Hidden Utilities and TEX

Headers

pdf

Pensieve header: Quantum sl_n following Rosso and Yamane.

pdf

Notes.
1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R-matrix for $U_\hbar(sl(N+1))$’’, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N”.
2. We call our algebra $U_q(\mathrm{sl}_{\$n})$, not $U_\hbar(\mathrm{sl}(N+1))$.
3. We dot not have a name for Rosso’s X_i and Y_j. Rosso’s $E_{\epsilon_i-\epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow$ xij. Likewise Rosso’s $\eta_{\epsilon_i-\epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow$ yij.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i.

pdf

In[∘]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
MakePDF[]
```

In[∘]:=
```
CF = Expand;
```

pdf

In[∘]:=
```
x[i_, j_] := x <> i <> j;
a[i_] := a <> i;
y[i_, j_] := y <> i <> j;
b[i_] := b <> i;
```

pdf

In[∘]:=
```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

pdf

The following ordering agrees with Rosso’s; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xayb.

pdf

In[∘]:=
```
$n = 5;
$PBWGens = Flatten@{
    Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
    Table[As[i, 1, $n - 1], a[i]],
    Table[As[i, 1, $n - 1], As[j, i + 1, $n], y[i, j]],
    Table[As[i, 1, $n - 1], b[i]]
   }
(#U = U@#) & /@ $PBWGens;
$PBWRule = Module[{k = 0}, Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, $PBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
```

Time: 0.00 seconds Out[∘]=

pdf

```
{x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3,
 a4, y12, y13, y14, y15, y23, y24, y25, y34, y35, y45, b1, b2, b3, b4}
```

pdf

In[∘]:=
```
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ι[s_Symbol] := Rest[Characters[SymbolName[s]]] /.
   {"0" → 0, "1" → 1, "2" → 2, "3" → 3, "4" → 4, "5" → 5, "6" → 6, "7" → 7, "8" → 8, "9" → 9};
```

In[∘]:=
```
{τ[x[3, 4]], ι[x[3, 4]]}
```

Time: 0.00 seconds Out[∘]=

```
{x, {3, 4}}
```

pdf

We take the reduction rules RR results from Yamane page 509:

(I) $i = m < j < n$, (II) $i < m < n < j$, (III) $i < m < j = n$,

(IV) $i < m < j < n$, (V) $i < j = m < n$, (VI) $i < j < m < n$.

$$q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{III})}.$$

$$[e_{ij}, e_{mn}] = 0 \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{II})} \bigcup C_{(\mathrm{VI})}.$$

$$[e_{ij}, e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{IV})}.$$

$$q^2 e_{ij}e_{mn} - e_{mn}e_{ij} = qe_{in} \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{V})}.$$

pdf

In[∘]:=
```
RR[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = ι@xij; {m, n} = ι@xmn;
  Which[
   i == m ∧ j < n, q^-2 U[xij, xmn], (* I *)
   i == m ∧ j == n, U[xij, xmn],
   i < m < j ∧ n < j, U[xij, xmn], (* II *)
   i < m < j ∧ n == j, q^-2 U[xij, xmn], (* III *)
   i < m < j ∧ n > j, U[xij, xmn] + (q^-2 - q^2) U[x[i, n], x[m, j]], (* IV *)
   j == m, q^2 U[xij, xmn] - q U[x[i, n]], (* V *)
   j < m, U[xij, xmn] (* VI *)
  ]
 ]
```

pdf

In[∘]:=
```
1_U = U[];
x_ ⊠ (c_. 1_U) := CF[c x]; (c_. 1_U) ⊠ x_ := CF[c x];
(a_. U[xx___, x_]) ⊠ (b_. U[y_, yy___]) := Module[{M},
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[A_, X_] := A X;
  If[x ≤ y, CF@M[a b, U[xx, x, y, yy]], U@xx ⊠ CF@M[a b, RR[x, y]] ⊠ U@yy ]]
```

In[∘]:=
```
Echo[U[x34] ⊠ U[x23]] ⊠ U[x12]
```

» $-q\,U[x24] + q^2\,U[x23, x34]$

Out[∘]=

$q^2\,U[x14] - q^3\,U[x12, x24] - q^3\,U[x13, x34] + q^4\,U[x12, x23, x34]$

In[∘]:=
```
U[x34] ⊠ Echo[U[x23] ⊠ U[x12]]
```

» $-q\,U[x13] + q^2\,U[x12, x23]$

Out[∘]=

$q^2\,U[x14] - q^3\,U[x12, x24] - q^3\,U[x13, x34] + q^4\,U[x12, x23, x34]$

In[∘]:=
```
Module[{Bas = Echo@Select[$PBWGens, τ[#] == "x" &]},
 DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    {X, Y, Z} → ((X_U ⊠ Y_U) ⊠ Z_U) - (X_U ⊠ (Y_U ⊠ Z_U))
   ]]
]
```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45}

Out[∘]=

{}

```
In[∘]:= Module[{Bas = Echo@Select[$PBWGens, τ[#] == "x" &]},
         DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
             {W, X, Y, Z} → W_U ⊠ ((X_U ⊠ Y_U) ⊠ Z_U) == W_U ⊠ (X_U ⊠ (Y_U ⊠ Z_U)) ==
               (W_U ⊠ X_U) ⊠ (Y_U ⊠ Z_U) == ((W_U ⊠ X_U) ⊠ Y_U) ⊠ Z_U == (W_U ⊠ (X_U ⊠ Y_U)) ⊠ Z_U
           ]]
        ]

     » {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45}

Out[∘]=
        {}
```