12} U_0[x_{12}] U_1[], \lambda \xi_{23} U_0[x_{23}] U_1[]]$

Time: 0.08 seconds Out[]=

$$\xi_{12} U_0[x_{12}] U_1[] + \lambda \xi_{12} \xi_{23} U_0[x_{13}] U_1[] + \xi_{23} U_0[x_{23}] U_1[]$$

pdf

```
In[*]:= O_i_[poly_] := Total[
  CoefficientRules[poly, #i & /@ $PBWGens] /.
  (p_ -> c_) -> c U_i @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
];
O_i_,is_[poly_] := O_i[O_is[poly]];
P_i_[e_] := e /. U_i[xs___] -> Times @@ (#i & /@ {xs});
P_i_,is_[poly_] := P_i[P_is[poly]];
```

In[*]:= $O_i[x_{12}^2 x_{13} + 7 x_{23}^{17}]$

Time: 0.06 seconds Out[]=

$$U_i[x_{12}, x_{12}, x_{13}] + 7 U_i[x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}]$$

In[*]:= $O_{i,j}[x_{12}^2 x_{13} x_{24} + 7 x_{23}^{17} x_{25}^3]$

Time: 0.03 seconds Out[]=

$$U_i[x_{12}, x_{12}, x_{13}] U_j[x_{24}] + 7 U_i[x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}] U_j[x_{25}, x_{25}, x_{25}]$$

In[*]:= $P_{i,j} @ O_{i,j}[x_{12}^2 x_{13} x_{24} + 7 x_{23}^{17} x_{25}^3]$

Time: 0.06 seconds Out[]=

$$x_{12}^2 x_{13} x_{24} + 7 x_{23}^{17} x_{25}^3$$

pdf

```
In[*]:= Seepage_i_[{ }, c_] = c;
Seepage_i_[eS_List, P_] := Module[{lx},
  lx = $PBWGens[[Length@eS]];
  CF@Collect[P, lx_i, P_i[Ad[Last[eS] U_i[lx]]][O_i[Seepage_i[Most@eS, #]]]]
  &];
Seepage_i_,is_[eS_List, eSs_List, P_] := Seepage_i[eS, Seepage_is[eSs, P]]
```

In[*]:= $\$k = 1$

Time: 0.00 seconds Out[]=

$$1$$

In[*]:= $\$PBWGens^*$

Time: 0.01 seconds Out[]=

$$\{\xi_{12}, \xi_{13}, \xi_{14}, \xi_{15}, \xi_{23}, \xi_{24}, \xi_{25}, \xi_{34}, \xi_{35}, \xi_{45}, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$

In[*]:= $\text{Seepage}_i[\$PBWGens^*, x_{35}_i]$

Time: 0.16 seconds Out[]=

$$e^{\frac{1}{2}(\alpha_2 - \alpha_3 - \alpha_4)} \gamma x_{35}_i + 2 e^{\frac{1}{2}(\alpha_2 - 3\alpha_4)} \gamma \gamma \in \xi_{45} \hbar x_{35}_i x_{45}_i$$

In[*]:= **Seepage**_{i,j}[**#i** & /@ **\$PBWGens***, **#j** & /@ **\$PBWGens***, **x35i x45j**]

Time: 0.25 seconds Out[]=

$$e^{\frac{1}{2} \gamma (\alpha_2 i - \alpha_3 i + \alpha_3 j - \alpha_4 i - 2 \alpha_4 j)} x_{35i} x_{45j} + 2 e^{-\frac{1}{2} \gamma (-\alpha_2 i - \alpha_3 j + 3 \alpha_4 i + 2 \alpha_4 j)} \gamma \in \hbar x_{35i} x_{45i} x_{45j} \xi_{45i}$$

pdf

```
In[*]:= Invertis__[ε_] := Module[{A, B, U1, j}, (* ε=A U[ ] + ε B *)
U1 = Times@@ (U#[ ] & /@ {is});
A = Coefficient[ε, U1] /. ε → 0;
B = UCF[ (ε - A U1) ];
UCF[
A-1 (U1 + Sum[As[j, 1, $k], (-1)j NonCommutativeMultiply @@ ConstantArray[B / A, j]])]
]
```

In[*]:= **\$k** = 3;

Invert_{i,j}[7 **U**_i[] **U**_j[] + **ε** **U**_i[x12] **U**_j[x23]]

Time: 0.09 seconds Out[]=

$$\frac{1}{7} U_i[] U_j[] - \frac{1}{49} \epsilon U_i[x12] U_j[x23] + \frac{1}{343} \epsilon^2 U_i[x12, x12] U_j[x23, x23] - \frac{\epsilon^3 U_i[x12, x12, x12] U_j[x23, x23, x23]}{2401}$$

{**\$k** = 1, **\$n** = 3}

Time: 0.01 seconds Out[]=

{2, 3}

pdf

$$\begin{aligned} L_\lambda &= \mathbb{C}\langle \lambda_{\lambda_i}^i x \rangle \mathbb{C}\langle \lambda_{\lambda_i}^i x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle f_i; x_i \rangle \mathbb{C}\langle \text{Seepage}(f_i, A) \rangle &= \mathbb{C}\langle e^{f_i; x_i} A \rangle \\ \text{TT}(L_\lambda) &= L'_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f_i\}, B). \\ R_\lambda &= \mathbb{C}\langle f+g \rangle, \quad f=f_i x_i, \quad g=\sum_{|w|=1} g_w w & e^f = F e^g = G \quad R_\lambda = \mathbb{C}\langle FG \rangle \quad \partial_\lambda F = F \partial_\lambda f \\ \text{TT}(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle' \mathbb{C}\langle F(\partial_\lambda f G + \partial_\lambda G) \rangle = S(G)' S(\partial_\lambda f G + \partial_\lambda G) \end{aligned}$$

pdf

```
In[*]:= sol[kk_, nn_] := PP"sol"[kk, nn] @ Block[{$k = kk, $n = nn}, Module[{(* TTL, *)},
ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
TTL = ES["TTL"] @ PP"TTL" @ TTλ @
Join[
Cases[$PBWGens, g_ := Which[
τ[g] === "x", λ (g*)i Uk[g],
τ[g] === "a", (g*)i Uk[g]
]],
Cases[$PBWGens, g_ := Which[
τ[g] === "x", λ (g*)j Uk[g],
τ[g] === "a", (g*)j Uk[g]
]]]
```

```

];
mons[-1] = {};
mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
ff = Table[
  Sum[εp fp,uk[λ], {p, 0, $k}],
  {u, $PBWGens}
] /. fp,u[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
fs = Flatten[Table[
  fp,ℙk[m],
  {p, 0, $k}, {m, mons[p]}
]];
gg = Sum[
  εp fp,ℙk[u][λ] ℙk[u],
  {p, 0, $k}, {u, Complement[mons[p], Uk/@$PBWGens]}
];
G = Series[egg, {ε, 0, $k}];
Si[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
SG = ES["SG"]@PP"SG"@Sk[G];
TTR = ES["TTR"]@PP"TTR"@UCF[Invertk[SG] ⊗ Sk[(∂λ(ff.(#k & /@$PBWGens)) G + ∂λG)]];
diff = UCF[TTL - TTR];
eqns = Flatten[Table[
  Coefficient[Coefficient[diff, m], ε, p] == 0,
  {p, 0, $k}, {m, mons[p]}
]];
inits = (#[0] == 0) & /@ fs;
PP"DSolve"@Module[{},
  ss = {}; reqns = eqns; rinit = inits; rfs = fs;
  While[rfs != {},
    pos = FirstPosition[reqns, eq_ /; (Count[eq, f, ∞, Heads → True] == 1), None, {1}];
    If[pos == None,
      Echo[rfs];
      {s} = DSolve[reqns ∪ rinit, rfs, λ];
      rfs = {},
      (* else *) {p} = pos;
      {s} = DSolve[reqns[[p]] ∧ rinit[[p]], rfs[[p]], λ];
      reqns = Delete[reqns, p] /. s /. e_ == 0 => CF[e] == 0;
      rinit = Delete[rinit, p];
      rfs = Delete[rfs, p]
    ];
    ss = Join[ss, s /. F_Function => ReplacePart[F, 2 -> CF[F[[2]]]]];
  ]
];
E[Total[fs] /. fp,u => εp u (fp,u /. ss)[1]] /. h | γ -> 1
]]

```

In[*]:= sol[1, 3]

$$\begin{aligned}
 & \gg \text{TTL} \left(e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} \xi_{12_i} + e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) U_k[x_{12}] + \\
 & \quad \ll 5 \gg + \left(2 e^{\ll 1 \gg} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} - 2 \ll 7 \gg + \ll 1 \gg + 2 \ll 6 \gg \xi_{23_j} \right) \ll 1 \gg \\
 & \gg \text{SG} U_k[] + \ll 3 \gg + U_k[x_{13}, x_{23}] \left(e^{-\frac{3}{2} \gamma (\alpha_2 + \alpha_2)} \in f_{\emptyset, x_{23_k}}[\lambda] f_{1, x_{12_k} x_{23_k}}[\lambda] + e^{-\frac{3}{2} \gamma (\alpha_2 + \alpha_2)} \in f_{1, \ll 3 \gg_k \ll 1 \gg}[\lambda] \right) \\
 & \gg \text{TTR} U_k[x_{12}] \left(e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} f_{\emptyset, x_{12_k}}'[\lambda] + e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\ll 1 \gg - \ll 1 \gg - \alpha_2)} \in f_{1, \ll 1 \gg \ll 1 \gg}'[\lambda] \right) + \\
 & \quad \ll 5 \gg + U_k[x_{13}, x_{23}] (\ll 1 \gg)
 \end{aligned}$$

Out[*]=

$$\begin{aligned}
 & E \left[x_{12_k} \left(\xi_{12_i} + e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) - 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{12_k} x_{13_k} \xi_{12_j} \xi_{13_i} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{13_k} \xi_{12_j} \xi_{23_i} + \right. \\
 & \quad 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{12_k} x_{23_k} \xi_{12_j} \xi_{23_i} - 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2)} \in x_{13_k} x_{23_k} \xi_{13_j} \xi_{23_i} + \\
 & \quad \left. x_{13_k} \left(\xi_{13_i} + e^{\frac{1}{2} (\alpha_1 + \alpha_2)} \xi_{13_j} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i} \right) + x_{23_k} \left(\xi_{23_i} + e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2)} \xi_{23_j} \right) \right]
 \end{aligned}$$

In[*]:= sol[1, 4]

» TTL <<1>>

» SG $U_k[\] + \langle\langle 16 \rangle\rangle +$

$$U_k[x_{24}, x_{34}] \left(e^{\frac{1}{2} \gamma (\alpha_1 + \alpha_1 - 3\alpha_3 - 3\alpha_3)} \in f_{\emptyset, x_{34}}[\lambda] f_{1, x_{23} x_{34}}[\lambda] + e^{\frac{1}{2} \gamma (\alpha_1 + \langle\langle 5 \rangle\rangle)} \in f_1 \langle\langle 1 \rangle\rangle x_{24} \langle\langle 1 \rangle\rangle \langle\langle 3 \rangle\rangle [\lambda] \right)$$

» TTR $U_k[x_{12}] \left(e^{\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} f_{\emptyset, x_{12}}'[\lambda] + e^{\frac{1}{2} \gamma (2\alpha_1 + 2\langle\langle 1 \rangle\rangle - \langle\langle 1 \rangle\rangle - \alpha_2)} \in f_{1, \langle\langle 3 \rangle\rangle}'[\lambda] \right) +$
 $\langle\langle 21 \rangle\rangle + U_k[x_{24}, x_{34}] (\langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle + \langle\langle 1 \rangle\rangle)$

» diff $U_k[x_{12}]$

$$\left(e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} \xi_{12_i} + e^{\langle\langle 1 \rangle\rangle} \xi_{12_j} - e^{\langle\langle 1 \rangle\rangle} \langle\langle 1 \rangle\rangle'[\lambda] - e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \langle\langle 2 \rangle\rangle - \alpha_2)} \in f_{1, \langle\langle 3 \rangle\rangle}'[\lambda] \right) +$$

$$\langle\langle 21 \rangle\rangle + \langle\langle 1 \rangle\rangle (\langle\langle 1 \rangle\rangle)$$

» eqns $\left\{ e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} \xi_{12_i} + e^{-\frac{1}{2} \langle\langle 1 \rangle\rangle} \langle\langle 1 \rangle\rangle (2\langle\langle 1 \rangle\rangle - \langle\langle 1 \rangle\rangle) \xi_{12_j} - e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} f_{\emptyset, x_{12}}'[\lambda] = 0, \right.$
 $\left. \langle\langle 1 \rangle\rangle = 0, \langle\langle 25 \rangle\rangle, \langle\langle 1 \rangle\rangle, \langle\langle 1 \rangle\rangle = 0 \right\}$

Out[*]=

$$\begin{aligned} & \mathbb{E} \left[x_{12} \left(\xi_{12_i} + e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) - 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{13} \in \xi_{12_j} \xi_{13_i} - \right. \\ & 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{14} \in \xi_{12_j} \xi_{14_i} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{13} \in \xi_{12_j} \xi_{23_i} + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{23} \in \xi_{12_j} \xi_{23_i} - \\ & 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} x_{13} x_{23} \in \xi_{13_j} \xi_{23_i} + x_{13} \left(\xi_{13_i} + e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i} \right) + \\ & x_{23} \left(\xi_{23_i} + e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} \xi_{23_j} \right) + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{24} \in \xi_{12_j} \xi_{24_i} - \\ & 4 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} x_{14} x_{23} \in \xi_{13_j} \xi_{24_i} - 2 e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} x_{23} x_{24} \in \xi_{23_j} \xi_{24_i} + \\ & x_{13} x_{14} \in \left(-2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{14_i} + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{14_i} \xi_{23_i} + 2 e^{\frac{1}{2} (3\alpha_1 - \alpha_3)} \xi_{12_j} \xi_{13_j} \xi_{24_i} \right) + \\ & 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} x_{13} x_{34} \in \xi_{13_j} \xi_{34_i} - 2 e^{\frac{1}{2} (\alpha_1 + \alpha_3)} x_{14} x_{34} \in \xi_{14_j} \xi_{34_i} - \\ & e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} x_{24} \in \xi_{23_j} \xi_{34_i} + 2 e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} x_{23} x_{34} \in \xi_{23_j} \xi_{34_i} - \\ & 2 e^{\frac{1}{2} (-\alpha_1 + \alpha_2 + \alpha_3)} x_{24} x_{34} \in \xi_{24_j} \xi_{34_i} + x_{14} \in \left(-e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{24_i} - e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{34_i} \right) + \\ & x_{14} \left(\xi_{14_i} + e^{\frac{1}{2} (\alpha_1 + \alpha_3)} \xi_{14_j} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{24_i} - e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{34_i} \right) + \\ & x_{24} \left(\xi_{24_i} + e^{\frac{1}{2} (-\alpha_1 + \alpha_2 + \alpha_3)} \xi_{24_j} - e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} \xi_{23_j} \xi_{34_i} \right) + \\ & x_{13} x_{24} \in \left(-2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i} \xi_{24_i} - 2 e^{\frac{1}{2} (3\alpha_2 - 2\alpha_3)} \xi_{13_j} \xi_{23_j} \xi_{34_i} \right) + \\ & x_{14} x_{24} \in \left(-2 e^{\frac{1}{2} (\alpha_1 + \alpha_3)} \xi_{14_j} \xi_{24_i} + 2 e^{\alpha_2} \xi_{14_j} \xi_{23_j} \xi_{34_i} + 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{24_i} \xi_{34_i} \right) + \\ & x_{34} \left(\xi_{34_i} + e^{\frac{1}{2} (-\alpha_2 + 2\alpha_3)} \xi_{34_j} \right) \Big] \end{aligned}$$

In[*]:= sol[2, 3]

» TTL $\left(e^{-\frac{1}{2} \gamma (2\alpha_{1i} + 2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} \xi_{12i} + e^{-\frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2j})} \xi_{12j} \right) U_k[x_{12}] +$
 $\ll 13 \gg + \left(2 e^{\frac{1}{2} \gamma (\ll 1 \gg)} \gamma^2 \epsilon^2 \lambda^2 \hbar^2 \xi_{13i} \xi_{23i}^2 + \ll 11 \gg + \ll 1 \gg \right) \ll 1 \gg$

» SG $U_k[] + \ll 20 \gg + U_k[x_{13}, x_{23}, x_{23}]$
 $\left(2 e^{\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{1j} - 5 \ll 2 \gg_{i-5} \alpha_{2j})} \gamma \epsilon^2 \hbar f_{\theta, x_{23k}}[\lambda] f_{1, x_{13k} x_{23k}}[\lambda] + \ll 1 \gg + e^{\frac{1}{2} \ll 1 \gg \ll 1 \gg (\ll 1 \gg)} \ll 1 \gg f_{\ll 1 \gg}[\lambda] \right)$

» TTR $\ll 1 \gg$

Out[*]=

$$\begin{aligned} & \mathbb{E} \left[x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon x_{12k} x_{13k} \xi_{12j} \xi_{13i} - \right. \\ & e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon x_{13k} \xi_{12j} \xi_{23i} - \frac{1}{2} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon^2 x_{13k} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon x_{12k} x_{23k} \xi_{12j} \xi_{23i} + \\ & 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon^2 x_{12k} x_{23k} \xi_{12j} \xi_{23i} + 2 e^{2\alpha_{1i} - \alpha_{2i}} \epsilon^2 x_{12k}^2 x_{23k} \xi_{12j}^2 \xi_{23i} - \\ & 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \epsilon x_{13k} x_{23k} \xi_{13j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon^2 x_{12k} x_{23k}^2 \xi_{12j} \xi_{23i}^2 + e^{2\alpha_{1i} - \alpha_{2i}} \epsilon^2 x_{13k}^2 \xi_{12j}^2 \xi_{23i}^2 - \\ & \frac{2}{9} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon^2 x_{13k}^3 \xi_{12j}^3 \xi_{23i}^3 + x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + \\ & \epsilon^2 x_{12k} x_{13k} \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} - 2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i} \right) + \\ & \epsilon^2 x_{12k}^2 x_{13k} \left(2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{13i} - \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i} \right) + \\ & \epsilon^2 x_{12k} x_{13k} x_{23k} \left(-4 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} \xi_{23i} - 4 e^{\frac{3\alpha_{1i}}{2}} \xi_{12j} \xi_{13j} \xi_{23i} \right) + \\ & \epsilon^2 x_{13k} x_{23k} \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^2 \right) + \\ & \epsilon^2 x_{12k} x_{13k}^2 \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i}^2 + \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i}^2 \right) + \\ & \epsilon^2 x_{13k} x_{23k}^2 \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i}^2 - \frac{2}{3} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^3 \right) + \\ & \left. \epsilon^2 x_{13k}^2 x_{23k} \left(2 e^{\alpha_{1i} + \alpha_{2i}} \xi_{13j}^2 \xi_{23i} + \frac{2}{3} e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i}^3 \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right) \right] \end{aligned}$$

In[*]:= sol[2, 4]

$$\begin{aligned}
& \gg \text{TTL} \left(e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \xi_{12i} + e^{\frac{1}{2}\gamma} (2\alpha_{1j}-\alpha_{2j}) \xi_{12j} \right) U_k[x_{12}] + \\
& \left(e^{\frac{1}{2}\gamma} (\ll 1 \gg) \xi_{13i} + \ll 15 \gg + \frac{1}{2} \ll 6 \gg \xi_{23j} \right) \ll 1 \gg + \ll 71 \gg + \\
& \left(2 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \epsilon^2 \lambda^2 \hbar^2 \xi_{24i} \xi_{34i}^2 + \frac{2}{3} e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \epsilon^2 \lambda^3 \hbar^2 \xi_{23i} \xi_{34i}^3 + \right. \\
& \left. \ll 10 \gg + \frac{2}{3} e^{-\frac{1}{2}\gamma} (\alpha_{1j}+\alpha_{2j}-5\alpha_{3j}) \gamma^2 \epsilon^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{34j}^3 \right) U_k[x_{24}, x_{34}, x_{34}] \\
& \gg \text{SG} U_k[] + \frac{1}{2} e^{\gamma} (3\alpha_{1i}+3\alpha_{1j}-\alpha_{3i}-\alpha_{3j}) \epsilon^2 U_k[x_{12}, x_{12}, x_{13}, x_{13}] f_{1,x_{12}x_{13}}[\lambda]^2 + \\
& \ll 165 \gg + U_k[x_{24}, x_{34}, x_{34}] \left(2 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma \epsilon^2 \hbar f_{\theta,x_{34}}[\lambda] f_{1,x_{24}x_{34}}[\lambda] + \right. \\
& \left. e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \epsilon^2 f_{\theta,x_{34}}[\lambda] f_{2,x_{23}x_{34}^2}[\lambda] + e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \epsilon^2 f_{2,x_{24}x_{34}^2}[\lambda] \right) \\
& \gg \text{TTR} U_k[x_{12}] \\
& \left(e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) f_{\theta,x_{12}'}[\lambda] + e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \epsilon f_{1,x_{12}'}[\lambda] + e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \epsilon^2 f_{2,x_{12}'}[\lambda] \right) + \\
& \ll 72 \gg + U_k[x_{24}, x_{34}, x_{34}] \left(\frac{2}{3} e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \epsilon^2 \hbar^2 f_{\theta,x_{34}}[\lambda]^3 f_{\theta,x_{23}'}[\lambda] + \right. \\
& \left. 2 e^{\ll 1 \gg} \ll 3 \gg \ll 1 \gg^2 f_{\ll 1 \gg}'[\lambda] + \ll 1 \gg + \ll 1 \gg + e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\ll 1 \gg + \ll 1 \gg -5\alpha_{3i}-5\alpha_{3j}) \epsilon^2 f_{2,\ll 1 \gg}'[\lambda] \right) \\
& \gg \text{diff} U_k[x_{12}] \left(e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \xi_{12i} + e^{\frac{1}{2}\gamma} (2\alpha_{1j}-\alpha_{2j}) \xi_{12j} - e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) f_{\theta,x_{12}'}[\lambda] - \right. \\
& \left. e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \epsilon f_{1,x_{12}'}[\lambda] - e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \epsilon^2 f_{2,x_{12}'}[\lambda] \right) + \\
& U_k[x_{13}] (e^{\ll 1 \gg} \ll 1 \gg + \ll 42 \gg) + \ll 70 \gg + \ll 1 \gg + U_k[x_{24}, x_{34}, x_{34}] \\
& \left(2 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\ll 2 \gg + \ll 1 \gg -5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \epsilon^2 \lambda^2 \hbar^2 \xi_{24i} \xi_{34i}^2 + \ll 1 \gg + \ll 21 \gg \right) \\
& \gg \text{eqns} \left\{ e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) \xi_{12i} + e^{\frac{1}{2}\gamma} (2\alpha_{1j}-\alpha_{2j}) \xi_{12j} - e^{\frac{1}{2}\gamma} (2\alpha_{1i}+2\alpha_{1j}-\alpha_{2i}-\alpha_{2j}) f_{\theta,x_{12}'}[\lambda] = 0, \right. \\
& e^{\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-\alpha_{3i}-\alpha_{3j}) \xi_{13i} + \ll 10 \gg = 0, \ll 1 \gg = 0, \ll 97 \gg, \\
& \ll 1 \gg = 0, -4 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\ll 7 \gg) \gamma^2 \lambda^2 \hbar^2 \xi_{24i} \xi_{24j} \xi_{34i} + \ll 47 \gg = 0, \\
& \left. 2 e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{1j}+\alpha_{2i}+\alpha_{2j}-5\alpha_{3i}-5\alpha_{3j}) \gamma^2 \lambda^2 \hbar^2 \xi_{24i} \xi_{34i}^2 + \frac{2}{3} e^{\ll 1 \gg} \gamma^2 \lambda^3 \hbar^2 \xi_{23i} \xi_{34i}^3 + \ll 21 \gg = 0 \right\}
\end{aligned}$$

Out[]=

$$\begin{aligned}
& \{ f_{\theta,x_{12}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{12i} + e^{-\frac{1}{2}\gamma} (2\alpha_{1i}-\alpha_{2i}) \lambda \xi_{12j}], \\
& f_{\theta,x_{23}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{23i} + e^{\frac{1}{2}\gamma} (\alpha_{1i}-2\alpha_{2i}+\alpha_{3i}) \lambda \xi_{23j}], \\
& f_{\theta,x_{13}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{13i} + e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{2i}-\alpha_{3i}) \lambda \xi_{13j} - e^{-\frac{1}{2}\gamma} (2\alpha_{1i}-\alpha_{2i}) \lambda^2 \xi_{12j} \xi_{23i}], \\
& f_{\theta,x_{34}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{34i} + e^{\frac{1}{2}\gamma} (\alpha_{2i}-2\alpha_{3i}) \lambda \xi_{34j}], \\
& f_{\theta,x_{24}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{24i} + e^{\frac{1}{2}\gamma} (\alpha_{1i}-\alpha_{2i}-\alpha_{3i}) \lambda \xi_{24j} - e^{\frac{1}{2}\gamma} (\alpha_{1i}-2\alpha_{2i}+\alpha_{3i}) \lambda^2 \xi_{23j} \xi_{34i}], \\
& f_{\theta,x_{14}} \rightarrow \text{Function}[\{\lambda\}, \\
& \lambda \xi_{14i} + e^{\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{3i}) \lambda \xi_{14j} - e^{-\frac{1}{2}\gamma} (2\alpha_{1i}-\alpha_{2i}) \lambda^2 \xi_{12j} \xi_{24i} - e^{-\frac{1}{2}\gamma} (\alpha_{1i}+\alpha_{2i}-\alpha_{3i}) \lambda^2 \xi_{13j} \xi_{34i}], \\
& f_{1,x_{12}} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x_{23}} \rightarrow \text{Function}[\{\lambda\}, 0], \\
& f_{1,x_{13}} \rightarrow \text{Function}[\{\lambda\}, -e^{-\frac{1}{2}\gamma} (2\alpha_{1i}-\alpha_{2i}) \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i}], f_{1,x_{34}} \rightarrow \text{Function}[\{\lambda\}, 0],
\end{aligned}$$

$$\begin{aligned}
f_{1,x24} &\rightarrow \text{Function}\left[\{\lambda\}, -e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{23j} \xi_{34i}\right], \\
f_{1,x14} &\rightarrow \text{Function}\left[\{\lambda\}, -e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{24i} - e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{34i}\right], \\
f_{1,x12 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i}\right], \\
f_{1,x12 \times 13} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13i}\right], \\
f_{1,x12 \times 24} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{24i}\right], \\
f_{1,x12 \times 14} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{14i}\right], \\
f_{1,x13 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{23i}\right], f_{1,x13^2} \rightarrow \text{Function}[\{\lambda\}, 0], \\
f_{1,x23 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{23j} \xi_{34i}\right], \\
f_{1,x13 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{34i}\right], \\
f_{1,x23 \times 24} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \gamma \lambda^2 \hbar \xi_{23j} \xi_{24i}\right], f_{1,x13 \times 24} \rightarrow \\
&\quad \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^3 \hbar \xi_{12j} \xi_{23i} \xi_{24i} - 2 e^{-\frac{1}{2}\gamma(3\alpha_2-2\alpha_3)} \gamma \lambda^3 \hbar \xi_{13j} \xi_{23j} \xi_{34i}\right], \\
f_{1,x14 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, -4 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{24i}\right], \\
f_{1,x13 \times 14} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{14i} + \right. \\
&\quad \left. 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^3 \hbar \xi_{12j} \xi_{14i} \xi_{23i} + 2 e^{-\frac{1}{2}\gamma(3\alpha_1-\alpha_3)} \gamma \lambda^3 \hbar \xi_{12j} \xi_{13j} \xi_{24i}\right], \\
f_{1,x24 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{\frac{1}{2}\gamma(\alpha_1-\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{24j} \xi_{34i}\right], \\
f_{1,x14 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_3)} \gamma \lambda^2 \hbar \xi_{14j} \xi_{34i}\right], \\
f_{1,x24^2} &\rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x14 \times 24} \rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_3)} \gamma \lambda^2 \hbar \xi_{14j} \xi_{24i} + \right. \\
&\quad \left. 2 e^{-\gamma\alpha_2} \gamma \lambda^3 \hbar \xi_{14j} \xi_{23j} \xi_{34i} + 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^3 \hbar \xi_{13j} \xi_{24i} \xi_{34i}\right], \\
f_{1,x14^2} &\rightarrow \text{Function}[\{\lambda\}, 0], f_{2,x12} \rightarrow \text{Function}[\{\lambda\}, 0], f_{2,x23} \rightarrow \text{Function}[\{\lambda\}, 0], \\
f_{2,x13} &\rightarrow \text{Function}\left[\{\lambda\}, -\frac{1}{2} e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{23i}\right], f_{2,x34} \rightarrow \text{Function}[\{\lambda\}, 0], \\
f_{2,x24} &\rightarrow \text{Function}\left[\{\lambda\}, -\frac{1}{2} e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{23j} \xi_{34i}\right], \\
f_{2,x14} &\rightarrow \text{Function}\left[\{\lambda\}, -\frac{1}{2} e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{24i} - \frac{1}{2} e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{13j} \xi_{34i}\right], \\
f_{2,x12 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{23i}\right], \\
f_{2,x12 \times 13} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{13i} - 2 e^{-\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{23i}\right], \\
f_{2,x12 \times 24} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{24i}\right], \\
f_{2,x12 \times 14} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^2 \hbar^2 \xi_{12j} \xi_{14i} - 2 e^{-\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{24i}\right], \\
f_{2,x13 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23i}^2\right], \\
f_{2,x13^2} &\rightarrow \text{Function}[\{\lambda\}, e^{-\gamma(2\alpha_1-\alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{23i}^2], \\
f_{2,x23 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{23j} \xi_{34i}\right],
\end{aligned}$$

$$\begin{aligned}
& f_{2,x13 \times 34} \rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{13j} \xi_{34i} \right], \\
& f_{2,x23 \times 24} \rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (-\alpha_1 + 2\alpha_2 - \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{23j} \xi_{24i} - 2 e^{\gamma (\alpha_1 - 2\alpha_2 + \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{23j}^2 \xi_{34i} \right], \\
& f_{2,x13 \times 24} \rightarrow \\
& \quad \text{Function} \left[\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23i} \xi_{24i} - 4 e^{-\frac{1}{2} \gamma (3\alpha_2 - 2\alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i} \right], \\
& f_{2,x14 \times 23} \rightarrow \text{Function} [\{\lambda\}, 0], f_{2,x13 \times 14} \rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{13j} \xi_{14i} + \right. \\
& \quad \left. 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{23i} \xi_{24i} - 2 e^{-\gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{34i} \right], f_{2,x24 \times 34} \rightarrow \\
& \quad \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 + \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{24j} \xi_{34i} - 2 e^{\frac{1}{2} \gamma (\alpha_1 - 2\alpha_2 + \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{34i}^2 \right], \\
& f_{2,x14 \times 34} \rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{14j} \xi_{34i} - 2 e^{\frac{1}{2} \gamma (-\alpha_1 - \alpha_2 + \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{34i}^2 \right], \\
& f_{2,x24^2} \rightarrow \text{Function} [\{\lambda\}, e^{-\gamma (-\alpha_1 + 2\alpha_2 - \alpha_3)} \gamma^2 \lambda^4 \hbar^2 \xi_{23j}^2 \xi_{34i}^2], \\
& f_{2,x14 \times 24} \rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_3)} \gamma^2 \lambda^2 \hbar^2 \xi_{14j} \xi_{24i} - \right. \\
& \quad \left. 2 e^{\frac{1}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{24i}^2 + 2 e^{\frac{1}{2} \gamma (-3\alpha_2 + 2\alpha_3)} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i}^2 \right], \\
& f_{2,x14^2} \rightarrow \text{Function} [\{\lambda\}, e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{24i}^2 + e^{-\gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^4 \hbar^2 \xi_{13j}^2 \xi_{34i}^2], \\
& f_{2,x12^2 \times 23} \rightarrow \text{Function} [\{\lambda\}, 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{23i}], \\
& f_{2,x12^2 \times 13} \rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{13i} - \frac{2}{3} e^{\frac{3}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^3 \xi_{23i} \right], \\
& f_{2,x12^2 \times 24} \rightarrow \text{Function} [\{\lambda\}, 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{24i}], \\
& f_{2,x12^2 \times 14} \rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j}^2 \xi_{14i} - \frac{2}{3} e^{\frac{3}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^3 \xi_{24i} \right], \\
& f_{2,x12 \times 23^2} \rightarrow \text{Function} [\{\lambda\}, 2 e^{\frac{1}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23i}^2], f_{2,x12 \times 13 \times 23} \rightarrow \\
& \quad \text{Function} \left[\{\lambda\}, -4 e^{\frac{1}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13i} \xi_{23i} - 4 e^{-\frac{1}{2} \gamma (3\alpha_1 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13j} \xi_{23i} \right], \\
& f_{2,x12 \times 13^2} \rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13i}^2 + \frac{2}{3} e^{\frac{3}{2} \gamma (-2\alpha_1 + \alpha_2)} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^3 \xi_{23i}^2 \right], \\
& f_{2,x12 \times 23 \times 34} \rightarrow \text{Function} [\{\lambda\}, 0], f_{2,x12 \times 13 \times 34} \rightarrow \text{Function} [\{\lambda\}, 0], f_{2,x12 \times 23 \times 24} \rightarrow \\
& \quad \text{Function} [\{\lambda\}, 4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23i} \xi_{24i} - 4 e^{-\frac{1}{2} \gamma (\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{23j} \xi_{24i}], \\
& f_{2,x12 \times 13 \times 24} \rightarrow \text{Function} [\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13i} \xi_{24i} - \\
& \quad 4 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{23i} \xi_{24i}], f_{2,x12 \times 14 \times 23} \rightarrow \\
& \quad \text{Function} [\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{14i} \xi_{23i} - 8 e^{-\frac{1}{2} \gamma (3\alpha_1 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13j} \xi_{24i}], \\
& f_{2,x12 \times 13 \times 14} \rightarrow \text{Function} [\{\lambda\}, 4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{13i} \xi_{14i} + 4 e^{-\frac{1}{2} \gamma (3\alpha_1 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \\
& \quad \xi_{13j} \xi_{14i} - 4 e^{-\gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j}^2 \xi_{14i} \xi_{23i} + \frac{4}{3} e^{-\frac{3}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^3 \xi_{23i} \xi_{24i}], \\
& f_{2,x12 \times 24 \times 34} \rightarrow \text{Function} [\{\lambda\}, 0], f_{2,x12 \times 14 \times 34} \rightarrow \text{Function} [\{\lambda\}, 0], \\
& f_{2,x12 \times 24^2} \rightarrow \text{Function} [\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{24i}^2], \\
& f_{2,x12 \times 14 \times 24} \rightarrow \text{Function} [\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{14i} \xi_{24i} - \\
& \quad 4 e^{\frac{1}{2} \gamma (-3\alpha_1 + \alpha_2 - \alpha_3)} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{14j} \xi_{24i} + 4 e^{\frac{1}{2} \gamma (-3\alpha_1 + \alpha_3)} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{24i} \xi_{34i}],
\end{aligned}$$

$$\begin{aligned}
f_{2, \times 12 \times 14^2} &\rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^3 \hbar^2 \xi_{12j} \xi_{14i}^2 + \frac{2}{3} e^{\frac{3}{2} \gamma (-2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^3 \xi_{24i}^2 \right], \\
f_{2, \times 13 \times 23^2} &\rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{23i}^2 - \frac{2}{3} e^{\frac{1}{2} \gamma (-2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{23i}^3 \right], \\
f_{2, \times 13^2 \times 23} &\rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{23i} + \frac{2}{3} e^{-\gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{23i}^3 \right], \\
f_{2, \times 13^3} &\rightarrow \text{Function} \left[\{\lambda\}, -\frac{2}{9} e^{-\frac{3}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^6 \hbar^2 \xi_{12j}^3 \xi_{23i}^3 \right], \\
f_{2, \times 23^2 \times 34} &\rightarrow \text{Function} \left[\{\lambda\}, 2 e^{\gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j}^2 \xi_{34i} \right], f_{2, \times 13 \times 23 \times 34} \rightarrow \\
&\text{Function} \left[\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{23i} \xi_{34i} + 4 e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i} \right], \\
f_{2, \times 13^2 \times 34} &\rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{34i} \right], f_{2, \times 23^2 \times 24} \rightarrow \\
&\text{Function} \left[\{\lambda\}, 2 e^{-\gamma (-\alpha_{1i} + 2 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j}^2 \xi_{24i} - \frac{2}{3} e^{\frac{3}{2} \gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{23j}^3 \xi_{34i} \right], \\
f_{2, \times 13 \times 23 \times 24} &\rightarrow \text{Function} \left[\{\lambda\}, 4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{23i} \xi_{23j} \xi_{24i} + \right. \\
&\left. 4 e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{23i} \xi_{23j} \xi_{34i} \right], f_{2, \times 13^2 \times 24} \rightarrow \\
&\text{Function} \left[\{\lambda\}, 2 e^{-\gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{23i}^2 \xi_{24i} - 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + 4 \alpha_{2i} - 3 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j}^2 \xi_{23j} \xi_{34i} \right], \\
f_{2, \times 14 \times 23^2} &\rightarrow \text{Function} [\{\lambda\}, 0], \\
f_{2, \times 13 \times 14 \times 23} &\rightarrow \text{Function} \left[\{\lambda\}, 4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{14i} \xi_{23i} + \right. \\
&\left. 8 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{24i} + 4 e^{-\frac{1}{2} \gamma (3 \alpha_{1i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{23i} \xi_{24i} \right], f_{2, \times 13^2 \times 14} \rightarrow \\
&\text{Function} \left[\{\lambda\}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j}^2 \xi_{14i} - 4 e^{-\frac{1}{2} \gamma (3 \alpha_{1i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{14i} \xi_{23i} + \right. \\
&\left. 2 e^{-\gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{14i} \xi_{23i}^2 - 2 e^{-\frac{1}{2} \gamma (4 \alpha_{1i} + \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j}^2 \xi_{24i} - \right. \\
&\left. \frac{2}{3} e^{\frac{3}{2} \gamma (-2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^6 \hbar^2 \xi_{12j}^3 \xi_{23i}^2 \xi_{24i} - \frac{2}{3} e^{-\frac{3}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j}^3 \xi_{34i} \right], \\
f_{2, \times 23 \times 34^2} &\rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (-\alpha_{1i} + 2 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{34i}^2 \right], \\
f_{2, \times 13 \times 34^2} &\rightarrow \text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{34i}^2 \right], \\
f_{2, \times 23 \times 24 \times 34} &\rightarrow \text{Function} \left[\{\lambda\}, -4 e^{\frac{1}{2} \gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{24i} \xi_{34i} - \right. \\
&\left. 4 e^{-\frac{1}{2} \gamma (-2 \alpha_{1i} + 3 \alpha_{2i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{24j} \xi_{34i} \right], f_{2, \times 13 \times 24 \times 34} \rightarrow \text{Function} [\{\lambda\}, \\
&\left. -4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{24j} \xi_{34i} - 4 e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i}^2 \right], f_{2, \times 14 \times 23 \times 34} \rightarrow \\
&\text{Function} \left[\{\lambda\}, -4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{23j} \xi_{34i} - 8 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{24i} \xi_{34i} \right], \\
f_{2, \times 13 \times 14 \times 34} &\rightarrow \text{Function} \left[\{\lambda\}, -4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{14i} \xi_{34i} - \right. \\
&\left. 4 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{14j} \xi_{34i} + 4 e^{-\frac{1}{2} \gamma (3 \alpha_{1i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{24i} \xi_{34i} \right], f_{2, \times 23 \times 24^2} \rightarrow \\
&\text{Function} \left[\{\lambda\}, 2 e^{-\frac{1}{2} \gamma (-\alpha_{1i} + 2 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{23j} \xi_{24i}^2 + \frac{2}{3} e^{\frac{3}{2} \gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{23j}^3 \xi_{34i}^2 \right], \\
f_{2, \times 13 \times 24^2} &\rightarrow \text{Function} \left[\{\lambda\}, -2 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{23i} \xi_{24i}^2 + \right. \\
&\left. 2 e^{\frac{1}{2} \gamma (\alpha_{1i} - 5 \alpha_{2i} + 3 \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j} \xi_{23j}^2 \xi_{34i}^2 \right],
\end{aligned}$$

$$\begin{aligned}
& f_{2, \times 14 \times 23 \times 24} \rightarrow \text{Function} \left[\{ \lambda \}, 4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{23j} \xi_{24i} + \right. \\
& \quad \left. 8 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{24i}^2 + 4 e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{23j} \xi_{24i} \xi_{34i} \right], \\
& f_{2, \times 13 \times 14 \times 24} \rightarrow \text{Function} \left[\{ \lambda \}, 4 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{14i} \xi_{23i} \xi_{24i} + \right. \\
& \quad 4 e^{\frac{1}{2} \gamma (-3 \alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{14j} \xi_{23i} \xi_{24i} + 4 e^{\frac{1}{2} \gamma (-3 \alpha_{2i} + 2 \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{14i} \xi_{23j} \xi_{34i} + \\
& \quad 4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + 3 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{14j} \xi_{23j} \xi_{34i} - 4 e^{\frac{1}{2} \gamma (-3 \alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j} \xi_{13j} \xi_{23i} \xi_{24i} \xi_{34i} - \\
& \quad \left. 4 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j} \xi_{13j} \xi_{23j} \xi_{24i} \xi_{34i} \right], f_{2, \times 14^2 \times 23} \rightarrow \text{Function} \left[\{ \lambda \}, 0 \right], f_{2, \times 13 \times 14^2} \rightarrow \\
& \text{Function} \left[\{ \lambda \}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{13j} \xi_{14i}^2 - 2 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{14i}^2 \xi_{23i} - \right. \\
& \quad 4 e^{-\frac{1}{2} \gamma (3 \alpha_{1i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{13j} \xi_{14i} \xi_{24i} + 2 e^{\frac{1}{2} \gamma (-5 \alpha_{1i} + \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{13j} \xi_{24i}^2 - \\
& \quad \frac{2}{3} e^{\frac{3}{2} \gamma (-2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^6 \hbar^2 \xi_{12j}^3 \xi_{23i} \xi_{24i}^2 + \frac{2}{3} e^{-\frac{3}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j}^3 \xi_{34i}^2 \left. \right], f_{2, \times 24 \times 34^2} \rightarrow \\
& \text{Function} \left[\{ \lambda \}, 2 e^{-\frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{24j} \xi_{34i}^2 - \frac{2}{3} e^{\frac{1}{2} \gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{23j} \xi_{34i}^3 \right], \\
& f_{2, \times 14 \times 34^2} \rightarrow \text{Function} \left[\{ \lambda \}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{34i}^2 - \frac{2}{3} e^{\frac{1}{2} \gamma (-\alpha_{1i} - \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{34i}^3 \right], \\
& f_{2, \times 24^2 \times 34} \rightarrow \text{Function} \left[\{ \lambda \}, 2 e^{-\gamma (-\alpha_{1i} + \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{24j}^2 \xi_{34i} + \frac{2}{3} e^{\gamma (\alpha_{1i} - 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{23j}^2 \xi_{34i}^3 \right], \\
& f_{2, \times 14 \times 24 \times 34} \rightarrow \text{Function} \left[\{ \lambda \}, 4 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{24i} \xi_{34i} + 4 e^{-\frac{1}{2} \gamma (\alpha_{2i} + 2 \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \right. \\
& \quad \left. \xi_{24j} \xi_{34i} - 4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^4 \hbar^2 \xi_{14j} \xi_{23j} \xi_{34i}^2 + \frac{4}{3} e^{-\frac{1}{2} \gamma (3 \alpha_{2i} - 2 \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j} \xi_{23j} \xi_{34i}^3 \right], \\
& f_{2, \times 14^2 \times 34} \rightarrow \text{Function} \left[\{ \lambda \}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j}^2 \xi_{34i} + \frac{2}{3} e^{\gamma (-\alpha_{1i} - \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j}^2 \xi_{34i}^3 \right], \\
& f_{2, \times 24^3} \rightarrow \text{Function} \left[\{ \lambda \}, -\frac{2}{9} e^{-\frac{3}{2} \gamma (-\alpha_{1i} + 2 \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^6 \hbar^2 \xi_{23j}^3 \xi_{34i}^3 \right], \\
& f_{2, \times 14 \times 24^2} \rightarrow \text{Function} \left[\{ \lambda \}, 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j} \xi_{24i}^2 - \frac{2}{3} e^{-\frac{1}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{12j} \xi_{24i}^3 - \right. \\
& \quad 4 e^{-\gamma \alpha_{2i}} \gamma^2 \lambda^4 \hbar^2 \xi_{14j} \xi_{23j} \xi_{24i} \xi_{34i} - 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{24i}^2 \xi_{34i} + \\
& \quad \left. 2 e^{\frac{1}{2} \gamma (\alpha_{1i} - 4 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{14j} \xi_{23j}^2 \xi_{34i}^2 - \frac{2}{3} e^{\frac{1}{2} \gamma (\alpha_{1i} - 5 \alpha_{2i} + 3 \alpha_{3i})} \gamma^2 \lambda^6 \hbar^2 \xi_{13j} \xi_{23j}^2 \xi_{34i}^3 \right], \\
& f_{2, \times 14^2 \times 24} \rightarrow \text{Function} \left[\{ \lambda \}, 2 e^{-\gamma (\alpha_{1i} + \alpha_{3i})} \gamma^2 \lambda^3 \hbar^2 \xi_{14j}^2 \xi_{24i} + \frac{2}{3} e^{-\gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^5 \hbar^2 \xi_{12j}^2 \xi_{24i}^3 - \right. \\
& \quad 2 e^{-\frac{1}{2} \gamma (\alpha_{1i} + 2 \alpha_{2i} + \alpha_{3i})} \gamma^2 \lambda^4 \hbar^2 \xi_{14j}^2 \xi_{23j} \xi_{34i} - 4 e^{-\frac{1}{2} \gamma (2 \alpha_{1i} + \alpha_{2i})} \gamma^2 \lambda^4 \hbar^2 \xi_{13j} \xi_{14j} \xi_{24i} \xi_{34i} + \\
& \quad \left. 2 e^{-\gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^5 \hbar^2 \xi_{13j}^2 \xi_{24i} \xi_{34i}^2 - \frac{2}{3} e^{-\frac{1}{2} \gamma (\alpha_{1i} + 4 \alpha_{2i} - 3 \alpha_{3i})} \gamma^2 \lambda^6 \hbar^2 \xi_{13j}^2 \xi_{23j} \xi_{34i}^3 \right], \\
& f_{2, \times 14^3} \rightarrow \text{Function} \left[\{ \lambda \}, -\frac{2}{9} e^{-\frac{3}{2} \gamma (2 \alpha_{1i} - \alpha_{2i})} \gamma^2 \lambda^6 \hbar^2 \xi_{12j}^3 \xi_{24i}^3 - \frac{2}{9} e^{-\frac{3}{2} \gamma (\alpha_{1i} + \alpha_{2i} - \alpha_{3i})} \gamma^2 \lambda^6 \hbar^2 \xi_{13j}^3 \xi_{34i}^3 \right] \}
\end{aligned}$$